On the generalization of the matroid parity problem

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Abstract. Let $T_1, T_2, \dots, T_t$ be disjoint k-element sets and let their union be denoted by S. Let $\mathcal{A}$ be a subset of $\{0, 1, 2, \dots, k\}$. For an integer $0 \le c \le t$, a subset $X \subseteq S$ is called $(\ge c)$-*legal* if $|X \cap T_i| \in \mathcal{A}$ holds for at least c subscripts and it is called c-*legal* if $|X \cap T_i| \in \mathcal{A}$ holds for exactly c subscripts. Let $\mathcal{M}$ be a matroid on S. In this paper we study problems like "Does there exist a $(\ge c)$-legal (or c-legal) independent set of given cardinality in $\mathcal{M}$?" Observe that if $\mathcal{A} = \{0, k\}$ and $c = t$ then both problems reduce to the matroid k-parity problem (in particular, to the classical matroid parity problem for $k = 2$). The problems have some motivations from engineering applications and are also related to the more recent theory of jump systems.

Mathematics Subject Classification (2000). Primary 05B35; Secondary 90C27, 05C70.

Keywords. Matroid parity.

1. Preliminaries

For brevity, we denote the set $\{0, 1, \dots, k\}$ by $[k]$. The problems stated in the abstract are trivial in the case when $\mathcal{A} = \emptyset$ and when $\mathcal{A} = [k]$. Hence, in what follows, we suppose that $\mathcal{A}$ is a proper subset of $[k]$ and we denote its complement by $\mathcal{B}$. Throughout, α and β denote the smallest elements of $\mathcal{A}$ and $\mathcal{B}$, respectively.

The two questions (existence of a $(\ge c)$-legal or a c-legal subset) may or may not be accompanied with a cardinality constraint. If the cardinality of the independent set may be arbitrary, we shall speak about the *weak version* of the problem while the *strong version* means that legal independent sets of a given size p are requested.

The complexity of the problems may also depend on the way how the matroid $\mathcal{M}$ is given. For example, the strong version of the existence of a t-legal set for $k = 2$ and $\mathcal{A} = \{0, 2\}$ (i.e. the classical matroid parity problem) is in **P**, i.e. is of polynomial time complexity if $\mathcal{M}$ is a *linear* matroid, and a representation with subspaces of a vector space over a field is given [Lo81], but is non-polynomial if $\mathcal{M}$ is given by an independence oracle [JeKo, Lo81]. Here we concern complexity issues so if $\mathcal{M}$ is represented over the field K then we tacitly assume that it is possible to perform elementary computations with the elements of K.

Hence for each prescription $\mathcal{A} \subseteq [k]$ we have the problem in 8 variations. Namely, given a matroid $\mathcal{M}$ by an independence oracle (or by a linear representation if $\mathcal{M}$ is linear) with ground set $S = T_1 \dot\cup \ldots \dot\cup T_t$, $|T_i| = k$, together with integers c and p, decide if there exists a ($\geq c$)-legal (or c-legal) independent set of arbitrary size (or of size p).

In what follows, full solution of six variations will be presented. If the matroid is given with a linear representation then we have only partial results for the strong version.

Remark 1.1. Although the questions concerning ($\geq c$)-legal or c-legal sets are clearly related, we shall consider them separately. Note that for a given $\mathcal{A}$, the complexity of the ($\geq c$)-legal version is at most t times the complexity of the c-legal version, hence if the latter is polynomial so is the former. The other implication is certainly false, see Statements 3.1 and 3.2 below for an example.

Similarly, for a given $\mathcal{A}$, the complexity of the weak version is at most $|S|$ times the complexity of the strong version.

If E is a set of edges in a graph then the set of vertices covered by E is denoted by $V(E)$. We will need the following observation.

Lemma 1.2. *If $\mathcal{M}$ is a linear matroid given by its representation on the ground set S and E is a set of undirected edges on the vertex set S, then we can find in polynomial time a matching $M \subseteq E$ of maximum size with the property that $V(M)$ is independent in $\mathcal{M}$.*

Proof. For each $v \in S$ replace v by $\deg_E(v)$ parallel elements such that E becomes a perfect matching on the new ground set S'. Note that the representation of the new matroid $\mathcal{M}'$ can easily be produced. For all new edges $uv = e \in E$ let $U_e = \{u, v\}$. By the matroid parity algorithm of Lovász [Lo81] we can find in polynomial time an independent set of $\mathcal{M}'$ of maximum size which is the union of some sets U_e. Such an independent set clearly corresponds to a matching M with the required property. $\square$

Finally, throughout the paper, we define the matroid $\mathcal{N}$ to be the partitional matroid [Re74] on S with $I \subseteq S$ independent if and only if $|I \cap T_i| \leq 1$ holds for all i.

2. The "weak version" of the problems

In this section we study the complexity of the problem formulated in Section 1 without the cardinality constraint. Recall that $\alpha = \min \mathcal{A}$ and $\beta = \min \mathcal{B}$. Note that the weak version is equivalent to the following: given a matroid $\mathcal{M}$ and an integer c, decide if there exists an independent set I of $\mathcal{M}$ with at least c (or exactly c in the c-legal case) subsets T_i with $|I \cap T_i| = \alpha$ and at most (or exactly) $t - c$ subsets T_i with $|I \cap T_i| = \beta$. Hence if $\alpha \geq 1$, both the c-legal and the $(\geq c)$-legal versions are equivalent to deciding if there exists an independent set I of $\mathcal{M}$ intersecting exactly c of the T_i's each in exactly α elements.

Hence our problem is closely related to the *matroid l-parity problem*, which is the following. Given a matroid $\mathcal{M}'$ on ground set $S' = U_1 \dot\cup \ldots \dot\cup U_t$ with $|U_i| = l$, determine the maximum size of an independent set I of $\mathcal{M}'$ with the property that for all $1 \leq i \leq t$ either $U_i \subseteq I$ or $U_i \cap I = \emptyset$ holds. $\mathcal{M}'$ may be given either by an independence oracle, or by a linear representation if it is linear. The case $l = 2$ is the classical matroid parity problem. Now we show that the matroid l-parity problem reduces to the weak version of our problem with respect to an arbitrary prescription $\mathcal{A}$ with $l = \alpha \geq 1$. First, add as a direct sum $t \cdot (k - l)$ loops to $\mathcal{M}'$ resulting in the matroid $\mathcal{M}$ on ground set S, and let the partition $S = T_1 \dot\cup \ldots \dot\cup T_t$, $|T_i| = k$ such that $U_i \subseteq T_i$ holds for $1 \leq i \leq t$. The independence oracle (or the representation) of $\mathcal{M}$ is immediate. Moreover, the largest c for which there exists a c-legal (or $(\geq c)$-legal) independent set in $\mathcal{M}$ with respect to $\mathcal{A}$ implies the answer to this instance of the matroid l-parity problem.

Here we cite the following results. Lovász [Lo81] proved that the matroid parity problem is polynomial for linearly represented matroids, and is of exponential order for matroids given by an independence oracle (proved also in [JeKo]). On the other hand, the matroid α-parity problem for $\alpha \geq 3$, even if restricted to graphic matroids, is known [We] to contain an **NP**-complete problem (determine whether a directed Hamiltonian path with given initial and terminal points exists in a directed graph).

These considerations will be used in proving the following statements.

Statement 2.1. *For matroids given by an independence oracle the problem concerning $(\geq c)$-legal sets is in* **P** *if and only if $\alpha \leq 1$.*

Proof. First, if $\alpha = 0$ then $\emptyset$ is always a $(\geq c)$-legal independent set. If $\alpha = 1$ then a $(\geq c)$-legal subset (of arbitrary size) which is independent in $\mathcal{M}$ exists if and only if there exists a common independent set of $\mathcal{M}$ and $\mathcal{N}$ with cardinality c, with the matroid $\mathcal{N}$ defined in the end of Section 1. This condition can be checked in polynomial time for any pair of matroids given by independence oracles. Finally, by our above observation, the case $\alpha \geq 2$ is not in **P**, because the matroid α-parity problem can be reduced to it. $\square$

Statement 2.2. *For represented linear matroids the problem concerning $(\geq c)$-legal sets is in* **P** *if $\alpha \leq 2$ and is* **NP***-complete for $\alpha \geq 3$.*

Proof. For the case $\alpha \leq 1$ we refer to the proof of Statement 2.1.

In case $\alpha = 2$ we can make the following reduction to the matroid parity problem. Add as a direct sum $2t$ coloops v_i', v_i'', $1 \leq i \leq t$, to $\mathcal{M}$ resulting in a matroid $\mathcal{M}'$ on ground set S'. Note that the representation of $\mathcal{M}'$ can easily be produced in linear time. Next we define a graph on vertex set S' with edge set

$E = \{v_i'v_i'' : 1 \le i \le t\} \cup \{v_i's,\ v_i''s : s \in T_i,\ 1 \le i \le t\}$. By Lemma 1.2 we can find in polynomial time a matching $M \subseteq E$ of maximum size with the property that $V(M)$ is independent in $\mathcal{M}'$. We can assume that $v_i',\ v_i'' \in V(M)$ for all $1 \le i \le t$ so it is easy to see that $|M| \ge c + t$ if and only if $\mathcal{M}$ has an independent set intersecting at least c of the T_i's each in exactly 2 elements. In other words, if and only if it has a $(\ge c)$-legal independent set.

Finally, by our above observation, the case $\alpha \ge 3$ is **NP**-complete because the matroid α-parity problem for represented matroids can be reduced to it. □

As for the c-legal version, note that a matroid has a c-legal independent set with respect to the prescription $\mathcal{A}$ if and only if it has a $(t-c)$-legal independent set with respect to the prescription $\mathcal{B}$. So the roles of $\mathcal{A}$ and $\mathcal{B}$ are symmetric, hence we may assume that $\alpha \ge 1$. Above we showed that in the case $\alpha \ge 1$ the c-legal and the $(\ge c)$-legal versions are equivalent in the weak version so the above statements imply the following results.

Statement 2.3. *For matroids given by an independence oracle the problem concerning c-legal sets is in* **P** *if and only if* $\max(\alpha, \beta) = 1$.

Statement 2.4. *For represented linear matroids the problem concerning c-legal sets is in* **P** *if* $\max(\alpha, \beta) \le 2$ *and is* **NP**-*complete for* $\max(\alpha, \beta) \ge 3$.

3. The "strong version" for matroids given by an oracle

In this section we consider the problem of deciding if there exists a $(\ge c)$-legal (or c-legal) independent set of cardinality p in the matroid $\mathcal{M}$, given the integers c, p and the matroid $\mathcal{M}$ by an independence oracle. Note that by truncating this problem is equivalent to finding a $(\ge c)$-legal (or c-legal) base of $\mathcal{M}$. Moreover, by dualizing, this problem is equivalent to finding a $(\ge c)$-legal (or c-legal) base with respect to $\mathcal{A}^{\mathsf{R}} = \{a :\ k - a \in \mathcal{A}\}$. The independence oracles for truncating and dualizing are straightforward to produce.

Statement 3.1. *For matroids given by an independence oracle the problem concerning* $(\ge c)$*-legal sets of cardinality p is in* **P** *if and only if* $\{1, 2, \dots, k-1\} \subseteq \mathcal{A}$.

Proof. An integer q is called a *gap* of $\mathcal{A}$ if $q \notin \mathcal{A}$ yet neither $\mathcal{A} \cap \{0, 1, \dots, q-1\}$ nor $\mathcal{A} \cap \{q+1, q+2, \dots, k\}$ is empty. In [Re83] it was proved that if $\mathcal{A}$ contains at least one gap then the above problem is not in **P** even in the special case $c = t$.

We can reduce the matroid α-parity problem to the case $\alpha \ge 2$ in exactly the same way as we did in the beginning of Section 2. Namely, extend U_i to T_i by adding $k - \alpha$ loops as a direct sum for all $1 \le i \le t$. Now, finding the largest c for which there exists a $(\ge c)$-legal independent set of cardinality $c \cdot \alpha$ solves the matroid α-parity problem. This is of exponential order for matroids given by an independence oracle when $\alpha \ge 2$ so this holds for our problem, too. Hence – by dualizing – we get that the case $\min \mathcal{A}^{\mathsf{R}} \ge 2$ is of exponential order as well. So $\{1, 2, \dots, k-1\} \subseteq \mathcal{A}$ is necessary for the polynomial solvability of this problem.

As for the sufficiency, we consider only the case $k \geq 2$, otherwise being trivial. For a base B of $\mathcal{M}$ let

$$s^+(B) = |\{i : T_i \subseteq B\}|,$$
$$s_-(B) = |\{i : T_i \cap B = \emptyset\}|$$

and $s^+_-(B) = s^+(B) + s_-(B)$. Moreover, let $s^+(\mathcal{M})$, $s_-(\mathcal{M})$ and $s^+_-(\mathcal{M})$ resp., denote the minima of the above values taken over all bases of $\mathcal{M}$. Note that one can determine $s_-(\mathcal{M})$ and $s^+(\mathcal{M})$ in polynomial time: $t - s_-(\mathcal{M})$ is just the maximum cardinality common independent set of $\mathcal{M}$ and $\mathcal{N}$ (defined in the end of Section 1), while $s^+(\mathcal{M}) = s_-(\mathcal{M}^*)$.

Omitting the reference to the matroid, one can observe that $s^+_- \geq s_- + s^+$. In fact, equality holds here. Though the linking property for polymatroids implies this, we include a short proof. Let B_0 be a base with $s_-(B_0) = s_-$ and let B be a base with $s^+(B) = s^+$ such that $B \cap B_0$ is maximal. We show that $s^+_-(B) = s_- + s^+$, proving the equality. Otherwise $s_-(B) > s_-$ so $B \cap T_i = \emptyset$, $B_0 \cap T_i \neq \emptyset$ holds for some T_i. For $u \in B_0 \cap T_i$ there exists $v \in B \setminus B_0$ such that $B + u - v$ is a base as well. Now $s^+(B) \leq s^+$ and $B \cap B_0$ increased, a contradiction. Hence $s^+_- = s_- + s^+$ can be determined in polynomial time as well.

First, let $\mathcal{A} = \{1, 2, \ldots, k\}$ (recall, that $\mathcal{A}$ is assumed to be a proper subset of $[k]$). Now $\mathcal{M}$ has a ($\geq c$)-legal base if and only if $s_-(\mathcal{M}) \leq t - c$. For $\mathcal{A} = \{0, 1, \ldots, k-1\}$ replace this condition with $s^+(\mathcal{M}) \leq t - c$ and for $\mathcal{A} = \{1, 2, \ldots, k-1\}$ with $s^+_-(\mathcal{M}) \leq t - c$. □

Statement 3.2. *For matroids given by an independence oracle the problem concerning c-legal sets of cardinality p is in* **P** *if and only if* $k = 1$.

Proof. By Remark 1.1 all the polynomially solvable cases satisfy $\{1, 2, \ldots, k-1\} \subseteq \mathcal{A}$. On the other hand, as we observed before Statement 2.3, the roles of $\mathcal{A}$ and $\mathcal{B}$ are symmetric in the c-legal version. So even $\{1, 2, \ldots, k-1\} \subseteq \mathcal{B}$ must hold in a polynomial case. This can happen only if $k = 1$, when the problem is trivial. □

4. The "strong version" for represented matroids

4.1. Necessary conditions for the solvability

Statement 4.1. *Deciding the existence of a* ($\geq c$)*-legal independent set of size* p *is* **NP***-complete for linearly represented matroids unless*

1. $\mathcal{A}$ *contains no adjacent gaps and*
2. $\mathcal{A}$ *intersects* $\{0, 1, 2\}$ *and* $\{k-2, k-1, k\}$.

Proof. If $\mathcal{A}$ contains adjacent gaps then even the special case $c = t$ is **NP**-complete, see [Re83].

Suppose that $\{0, 1, 2\} \cap \mathcal{A} = \emptyset$, i.e. $\alpha \geq 3$. As usual, we make a reduction of the matroid α-parity problem: extend U_i to T_i by adding $k - \alpha$ loops as a direct sum for all $1 \leq i \leq t$. Now, finding the largest c for which there exists a ($\geq c$)-legal independent set of cardinality $c \cdot \alpha$ solves the matroid α-parity problem, which is **NP**-complete for linearly represented matroids when $\alpha \geq 3$.

Suppose now that $\{k-2, k-1, k\} \cap \mathcal{A} = \emptyset$. So $\alpha' \geq 3$ with the notation $\alpha' = k - \max \mathcal{A}$. We reduce the matroid α'-parity problem of the matroid $\mathcal{M}'$ to this problem: extend U_i to T_i by adding $k - \alpha'$ *coloops* as a direct sum for all $1 \leq i \leq t$ resulting in the new matroid $\mathcal{M}$. A linear representation of $\mathcal{M}$ is straightforward to produce. Now there exists a $(\geq c)$-legal independent set of size $(t-c) \cdot \alpha' + t(k - \alpha')$ in $\mathcal{M}$ if and only if $\mathcal{M}'$ has an independent set consisting of $t - c$ of the U_i's. Hence finding the smallest such c solves the matroid α'-parity problem. □

As we noted before Statement 2.3, in the c-legal version the roles of $\mathcal{A}$ and $\mathcal{B}$ are symmetric, hence we get the following corollary.

Corollary 4.2. *Deciding the existence of a c-legal independent set of size p is* **NP**-*complete for linearly represented matroids unless*

1. *$\mathcal{A}$ and $\mathcal{B}$ contain no adjacent gaps (hence both are jump systems* [BoCu, Lo97]*) and*
2. *both $\mathcal{A}$ and $\mathcal{B}$ intersect $\{0,1,2\}$ and $\{k-2, k-1, k\}$.*

For example, if $k = 3$ then $\mathcal{A} = \{0,3\}$ and $\mathcal{A} = \{1,2\}$ are excluded by 1. and $\mathcal{A} = \{0\}$, $\mathcal{A} = \{3\}$, $\mathcal{A} = \{0,1,2\}$ and $\mathcal{A} = \{1,2,3\}$ are excluded by 2.

4.2. Sufficient conditions for the solvability

Statement 4.3. *If $\mathcal{A}$ is the set of even integers then it is polynomial to decide whether a $(\geq c)$-legal independent set of size p exists in a matroid represented over the reals, given the integers p and c.*

Proof. We may assume that $p \equiv t - c \pmod 2$. Let us extend the underlying set S of the matroid $\mathcal{M}$ by t new elements $s_1, s_2, \ldots, s_t$, and define $\mathcal{M}'$ on this extended set S' as the direct sum of $\mathcal{M}$ and the uniform matroid $\mathcal{U}_{t,t-c}$. A linear representation of $\mathcal{M}'$ over the reals can easily be produced. Next we define a graph on vertex set S' with edge set

$$E = \{s_i s_j : 1 \leq i < j \leq t\} \cup \{v s_i : v \in T_i, 1 \leq i \leq t\} \cup \{vv' : v, v' \in T_i, 1 \leq i \leq t\}.$$

By Lemma 1.2 we can find in polynomial time a matching $M \subseteq E$ of maximum size with the property that $V(M)$ is independent in $\mathcal{M}'$. Let us call such a matching *independent*.

Now we prove that the existence of a $(\geq c)$-legal independent set of size p is equivalent to the existence of an independent matching covering at least $p + t - c$ elements. One direction is clear using that $p \equiv t - c \pmod 2$. So suppose that $M \subseteq E$ is an independent matching covering at least $p + t - c$ elements. Since $\mathcal{U}_{t,t-c}$ is a direct component in $\mathcal{M}'$ we may assume that M covers exactly $t - c$ of the elements s_i. Hence deleting the edges of M incident to some s_i results in a $(\geq c)$-legal independent set I of $\mathcal{M}$ such that $|I| - p \geq 0$ is even. If $|I| > p$ then it is possible to delete either two elements from I contained in one subset T_i or two elements from two subsets T_i, T_j with $|I \cap T_i|$, $|I \cap T_j|$ odd. In this way we do not decrease the number of subscripts i with $|I \cap T_i|$ even. Hence there exists a $(\geq c)$-legal independent set of size p.

By Lemma 1.2, the existence of an independent matching covering at least $p + t - c$ elements can be decided in polynomial time by Lovász' matroid parity algorithm. $\square$

We do not know the complexity of the remaining cases, they may include both polynomial and **NP**-complete problems. We mention that deciding the existence of a c-legal (or $(\geq c)$-legal) *base* is much easier. This would be the case if $p = r(\mathcal{M})$ or if we could represent the p-truncation of the given matroid over some field in polynomial time (the complexity of such a representation is unknown). In this case we could decide the existence of a c-legal base for $\mathcal{A} = \{\text{even numbers}\}$ (and hence for $\mathcal{A} = \{\text{odd numbers}\}$) and for $\mathcal{A} = \{0, 1\}$, $k = 3$. We do not go into details.

It may be interesting to show a graph theoretic problem, which is similar in flavor. Both the degree-sequences of the subgraphs of an undirected graph, and the vectors $(|I \cap T_1|, \ldots, |I \cap T_t|)$ for all independent sets I of a matroid with ground set $S = T_1 \dot\cup \ldots \dot\cup T_t$ form a jump system [BoCu]. Hence it is tempting to formulate the related problem for degree-sequences: given an undirected graph G, an integer c and a degree-prescription $\mathcal{A} \subseteq \mathbb{N}$, does there exist a subgraph $F \subseteq E(G)$ such that $\deg_F(v) \in \mathcal{A}$ holds for at least (or exactly) c vertices v? Besides some trivially polynomial cases (e.g. when $\mathcal{A}$ consists of the odd numbers, which is an instance of the degree prescribed factor problem introduced by Lovász, see [Lo72]), one can prove by a reduction to the set cover problem that the $(\geq c)$-legal case is **NP**-complete even for the simple prescription $\mathcal{A} = \{1\}$ [Be].

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