Applications of combinatorics to statics – rigidity of grids

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1 Summary

The infinitesimal rigidity (or briefly rigidity) of a bar-and-joint framework (in any dimension) can be formulated as a rank condition of the so called rigidity matrix. If there are n joints in the framework then the size of this matrix is $O(n)$, so the time complexity of determining its rank is $O(n^3)$. But in special cases we can work with graph and matroid theoretical models from which very fast and effective algorithms can be obtained. At first the case of planar square grids will be presented where they can be made rigid with diagonal rods and cables in the squares, and with long rods and cables which may be placed between any two joints of the grid. Then we will consider the one- and multi-story buildings, and finally some other results and algorithms.

2 Planar square grids with diagonal and long rods

Let us consider a $k \times l$ square grid which consists of rigid rods and rotatable joints (in the grid points). The motions of such a planar square grid framework have a very simple description. Since the opposite rods of a square will be parallel after any planar motions, all the deformations can be described with the rotations of the rows and columns of the grid. The extra diagonal rod in Figure 1 ensures $x_2 = y_1$ (that is, the rotations of row 2 and column 1 are identical, preserving the shape of the square in their intersection).

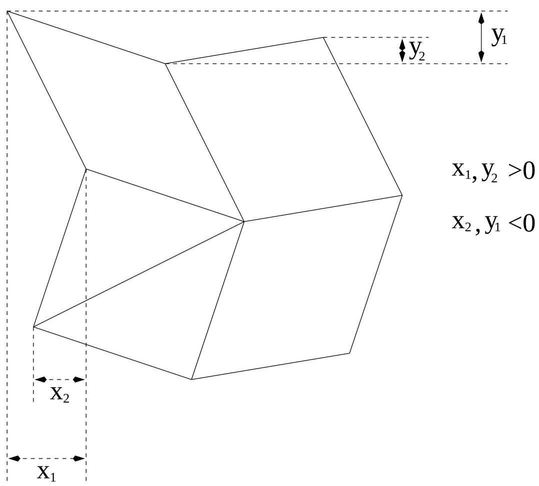


Figure 1

We can define a bipartite graph where the vertices of the graph correspond to the rows and columns, respectively, and there is an edge between two vertices if and only if there is a diagonal rod in the intersection of the corresponding row and column, ensuring that the deformations of the corresponding row and column must be equal. This graph theoretical model of square grid frameworks was given by Bolker and Crapo [3]. They proved that a planar square grid framework will be rigid if and only if the corresponding bipartite graph is connected. In this case all the x_i and y_j quantities are equal and hence any motion is necessarily a congruent motion (a rotation) of the whole framework. As a corollary one can deduce that the minimal rigid systems of diagonal rods in a $k \times l$ square grid consist of $k + l - 1$ diagonals (forming a spanning tree in the graph) and the rigidity of the system can be recognized in $O(k + l)$ time (we only have to check the connectivity of the corresponding graph). Observe that using the rigidity matrix (see Section 1) we should need $O((k + l)^6)$ time.

The reader can immediately observe the difference between a "real" deformation (like at the right hand side of Figure 4) and an "infinitesimal" one (like in Figure 8c). Throughout, rigidity will mean that even infinitesimal deformations are excluded. Most authors call this concept infinitesimal rigidity.

The natural generalization of this problem is if we use long rods in the square grid which can be placed between any two joints of the grid. The graph theoretical model does not work in this case. Long rods, parallel to the rows or columns, have no effect to the infinitesimal motions of the grid so they are ignored. The effect of a general long rod can be described with a linear equation [7]. These equations have very simple structure: they have the rotations of the rows and columns as variables and the coefficients are 0 and ± 1. In Figure 2 we show an example, but we have to emphasize that the possible motions to be prevented by the long rod are infinitesimal only. Of course "short" diagonal rods are special long rods, so if we have a $k \times l$ square grid with s pieces of long rods we have to consider the system of equations with $k + l$ variables and s equations.

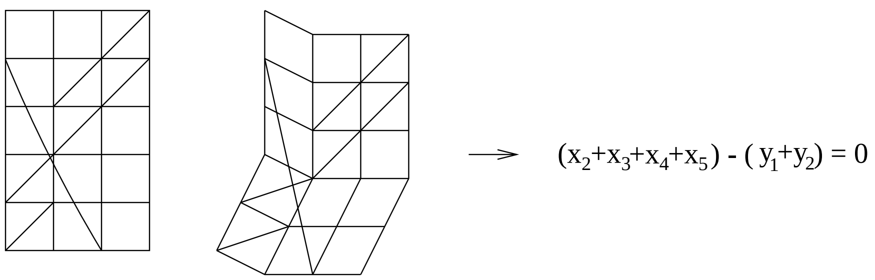


Figure 2

The planar frameworks are rigid if and only if they have only the trivial congruent motions in the plane, where the rotations of rows and columns are the same. In the $k + l$-dimensional linear space all of the hyperplanes corresponding to the equations contain this one-dimensional subspace of the trivial solutions. To obtain the line of the trivial solutions as the intersection of the hyperplanes we need at least $k + l - 1$ hyperplanes.

Theorem 2.1 *[7] The square grid framework with long rods is infinitesimally rigid if and only if the corresponding system of equations has only a one-dimensional set of solutions (the congruent motions). So it requires at least $k + l - 1$ long rods to make the square grid rigid and so many rods are sufficient. (see Figure 3)*

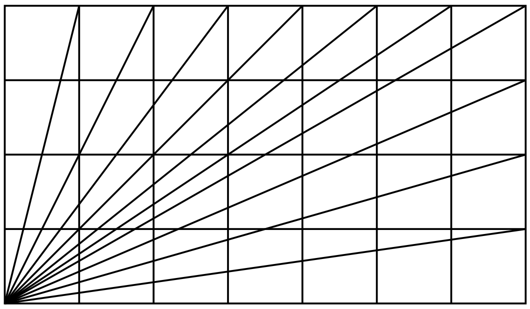


Figure 3

3 Planar square grids with diagonal and long cables

Physically realizable rods, unfortunately, are less reliable against compression than tension. If we want to model the physically constructible frameworks and wish to permit only tension in the diagonals, we have to introduce the concept of tensegrity frameworks. Here we can use three kind of elements between joints: rods (which are rigid both under tension and compression), cables (which are reliable against only tension) and struts (which are reliable against only compression). Since a diagonal cable and a diagonal strut in the opposite (that is, perpendicular) position in a square framework have the same effect, we may disregard struts in our model. In what follows, rods and cables will be drawn by continuous and by broken lines, respectively.

If we use diagonal cables in the squares the problem will be similarly very simple. A diagonal cable can prevent the deformation of the square only in one direction which means that an inequality will hold between the rotations of the corresponding row and column. Since the effect of the cable depends on its position we have to use directed edges in the graph to indicate the direction of the cable (and also the inequality). Baglivo and Graver [1] showed that the square grid framework with diagonal cables is rigid if and only if the corresponding digraph is strongly connected. (Undirected edges, indicating diagonal rods, are considered as pairs of oppositely oriented directed edges.) For example, there is no directed edge from $\{y_3, y_5, x_4\}$ to the other vertices, hence the tensegrity framework of Figure 4 is nonrigid. So we need at least

$2\max(k,l)$ diagonal cables to make the $k \times l$ grid rigid.

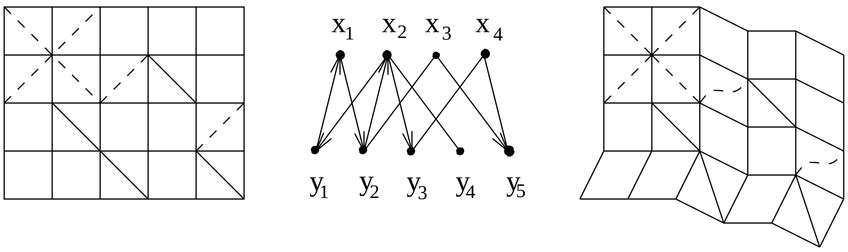


Figure 4

To describe the effect of long cables we need linear inequalities because a cable can prevent motions in one direction only.

Theorem 3.1 *[7] The square grid framework with long cables is infinitesimally rigid if and only if the corresponding system of inequalities has only a one-dimensional set of solutions (the congruent motions). So it requires at least $k+l$ long cables to make the $k \times l$ square grid rigid and so many cables are sufficient (see Figure 5).*

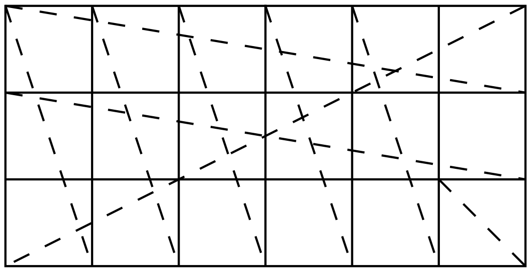
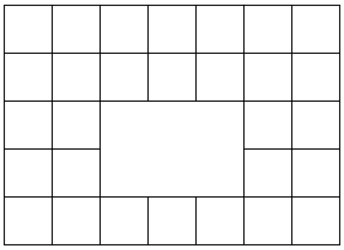


Figure 5

We can observe that using long rods instead of "short" diagonal rods the required number $k+l-1$ of rods did not change. But in the case of cables this number decreased, sometimes significantly: the required number of "short" diagonal cables (in a $k \times l$ square grid) was $2\max(k,l)$, while $k+l$ long cables are sufficient. For example in Figure 5 this number is 9 instead of 12.

4 Planar square grids with holes

So far the bases of the frameworks were complete rectangular parts of the infinite square grid. It is easy to see that using "convex" square grids, where each row and column consist of one connected sequence of squares, the previous theorems remain valid without any changes, only the number of possible rods and cables will decrease. If the square grid is not "convex" but there is no hole in it then we have to increase the number of variables introducing different variables for the independent segments of rows and columns. So the previous theorems will hold if we substitute the number of row and column segments for k and l, respectively. But what is the situation if we have hole(s) in the grid? It was shown in [8] that each hole forces two more equations, one among the rotations of the row segments and one among the column segments, respectively (see Figure 6).



$$\longrightarrow \quad x'_3+x'_4-x''_3-x''_4 = 0$$

$$\longrightarrow \quad y'_3+y'_4+y'_5-y''_3-y''_4-y''_5 = 0$$

Figure 6

This observation implies that at least

$$(\#\{\text{row and column segments}\}) - 2(\#\{\text{holes}\}) - 1$$

pieces of rods, or at least one more of cables, are required to infinitesimally rigidify the square grid framework with holes. For example the framework of Figure 7a is nonrigid (an infinitesimal deformation is indicated in Figure 7c), but changing the direction of a cable (Figure 7b) it becomes rigid.

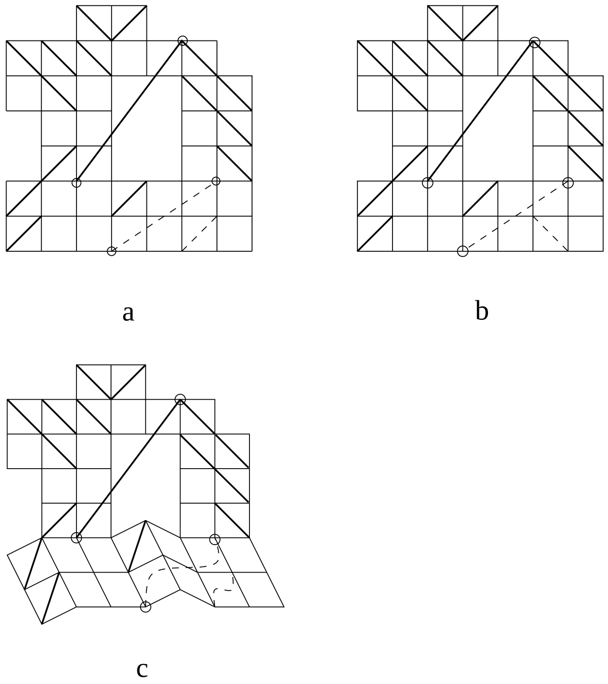


Figure 7

5 One-story buildings with diagonal and long rods

A one-story building is a square grid whose joints are connected to the ground via joints by rods of uniform length. The first observation about one-story buildings was that a diagonal rod in a vertical wall prevents the motions of the wall along its plane, so putting diagonal rods into two intersecting vertical walls the vertical rod in the intersection will be fixed. If we put four diagonal rods into the four external vertical walls then the problem can be reduced to the planar problem of a square grid with its four corners fixed to the plane. That is, the sum of the rotations of the rows and the sum of the rotations of the columns must be equal to zero.

$$x_1 + x_2 + \ldots + x_k = 0 \qquad y_1 + y_2 + \ldots + y_l = 0. \tag{1}$$

These equations result that in the case of $k \times l$-sized one-story buildings the required number of diagonal rods is $k + l - 2$ (in addition to the four rods in the vertical walls).

Theorem 5.1 *[5] The framework of a one-story building which has rods in the external vertical walls is made infinitesimally rigid by certain diagonal rods of the ceiling if and only if the corresponding bipartite graph is either connected or is an asymmetric 2-component graph. Asymmetric means that*

$$\begin{vmatrix} |V_1 \cap A| & |V_1 \cap B| \\ |V_2 \cap A| & |V_2 \cap B| \end{vmatrix} \neq 0$$

where V_1 and V_2 are the vertex sets of the two connected components of G, while A and B are the two subsets of the original bipartition of the bipartite graph.

For example Figure 8 shows two square grids and the corresponding 2-component forests. The ratios of rows and columns in the components of the first graph are 3:3 and 3:6, respectively, the graph is asymmetric, so the first framework is rigid. In the second graph these ratios are 2:3 and 4:6 which are the same, the second graph is symmetric and we can see a deformation of the framework in Figure 8c. (It is easy to see that these systems make the framework rigid if and only if, after a suitable permutation among the rows and after another one among the columns to make the grid "block-diagonal", the three special points of the grid u, v and w, determined by the two components of the forest, are not collinear.) One can prove that the asymmetric 2-component forests of a graph form the base set of a matroid, this result can be generalized to non-bipartite graphs as well [17].

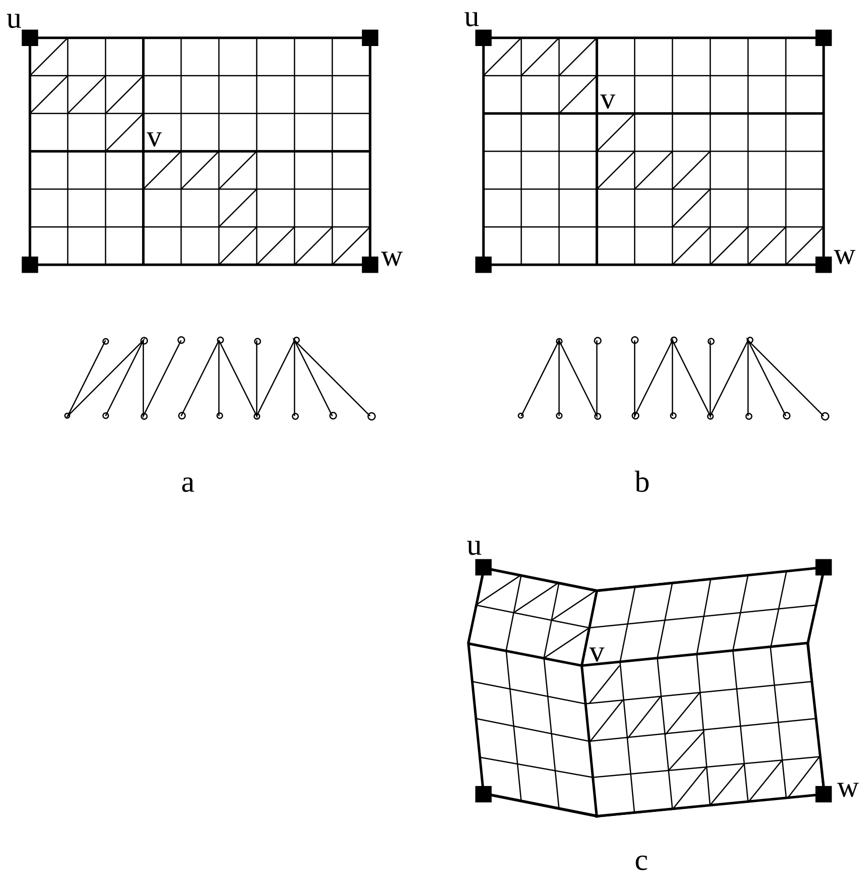


Figure 8

In the general case of one-story buildings it is easy to see that at least three vertical walls, not all parallel, must contain diagonal rods. In such a good situation we have to consider only the horizontal motions of the joints because preventing the horizontal motions the framework will be rigid (the joints cannot move vertically). In this case the effect of a diagonal rod in a vertical wall can be described with an equation where the variables are the rotations of rows and columns and two further auxiliary variables:

Theorem 5.2 *[14] Let us consider a $k \times l$ -sized one-story building with some diagonal rods of certain (horizontal or vertical) squares. The building is infinitesimally rigid if and only if the rank of the coefficient matrix of the corresponding system of equations is $k + l + 2$. Hence this is the minimum number of diagonal rods to make the building rigid.*

The minimal rigid systems of diagonal rods form the base set of a matroid (the coefficient matrix of the equation system is a representation). In the simplest case, where $k = l = 1$ this matroid is isomorphic to the cycle matroid of the graph C_5 (cycle of length 5). In general the matroids have the following property:

Theorem 5.3 *[14] The matroid of a $k \times l$ -sized one-story building is binary if and only if $k = l = 1$.*

Making the one-story building rigid with long rods is solved only in the special case mentioned above, when we put diagonal rods into the external vertical walls. In this case, besides the linear equations as the effect of long rods, we have to put the two additional equations (1) into the system. The one-story building will be rigid if and only if the intersection of the hyperplanes corresponding to the equations of the long rods and the pinned points consists of the origin only.

Theorem 5.4 *[7] The one-story building based on a $k \times l$ square grid is made infinitesimally rigid by (neither horizontal nor vertical) long rods if and only if the solution of the corresponding system of linear equations is unique: the zero vector.*

So we need $k + l - 2$ long rods to make the building rigid (of course in addition to the rods in the four vertical walls).

6 One-story buildings with diagonal and long cables

The general problem of the one-story buildings with diagonal cables seems to be much more difficult than the problem with diagonal rods (see the previous section). Only the special case of the building with four external vertical walls braced with diagonal rods was solved, but the minimal rigid systems of diagonal cables have various structures.

The first observation is that the difference between the required numbers of diagonal cables or rods is not so big as it was in the planar case ($2\max(k, l)$ versus $k + l - 1$).

Theorem 6.1 *[4] Let $k, l \geq 2$ and $k+l \geq 5$. Then at least $k+l-1$ diagonal cables are required to make the one-story building (with braced vertical walls) infinitesimally rigid, and that number will always do.*

Let us presume that the bipartite graph $G(A, B)$ of the system of cables is connected (where A and B are the vertex classes corresponding to the rows and columns, respectively). Then there is a necessary and sufficient condition for the graph to make the one-story building rigid:

Theorem 6.2 *[19] Let the graph $G(A, B)$ of the cables be connected. Then this system of cables makes the one-story building infinitesimally rigid if and only if*

$$|N^*(X)| \cdot k > |X| \cdot l$$

for all proper subsets X of A or

$$|N^*(Y)| \cdot l > |Y| \cdot k$$

for all proper subsets Y of B, where $N^*(Z)$ denotes the set of vertices in the other vertex class which can be reached with directed paths from Z.

There is an interesting question implied by this theorem: what kind of cable systems can occur as minimal rigid cable systems. The first special case is when we prescribe that all the cables must be parallel. Then we have a simple necessary and sufficient condition:

Theorem 6.3 *[18] Consider a system of $k+l-1$ diagonal cables in the $k \times l$ square grid where the corners are pinned down, and suppose that all the diagonals are parallel. Let $G(A, B)$ be the corresponding bipartite graph. Then the system makes the grid infinitesimally rigid if and only if $|N(X)| > \frac{l}{k}|X|$ holds for every proper subset X of A, where $N(X)$ denotes the set of those vertices of B which are adjacent to at least one vertex of X.*

Recall that a bipartite graph, with bipartition subsets A, B of cardinalities k and l, respectively, has a perfect-matching if and only if $k = l$ and $|N(X)| \geq |X|$ holds for every subset X of A. This theorem of Hall has a strengthening which is in complete formal analogue of the condition of Theorem 6.3, namely that every edge of a connected bipartite graph is contained in some perfect matching if and only if $k = l$ and $|N(X)| > |X|$ holds for every proper subset X of A (see Hetyei [9]).

On the other hand, the graph of a minimal system, since the required number of cables is $k+l-1$, can be a directed tree. The natural questions are: which trees can also be the graphs of a minimal system, and are there any other possibilities as well? The answer to the second question is in the affirmative (i.e. the graph need not be a tree) but there is only one exception:

Theorem 6.4 *[18] Consider a system of $k+l-1$ diagonal cables in the $k \times l$ square grid where the corners are pinned down. Suppose the corresponding graph is not a tree. Then the system of cables makes the grid infinitesimally rigid if and only if $k-l=\pm 1$ and the corresponding graph consists of an isolated vertex and a directed circuit with $2\min(k,l)$ vertices (like in Figure 9).*

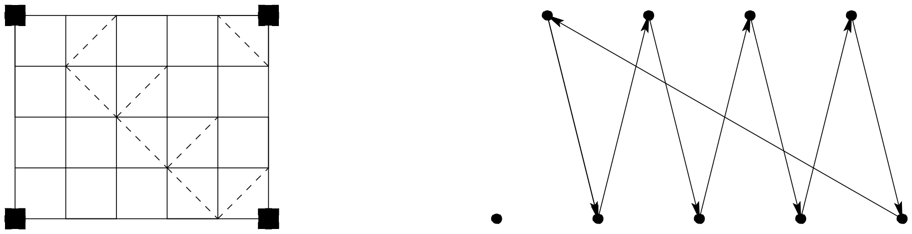


Figure 9

As we saw the graph of most of the rigid minimal systems of cables is a directed tree. Let us consider the reverse problem: if we have an undirected tree as the graph of the cables (we have information only about the squares in which the cables are placed but the positions of the cables are unknown) is there a good orientation of the edges (and that of the cables in the squares) which makes the one-story building rigid? There is a simple characterization of these "rigid" trees. Call an edge e of the tree F *critical*, if $F-\{e\}$ is a symmetric 2-component forest (cf. Theorem 5.1).

Theorem 6.5 *[19] F has a rigid orientation if and only if F has no critical edges. The rigid orientation, if it exists, is essentially (up to inversion of the whole orientation) unique.*

The necessity of the condition is obvious: the remark about the collinearity of the points u, v and w following Theorem 5.1 shows that if $F-\{e\}$ is symmetric then a single cable cannot prevent a deformation like that of Figure 8c.

In [19] we can find an algorithm which provides a good orientation if it exists: Let V_1, V_2 denote the vertex sets of the two connected components of $F - \{e\}$. The edge e has tail in V_i and head in V_j if and only if $k_i l_j > k_j l_i$, where k_i denotes the number of vertices in V_i which correspond to rows of the grid, and l_i denotes the number of vertices in V_i which correspond to columns.

7 Results in 3-dimension

The 3-dimensional problem of cubic grids is much more difficult than the planar problem, we have no such effective methods in the space as the graph theoretic model was in the plane. But the problem is very interesting so partial results have been already published about rigidity of special cubic grids [10] or general observations about d-cube grids [12]. However, the problem of the t-story buildings with diagonal rods, as the simplest 3-dimensional case is solved.

The description of the one-story building can easily be generalized to the case of higher buildings. It is easy to see that in each floor at least three vertical walls, not all parallel, must contain diagonal rods, hence we have to consider only the horizontal motions of the floors. Describing the effect of a diagonal rod we will obtain similar linear equations as in the case of one-story buildings [14]. In this system of equations the number of variables are $t(k+l+2)$ (in the case of a $k \times l$-sized t-story building), while the number of equations is equal to the number of diagonal rods.

Theorem 7.1 *[14] A $k \times l$ -sized t-story building is infinitesimally rigid if and only if there are at least three vertical walls braced in each floor and these are not all parallel, and the system of equations obtained from the rods has the zero vector as the only solution.*

This means that we need at least $t(k + l + 2)$ diagonal rods to make a $k \times l$ -sized t-story building rigid.

Similarly to the one-story case we can define the matroid of a t-story building. Of course this matroid will be a representable matroid but from Theorem 5.3 it is obvious that such a matroid can be graphic only if $k = l = 1$ (that is, for a $1 \times 1 \times t$ building). However, in this case the structure of these matroids is very simple:

Theorem 7.2 *[14] The matroid of a $1 \times 1 \times t$ building is always graphic and the corresponding graph is a "chain" of a pentagon (corresponding to the first floor) and $t-1$ pieces of hexagons, like in Figure 10.*

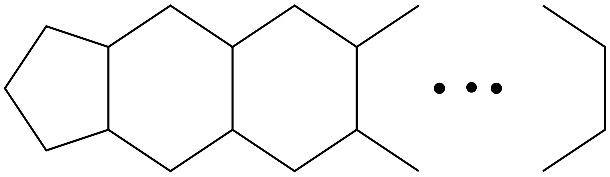


Figure 10

8 Other gridlike structures

It is easy to see that all of the results in Section 2 through 4 are almost the same if we have a planar grid of parallelograms [20]. The only changes will arise when we construct the linear equations or inequalities because the coefficients depend on the size of the paralelograms.

The graph theoretical method can be applied for other gridlike structures as well, see [11] for the Archimedian semiregular grids in the plane. The semiregular grids (33344) and (33434) are isomorphic to square grids with diagonal rods of certain squares (see Figure 11) so we can use the graph theoretical method of square grids to make these semiregular grids rigid. But if there are hexagons or larger faces in the grid then this method can work only with special further assumptions about the motions of the rods.

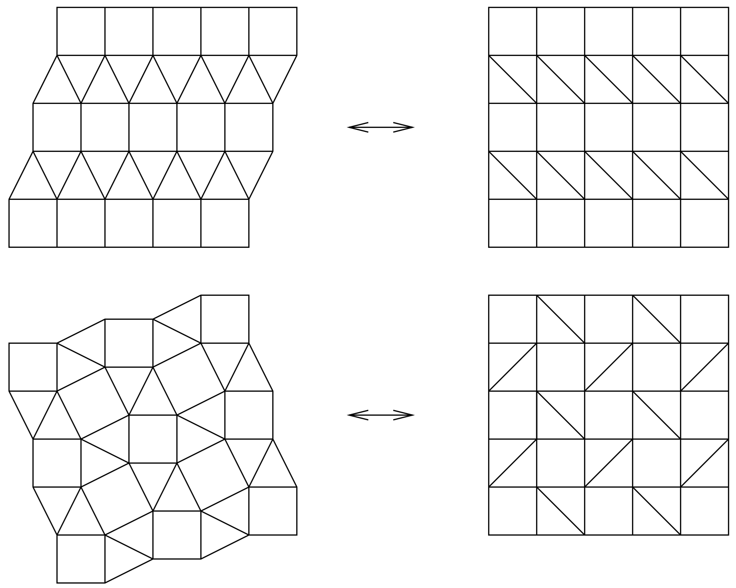


Figure 11

9 Algorithmic aspects

As it was mentioned, using the rigidity matrix the number of operations required to determine the rigidity of a framework is proportional to the cube of the number of joints, hence can be bounded from above by $c \cdot n^q$ where $n = k + l$ ($k + l + t$ in Section 7) and c, q are constants. But using the new results all the above problems require less operations. The decrease is significant in every case except for the grid with holes: if we have a large number of holes, the new method decreases only the coefficient c in the bound for the number of operations, as compared to the original method. In all the other cases the exponent q is reduced in the upper bound.

In all cases the set of minimal rigid systems form the base set of a matroid. It means that the greedy algorithm can work in these cases, so we can make these types of frameworks rigid even if we have special requirements about placing the rods or cables. For example, if we wish to find a minimum system of diagonals to make a system rigid then the "user" may specify priorities for certain diagonals.

Another problem is how can one extend a given set of diagonal rods or cables in a planar square grid to obtain a rigid system or to increase the reliability of the system (if it is not only rigid but also remains rigid if one diagonal is "broken"). [2] contains a linear time algorithm for the connectivity augmentation problem for graphs with special requirements. A special case of this problem – augmenting the connectivity of bipartite graphs while preserving bipartiteness – gives a linear time algorithm for our problem. The analogous problem concerning strong connectivity of digraphs was solved in [6]. Their algorithm solves, still in linear time, the problem of completing square grid frameworks with diagonal cables.

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