BOLYAI SOCIETY
MATHEMATICAL STUDIES, X

Conference on Finite and Infinite Sets
Budapest, pp. 1–12.

The Evolution of an Idea – Gallai's Algorithm

A. RECSKI and D. SZESZLÉR

Vera T. Sós is probably the single most influential person for orienting the research interest of many of the participants of this conference towards discrete mathematics. It is appropriate to recall that the single most influential person for orienting *her* interest towards discrete mathematics was his secondary school math teacher, Tibor Gallai who achieved outstanding results in several areas of graph theory. In this note the first forty years of the influence on VLSI design of a classic result of Gallai about the perfectness of interval graphs is described.

1. Introduction

The first classic result in the topic of VLSI (Very Large Scale Integrated) routing is probably Gallai's linear time algorithm. From a graph-theoretical viewpoint it is nothing else but an alternative proof of the fact that interval graphs are perfect.

The design of VLSI circuits is a broad area, it covers a wide range of substantially different problems that arise during the design process. One of these problems is *detailed routing* which can be formulated in the following way. Assume that the devices of the electric equipment have already been placed on the four boundaries of a rectangular circuit board. Our task is to interconnect certain given subsets (or *nets*) of the pins (or *terminals*) of these devices by wires. Wires belonging to different nets must not intersect or get closer to each other than a given distance. To this end, it is mostly assumed that the wires must go on a given 3-dimensional rectangular grid consisting of a number of planar layers, each of them parallel with the circuit board. (In the 1-layer, that is, 2-dimensional case the problem is unsolvable in most cases.) Wires can leave a layer for a consecutive one at any gridpoint. To sum it up from a graph-theoretical viewpoint, the detailed routing problem consists of finding vertex-disjoint Steiner-trees (trees with a given terminal vertex set) in a 3-dimensional rectangular grid graph. In this context, the given vertex sets of the trees are the nets and the Steiner-trees themselves are called *wires*.

Traditionally, detailed routing was considered a 2-dimensional problem because the number of layers was very small compared to the length and the width of the board. (Originally, in the ancient times of printed circuit technology there were only two layers: the two sides of the board. Later the number of layers was gradually extended to $3, 4, \dots$) Since recent technology permits more and more layers (6, 8 or even more) a 'real' 3-dimensional approach becomes reasonable. In this paper we first give a brief survey of 2-dimensional results with a special emphasis on those that use Gallai's algorithm or an idea similar to it. Then we turn our attention to 3-dimensional routing and we survey a few related results.

2. 2-dimensional Routing

2.1. Single Row Routing

Within detailed routing, the easiest special case is *single row routing*. In a single row routing problem all the terminals of each net are situated on one boundary (say, the upper boundary) of the circuit board. Hence the specification of such a problem only fixes the number of columns of the grid (the *length* of the problem). Therefore the usual formulation of single row routing is to fix the number of layers and ask for the minimum *width* routing, that is, a routing that occupies the minimum number of rows in the grid.

A solution of a (not necessarily single row) routing problem is said to belong to the *Manhattan model* if consecutive layers contain wire segments of different directions only. That is, layers with horizontal (parallel with the upper boundary) and vertical (perpendicular to the upper boundary) wire segments alternate. This notion is motivated by the fact that for certain technologies it is advantageous not to have long parallel wire segments on two consecutive layers. Therefore there are many results that provide routings in the Manhattan model. If a solution does not belong to the Manhattan model, it is said to be in the *unconstrained model.*

Gallai's algorithm solves the single row routing problem with optimal width in the 2-layer Manhattan model. Such a routing problem together with a possible solution is shown in Fig. 1. In Fig. 1 solid dots denote the terminals and sets of terminals marked with a common number form the nets. Wire segments of the two layers are denoted by continuous and dashed lines, respectively. Empty dots denote the edges of the wires that join adjacent vertices of the two layers (these are called *vias*).

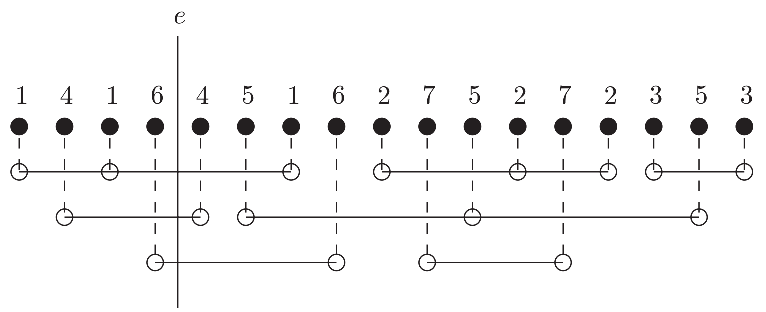

Fig. 1

For every vertical line e that cuts the grid into two we define its *congestion* $c(e)$: it is the number of nets that are divided into two by e (that is, the number of nets that have terminals both left to e and right to e). For example, the congestion of the line e in Fig. 1 is $c(e) = 3$. The maximum congestion of all vertical lines that cut the grid into two is called the *density* of the problem. It is straightforward that the density is a lower bound on the width of any routing (again, in the 2-layer Manhattan model).

Theorem 1 (T. Gallai) [3]**.** *The minimum width of a solution of a single row routing problem in the* 2*-layer Manhattan model is equal to the density of the problem.*

The proof involves a linear time algorithm (the 'left edge algorithm'). The connection between the above result and the perfectness of interval graphs is almost straightforward. A horizontal interval is associated with every net, stretching from its leftmost terminal to its rightmost terminal. The density is equal to the clique number of the corresponding interval graph. A colouring with an equal number of colours can easily be transformed into an optimal width routing: nets belonging to a common colour class can be routed in a common row. In Fig. 2 the interval graph corresponding to the routing problem of Fig. 1 is coloured using three colours; the solution of the routing problem obtained from this colouring is shown in Fig. 1.

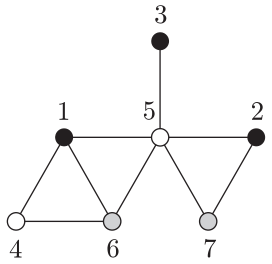

Fig. 2

We mention that no polynomial time algorithm is known to find an optimum width solution for a single row routing problem in the unconstrained 2-layer model.

2.2. Channel Routing

By the *channel routing problem* we mean the special case of detailed routing in which all the terminals of each net are situated on two opposite boundaries of the grid (say, the upper and lower boundaries). Again, the usual setting of the problem is to fix the number of layers and ask for the minimum width routing, if at all a routing exists. However, channel routing is much more complicated than single row routing as it is shown by the following theorem.

Theorem 2 (T. G. Szymanski, 1985) [12]**.** *It is NP-complete to decide whether a channel routing problem is solvable in the* 2*-layer Manhattan model with width at most* w *(where* w *is part of the input).*

Therefore it is worthwhile to look at this problem under less strict conditions: either in the 2-layer unconstrained model, or in the multilayer Manhattan model.

It is true that every channel routing problem is solvable in the 2-layer unconstrained model in polynomial time (with a sufficiently large width). This was first proved by M. Marek-Sadowska and E. Kuh [6]. Later A. Recski and F. Strzyzewski found a linear time algorithm which also uses Gallai's algorithm as a 'subroutine'.

Theorem 3 (A. Recski and F. Strzyzewski, 1990) [8]**.** *Every channel routing problem can be solved in linear time in the* 2*-layer unconstrained model.*

Their algorithm does not give an optimal width solution. The complexity of the naturally arising question of finding a minimum width routing is not known, but according to the widely accepted conjecture of D. S. Johnson [5] it is NP-hard.

It is also true that every channel routing problem is solvable in the k-layer Manhattan model for every $k \geq 3$. This again can be proved by a simple modification of Gallai's algorithm. The complexity of finding a minimum width routing is in this case known to be NP-hard [7].

2.3. Switchbox Routing

In the *switchbox routing problem* terminals may occur on all four boundaries of the circuit board. Since the specification of such a problem fixes both the length and the width of the board, the number of layers is to be optimized. We suppose that the corners of the board are not occupied by a terminal and routings must not use them either. We also suppose that the wires can access the terminals on any layer.

We have seen that in case of single row and channel routing two layers were always sufficient to solve any problem (and if we restrict ourselves to the Manhattan model, three layers were needed in case of channel routing). This, however, is not true for switchbox routing. Moreover, no fixed number of layers suffice, which is shown by the following theorem.

Theorem 4 (S. E. Hambrusch, 1985) [4]**.** *For every positive integer k there exists a switchbox routing problem that cannot be solved on k layers in the unconstrained model.*

Proof. Consider the switchbox routing problem of Fig. 3. The congestion of the line e is $n + w$, that is, each of the $n + w$ nets have terminals on both sides of e. Therefore the existence of a routing on k layers implies $n + w \leq kw$ since there are w rows on every layer. From this we have $\frac{n}{w} + 1 \leq k$. The value of n and w can be chosen such that this inequality does not hold, which proves the theorem. ■

Obviously, the background of the phenomenon involved in the above theorem is the fact that in case of switchbox routing both the length n and the width w are given by the specification. Denote the ratio $\max\left(\frac{n}{w}, \frac{w}{n}\right)$ by

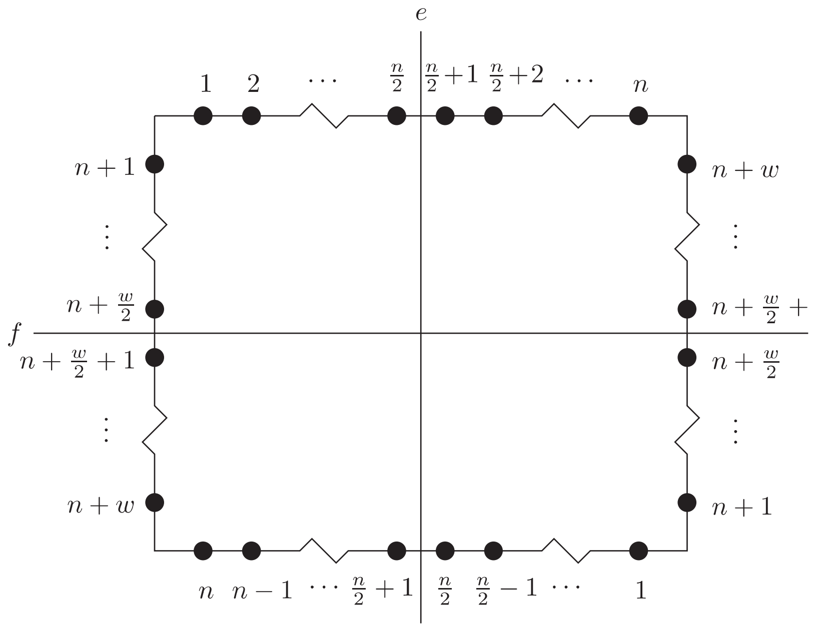

Fig. 3

m. The proof of the above theorem also includes the following statement: $\lceil m \rceil + 1$ is a lower bound on the minimum number of layers of a solution in the worst case. A slight modification of the proof shows that if we restrict ourselves to the Manhattan model then at least $2\lceil m \rceil + 1$ layers are needed in the worst case if $m > 1$ and 4 layers are needed if $m = 1$.

It is a natural question whether there is also an upper bound for the necessary number of layers as a function of m? The following theorem answers this question in the affirmative.

Theorem 5 (E. Boros, A. Recski and F. Wettl, 1995) [2]**.** *Any switchbox routing problem can be solved in linear time on at most 18 layers if* $m \leq 2$ *and on at most* $2m + 14$ *layers if* $m > 2$ *in the unconstrained model.*

Later the bounds given in the above theorem were improved. The construction of the following result can also be regarded as a generalization of Gallai's method.

Theorem 6 (D. Szeszlér, 1997) [11]**.** *Any switchbox routing problem can be solved in linear time on at most* $2\lceil m \rceil + 4$ *layers in the Manhattan model.*

3. 3-dimensional Routing

Although the solution of a switchbox problem can require arbitrarily many layers, switchbox routing can still be regarded as a 2-dimensional problem: the input consists of four sequences (the terminals on the four boundaries) and the output consists of a fixed number of planar layers (provided that the value of m defined in the previous section is fixed).

Due to the quick improvement of routing technology, research has recently turned towards 'real' 3-dimensional routing. In the *single active layer routing problem* (or *SALRP* for short) the terminals to be interconnected are situated on a rectangular planar grid of size $w \times n$ and the routing should be realized in a cubic grid of height h above the original grid that contains the terminals. Evidently, the height h is to be optimized. Henceforth we will use the term 'vertical direction' to refer to the direction of h (that is, the direction perpendicular to the $w \times n$ rectangle) and not for the direction of w.

One can easily see even in small instances like 4×1 or 2×2 that a routing is usually impossible unless either the length n or the width w may be extended by introducing extra rows or columns between rows and columns of the original grid.

By a *spacing of* s_n *in direction* n we are going to mean that we introduce $s_n - 1$ pieces of extra columns between every two consecutive columns (and also to the right hand side of the rightmost column) of the original grid. A *spacing of* s_w *in direction* w is defined analogously. This way the length of the grid is extended to $n' = s_n \cdot n$ and the width is extended to $w' = s_w \cdot w$.

A very similar argument to that of Theorem 4 provides a lower bound on the height h in the worst case.

Lemma 1. *For any given n and s_w there exists a routing problem that cannot be solved with height h smaller than $\frac{n}{2s_w}$.*

Proof. Let, for simplicity, the width and the length be even, let $w = 2a$ and $n = 2b$. Consider the following example. Suppose that each net consists of two terminals in central-symmetric position as shown in Fig. 4.

The number of nets is an. Since each net is cut into two by the central vertical line e, any routing with width $w' = s_w \cdot w$ and height h must satisfy $w'h \geq an$. Therefore $h \geq (w/2w')n$, hence $h \geq \frac{n}{2s_w}$. ■

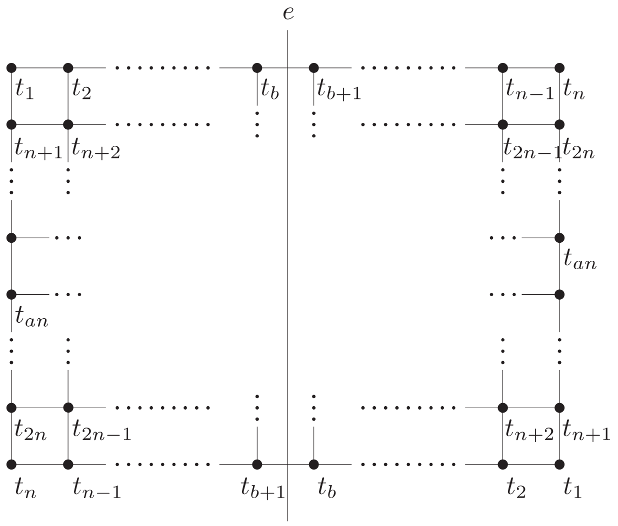

Fig. 4

The following lemma, on the other hand, provides a trivial upper bound for the height.

Lemma 2. *If $s_w \geq 2$ and $s_n \geq 2$ then every routing problem can be solved with height $h \leq \frac{wn}{2}$.*

Proof. We assign a separate layer to each net. For every terminal we introduce a vertical (parallel with the height) wire segment to connect the terminal with the layer of its net. The interconnection of the terminals of each net can now be performed trivially on its layer using the extra rows and columns guaranteed by the spacing in both directions.

Since 1-terminal nets can be disregarded, the number of nets is at most $\frac{1}{2}nw$ thus $h \leq \frac{wn}{2}$ follows immediately. ■

The above lemma is a partial explanation for the phenomenon that the nature of single active layer routing seems to depend fundamentally on whether only one of the quantities s_w and s_n is at least 2 or both of them.

3.1. The $s_n = 1$ case

An alternative interpretation of Lemma 2 is that if we fix w then there is a routing of height $h = O(n)$, provided that $s_w, s_n \geq 2$. The truth of the same statement is not at all obvious in the $s_n = 1$ case. However, the following result shows that such a statement is true if $s_w \geq 8$ holds.

Theorem 7 [9]**.** *If $s_w \geq 8$ then for any fixed value of w and for any n a single active layer routing problem can always be solved in time $t = O(n)$ and with height $h = O(n)$ such that the length n is preserved or increased by at most one. Both linear bounds are best possible.*

Our algorithm gives $t = O(w^3 n)$ and $h = O(wn)$. (The straightforward lower bound for the time is the length of the input, that is, $t = \Omega(wn)$.)

The proof of the above theorem is somewhat lengthy and highly technical. Nevertheless, the basic idea is again a 3-dimensional modification of the Gallai algorithm. This is only illustrated by the following remarks.

Suppose at first that $w = 1$. Then what we have is essentially a single row routing problem with density d. Each net determines an interval of length at most n and these intervals can be packed in a vertex-disjoint way into d parallel lines, usually called *tracks*, using the Gallai algorithm. Using the classical 2-layer Manhattan model, we can arrange the tracks in a horizontal plane, as shown in the top of Fig. 5, thus realizing a routing with $w' = d$ and $h = 2$. However, alternatively these tracks can occupy either a vertical plane, leading to $w' = 2$ and $h = d$, or two vertical planes, leading to $w' = 3$ and $h = \lceil d/2 \rceil$, see the middle and the bottom drawing of Fig. 5, respectively. (Theoretically one can pack the tracks to more vertical planes and thus ensure $h = \left\lceil 3d/(2w') \right\rceil$ for larger values of w' as well but it does not seem to be interesting.) Throughout in Figures 5 and 6 continuous lines denote wires while dotted lines are for the indication of coplanarity only.

Similarly, if $w = 2$ then we have a channel routing problem with density d and using the same linear time algorithm we can always realize a routing with $w' = d + 1$ and $h = 3$ or with $w' = 3$ and $h = d + 1$, see Fig. 6.

Actually, it is the right hand side of Fig. 6 that shows the essential idea of the proof of Theorem 7.

3.2. The $s_n, s_w \geq 2$ case

We have seen in Lemma 2 that every SALRP problem can trivially be solved with height $h = \frac{wn}{2}$ if $s_w, s_n \geq 2$. This provides an upper bound of $h = O(n^2)$ in the $n = w$ case. However, in 2000 Aggarwal et al. [1] proved the following theorem using elaborate probabilistic methods.

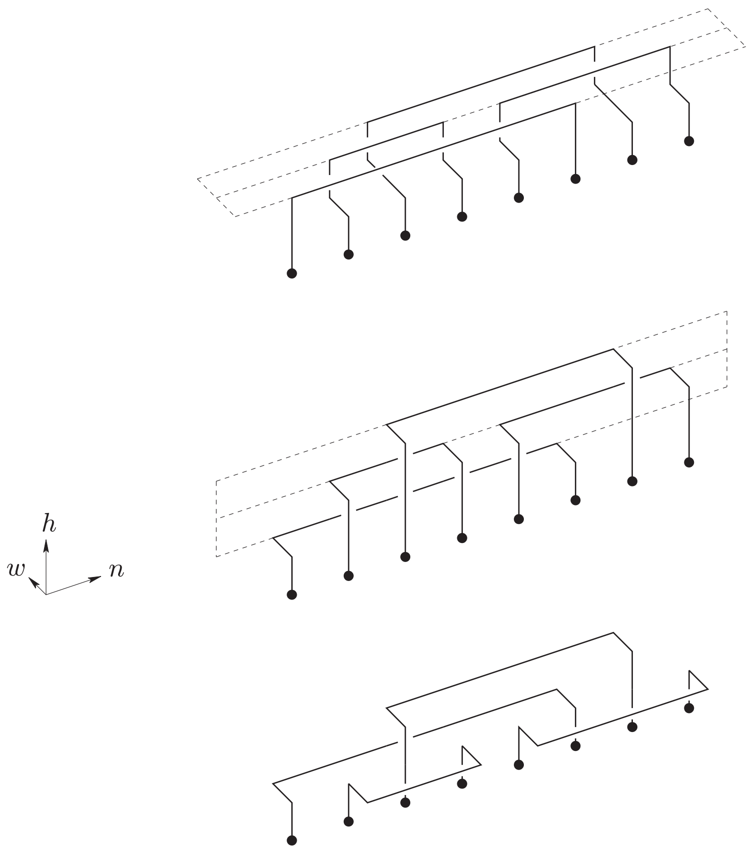

Fig. 5

Theorem 8 (A. Aggarwal, J. Kleinberg and D. P. Williamson, 2000) [1]**.** *If each net consists of two terminals only then the nets of an $n \times n$ SALRP can be partitioned into $O(n \log^2 n)$ classes such that each class of nets can be routed on a copy of the grid (of size $n \times n$).*

An easy corollary of this theorem is that if $s_w = s_n = 2$ and each net consists of two terminals only than every SALRP can be solved with height $h = O(n \log^2 n)$. The following result shows that actually $h = O(n)$ also suffices, even if multiterminal nets are also allowed.

Theorem 9 [10]**.** *Any SALRP can be solved with $s_n \geq \lceil \frac{w}{2n} \rceil + 1$, $s_w = 2$ and height $h = 6n$. Furthermore, if each net consists of two terminals only then $h = 3n$ also suffices.*

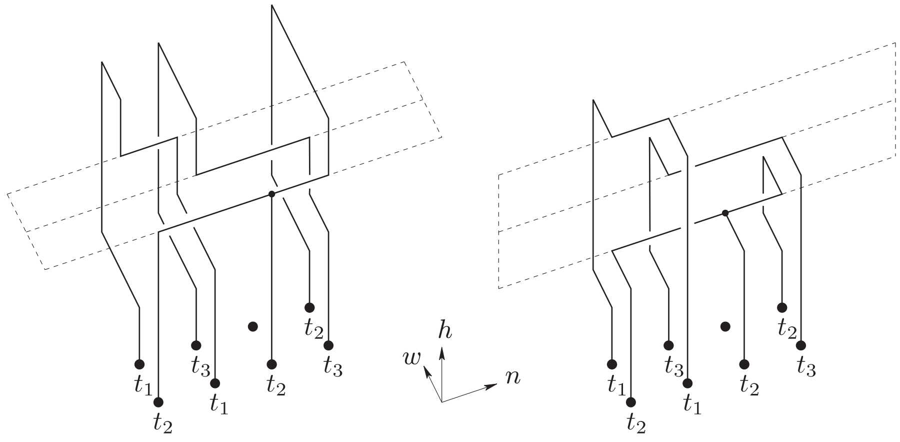

Fig. 6

Since the role of n and w is symmetric, $n \geq w$ can be assumed without loss of generality and thus we obtain the following corollary.

Corollary 1 [10]. *Any SALRP can be solved with* $s_w = s_n = 2$ *and height* $h = 6 \max(n, w)$. *If each net consists of two terminals only then* $h = 3 \max(n, w)$ *also suffices.*

The constructions of the above results can be performed by a polynomial algorithm. (If $w = \Theta(n)$ is assumed then the algorithms work in $O\big(A^{\frac{3}{2}}\big)$ time, where $A = w \cdot n$ is the size of the input.)

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András Recski

Department of Computer Science and Information Theory
Budapest University of Technology and Economics
H-1521 Budapest
Hungary

recski@cs.bme.hu

Dávid Szeszlér

Department of Computer Science and Information Theory
Budapest University of Technology and Economics
H-1521 Budapest
Hungary

szeszler@cs.bme.hu