Stable exchange of indivisible goods with restrictions

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Abstract: We survey some stable exchange problems under different restrictions and various stability concepts. Beside describing well-known complexity results from a general point of view, we present two new theorems on the NP-hardness of some basic 3-way stable 3-way exchange problems. As a relevant application, we introduce the kidney exchange problem.

Keywords: indivisible goods, stable matching, cooperative game theory, core, computational complexity, inapproximability

Introduction

Shapley and Scarf [33] described the exchange problem of indivisible goods as an NTU-game. They supposed that each agent has one good, like a house, and preferences over the other's goods. An outcome of the game is an exchange without transfers. It is in the core if there is no *blocking coalition*: some agents that would be strictly better off by simultaneously trading between each other. Shapley and Scarf proved that such a market always has nonempty core.

Another crucial starting point in the literature of NTU-games is the famous article of Gale and Shapley [10], where the stable marriage and stable roommates problems are defined. One connection between these models, that a stable matching can be considered as a pairwise stable pairwise exchange.

As a common generalization of these problems, we study the question of stable exchanges, where the length of the cycles in the exchange may be restricted, and the size of the blocking coalitions can be also bounded, independently. We consider the problem of finding weakly, strongly and super-stable exchange in the case, where ties are allowed in the preference lists. We recall the definition of $\mathscr{L}$-preferences introduced in [5]. There, the preferences of the agents are lexicographic, primarily the agents care about the goods that they recieve, secondarily they want to minimize the length of their trading cycle. Beside finding a stable solution, another natural goal is to maximize the number of agents involved in the exchange.

The trade of goods without transfer is not usual. However, in some countries the exchange of tenancies [9] or residences [35] is only allowed in this way. Another exchange-market can be developed soon on the popular idea of home-exchange, where families swap flats between each other for holiday without payments. Finally, the most serious recent application of our model is the kidney exchange. Here, to find an optimal solution by a centralized program is not just possible, but also required.

Currently, living donation is the most effective currently treatment for kidney failure. But patients needing transplants may have donors who cannot donate them because immunological incompatibility. So these incompatible patient-donor pairs may want to exchange kidneys with other pairs. Kidney exchange programs have already been established in several countries [18], [16], [21]. The most important question, what the goal of the program is. As a first priority, the most of the current models want to maximize the number of patients that receives a suitable kidney in the exchange (see [26], [27], [28], [29], [30]) by considering only the suitability of the kidneys. Some more sophisticated models [32], [3] consider the difference between suitable kidneys and try to find a solution where the sum of benefits is maximal. A third concept, introduced in [25] and developed in [5], [8], [2] require first the stability of the solution under various criteria.

In some models, the difficulty of the according problem is due to the fact that the length of the cycles in the exchanges is bounded. The reason of this is that all operations along a cycle have to be done simultaneously. So most programs allow only pairwise exchanges. Sometimes 3-way exchanges are also possible, like it is possible currently in the New England Program [21], and it may be also allowed in the national program of the USA (as it is declared to be a goal of the system in the future in the Proposal for National Paired Donation Program [34]).

1 Definitions, preliminaries

Given a simple digraph $D = (V, A)$, where V is the set of agents. Suppose that each agent has exactly one indivisible good, and $(i, j) \in A$ if the good of agent i is suitable for agent j. An *exchange* is a permutation π of V such that, for each $i \in V$, $i \neq \pi(i)$ implies $(i, \pi(i)) \in A$. We denote by $C^\pi(i)$ the cycle of π containing i. If $C^\pi(i)$ has length at least 2, then the agent is said to be *covered*.

Let each agent have preferences over the goods, that are suitable for him. These orderings can be represented by preference lists. As in an exchange π the agent i receives the good of his *predecessor*, $\pi^{-1}(i)$, the agent i prefers an exchange π to another exchange σ, if he prefers $\pi^{-1}(i)$ to $\sigma^{-1}(i)$. In the first approach, we say that an exchange π is *stable* if there is no such a blocking coalition B and a permutation σ of B, that each agent $i \in B$ prefers σ to π.

1.1 Stable exchange with ties

In this paper we consider the case of strict preferences and also the case of preferences with *ties*. Strict preference means linear ordering. If some goods are tied in a preference list of agent i, then agent i is indifferent between them. In the *stable exchange problem* (SE) we suppose that the preferences are strict. The stable exchange problem with ties is denoted by SE+T.

Here, we study three stability concepts. An exchange is *weakly stable* is there exist no such a blocking coalition B and a permutation σ of B, that each agent $i \in B$ *strictly* prefers σ to π. An exchange is *strongly stable* is there exist no such a blocking coalition B and a permutation σ of B, that one of the agent from B strictly prefers σ to π, and each other agent $i \in B$ either strictly prefers σ to π or is indifferent between them. Finally, an exchange is *super-stable* is there exist no such a blocking coalition B and a permutation σ of B, that σ is not equal to π on B and each agent $i \in B$ either strictly prefers σ to π or is indifferent between them.

Proposition 1 *Suppose that in an instance I of SE+T the exchange π is*

1/a) strongly stable, then it is also weakly stable.

1/b) super-stable, then it is also strongly stable.

Given an instance I of SE+T. If an instance I' of SE is obtainable from I by breaking the ties, then I' is a *derived* instance from I.

Proposition 2 *Let I be an instance of SE+T.*

2/a) An exchange π is weakly stable if it is stable in at least one *instance I' of SE which can be derived from I.*

2/b) An exchange π is super-stable if it is stable in every *instance I' of SE which can be derived from I.*

1.2 Stable exchange under $\mathscr{L}$-preferences

Under *$\mathscr{L}$-preferences*, an agent i prefers a permutation π to another permutation σ if either he prefers $\pi^{-1}(i)$ to $\sigma^{-1}(i)$ or he is indifferent between them, but the length of $C^{\pi}(i)$ is smaller than the length of $C^{\sigma}(i)$. This notion was defined by Cechlárová *et al.* in [5]. They called the NTU-game related to the problem of SE under $\mathscr{L}$-preferences as *kidney exchange game*. (To distinguish between $\mathscr{L}$-preferences from and the original ones, the latter will referred as *normal preferences* hereafter.)

We remark, that a similar lexicographic ordering, the $\mathscr{B}$-preference was defined earlier by Cechlárová and Romero-Medina [7]. Here, an agent i prefers a coalition C to another coalition D, if either he prefers the best member of C to the best member of D, or he is indifferent between them, but the size of C is smaller than the size of D.

Proposition 3 *Given an instance of SE+T. If an exchange π is*

3/a) (weakly) stable under $\mathscr{L}$-preferences, then it is also (weakly) stable under normal preferences.

3/b) super-stable under normal preferences, then it is aslo super-stable under $\mathscr{L}$-preferences.

1.3 Restrictions on the lengths

In some applications the length of the possible cycles is bounded by some constant l. In this case we consider an *l-way exchange problem*. On the other hand, the size of the blocking coalitions can be also restricted. We say that an exchange is *b-way stable* if there exist no blocking coalition of size at most b. Obviously, the most relevant problems due to the constants 2 and 3. If $b = l$ then a stable exchange corresponds again to a core-solution of some related NTU-game, because the possible coalitions, those that can form and those that can block, are the same.

Proposition 4 *Suppose that π is a b-way stable l-way exchange, then it is also a*

4/a) $(b-1)$*-way stable* l*-way exchange.*

4/b) b*-way stable* $(l+1)$*-way exchange.*

A 2-way exchange can be equivalently called *pairwise exchange*, that is actually a *matching* of the agents.

Proposition 5 *Given an instance of SE+T. If a* pairwise *exchange* π *is*

5/a) (weakly) stable under normal preferences, then it is also (weakly) stable under $\mathscr{L}$*-preferences.*

5/b) strongly stable under normal preferences, then it is also strongly stable under $\mathscr{L}$*-preferences.*

The weakest stability condition is the 2-way stability (or in other words, the *pairwise stability*), where no pair of agents can block a stable solution.

Proposition 6 *Given an instance of SE+T. If an exchange* π *is*

6/a) pairwise *super-stable under* $\mathscr{L}$*-preferences, then it is also* pairwise *super-stable under normal preferences.*

6/b) pairwise *strongly stable under* $\mathscr{L}$*-preferences, then it is also* pairwise *strongly stable under normal preferences.*

Corollary 7 *Given an instance* I *of SE+T. A pairwise exchange* π *is pairwise {weakly, strongly, super-} stable under* $\mathscr{L}$*-preferences if and only if* π *is* pairwise *{weakly, strongly, super-} stable under normal preferences, respectively.*

If we consider strict preferences, then some further statements can be verified.

Proposition 8 *Given an instance of SE. If an exchange* π *is*

8/a) strongly stable, then π *is also super-stable.*

8/b) strongly stable under $\mathscr{L}$*-preferences, then* π *is also super-stable under* $\mathscr{L}$*-preferences.*

Proposition 9 *Given an instance of SE.*

9/a) If a pairwise *exchange* π *is strongly stable under* $\mathscr{L}$*-preferences, then* π *is also weakly stable under* $\mathscr{L}$*-preferences.*

9/b) If an exchange π *is* pairwise *strongly stable under* $\mathscr{L}$*-preferences, then* π *is also* pairwise *weakly stable under* $\mathscr{L}$*-preferences.*

Corollary 10 *In an instance* I *of SE the same* pairwise *exchanges are {weakly, strongly, super-} stable under* $\mathscr{L}$*-preferences, and {strongly, super-} stable under normal preferences. Moreover, the same exchanges are pairwise {weakly, strongly, super-} stable under* $\mathscr{L}$*-preferences, and weakly stable under normal preferences. Thus, in case of pairwise stable pairwise exchanges, these stability concepts are equivalent if the preferences are strict.*

1.4 Problems

Beside finding a stable exchange, we may want to find such a stable solution, where the number of covered agents is maximal. We denote the problem of finding such a maximal solution for a stable exchange problem by

- MAXCOVER-SE in the basic case, (i.e. normal preferences, no ties, no restrictions)
- MAXCOVER-{W,SU,ST}-SE+T for {weakly, strongly, super-} stable exchanges with ties,
- MAXCOVER-$\mathscr{L}$SE under $\mathscr{L}$-preferences, and
- MAXCOVER-S_bE_l for b-way stable l-way exchanges.

2 Complexity results

We present some complexity results on exchange problems, matching problems and 3-way exchange problems. Some important results are collected in a table at the end of the paper. To clearify the connections, we give a reference index [Ri] to each problem contained in the table of results.

2.1 Exchange problems

Shapley and Scarf [33] showed in their paper, that there always exists a (weakly) stable exchange in an instance I of SE+T [R1]. Moreover, they showed that a stable exchange can always be found by the Top Trading Cycle (TTC) algorithm, proposed originally by Gale. Roth and Postlewaite [24] proved that the exchange obtained by the TTC algorithm is super-stable for instances of SE. Moreover, this is the only possible super-stable solution. We remark that this uniqueness holds also for strongly stable exchanges by Proposition 1/a, but obviously, not for the weakly stable exchanges. Thus here, the MAXCOVER-SE problem is nontrivial. In the case of $\mathscr{L}$-preferences Biró and Cechlárová [2] proved recently the following theorem.

Theorem 11 (Biró-Cechlárová, 2007) MAXCOVER$\mathscr{L}$-SE *is not approximable within* $n^{1-\varepsilon}$ *for any* $\varepsilon > 0$ *unless* $P = NP$.

2.2 Matching problems

To find a pairwise stable pairwise exchange in an instance of SE with complete preference lists is equivalent to finding a solution for the according *stable roommates problem* (SR). This problem was defined by Gale and Shapley [10]. They showed by an example that stable matching may not exist for an instance of SR. For the bipartite case, they gave a natural algorithm that always finds a stable matching in an instance of so-called *stable marriage problem* (SM). Irving [12] constructed the first polynomial time algorithm which determines whether a given instance of SR admits a stable matching, and if so finds one [R2].

Considering the stable roommates problem with ties (SRT), the problem of finding a (weakly) stable matching in an instance of SRT is NP-hard. This was proved first by Ronn [23]. Later, Irving and Manlove [15] verified the same result by a different proof for that more general case, where the lists can be incomplete (SRTI). According our definitions, this theorem is the following:

Theorem 12 (Ronn, 1990; Irving-Manlove, 2002) *The decision problem of finding a (weakly) pairwise stable pairwise exchange in an instance of SE+T is NP-complete [R3].*

Manlove *et al.* [20] proved that the decision problem related to finding the maximum size of weakly stable matching for a given instance of SMTI is NP-complete.

Theorem 13 (Manlove *et al.*, 2002) *The decision problem related to* MAXCOVER-S_2E_2+T *is NP-complete for bipartite graphs as well [R4].*

On the other hand, Irving [13] constructed two polynomial algorithms which determine whether a given instance of SMT admits a {strongly, super-} stable matching, and if so find one. Manlove [19] proved similar results for SMTI problems. The same questions are tractable for SRTI problems as well. This was verified by Irving and Manlove [15] in case of super-stability and by Scott [31] in case of strong-stability. The MAXCOVER problems for strongly and super-stable matchings are also solvable, since Manlove [19] proved that for a given instance of SMTI, the same agents are matched in each strongly stable matching, and similarly, the same agents are matched in each super stable-matching. Same results were proved in the roommates case by Irving and Manlove [15] for super-stable matchings and by Scott [31] for strongly stable matchings.

Recently, Irving [14] showed that the decision problem of finding a *cycle stable matching* in an instance of *cycle stable roommates problem* (SCR) is NP-complete. Moreover, the length of each possible blocking cycle is at most 3 in his construction, so he proved the following:

Theorem 14 (Irving, 2006) *The problem of finding a stable pairwise exchange in an instance of SE is NP-complete [R5]. The same result holds for* 3-way *stable pairwise exchanges [R6].*

2.3 Exchange with restricted lengths

As in some applications the 3-way exchanges are also allowed, it is worth to study these models. We present the following theorem without proof.

Theorem 15 *The decision problem of finding a 3-way stable 3-way exchange in an instance of SE is NP-complete [R7].*

As a natural generalization of the SM problem, Knuth [17] defined the *three-sided stable matching problem.* In this special coalition formation game the possible coalition are (m, w, c) triples (i.e. families) from $M \times W \times C$ (i.e. men, women, cats), where $|M| = |W| = |C| = n$, and everybody prefers beeing in a family to remaining single. Alkan [1] showed that a stable solution may not exist if the preferences of the agents can be arbitrary over the pairs from the two other sides. Moreover, Ng and Hirschberg [22] proved that the problem of determining whether a stable solution does exist is NP-complete.

Boros *et al.* [4] showed that the core of this coalitional formation game can be nonempty even if the preferences of the agents are *lexicographically cyclic* (i.e. men primarily care about women, the women primarily care about cats, and the cats primarily care about men). They raised the same question for *purely cyclic preferences*, in which the men only care about women, so a man is indifferent between two families if he has the same wife in both of them, (same conditions for women and cats). This problem is equivalent to the 3-way stable 3-way exchange problem for *three-sided cyclic digraphs*, (i.e. $V(D) = M \cup W \cup C$ and every arc $(i, j) \in A(D)$ is from either $M \times W$ or $W \times C$ or $C \times M$), in that special case where the sizes of the three sides are the same, and the digraph contains all possible edges. Here, it can be proved that without the two latter restrictions, the general problem of finding a maximum size 3-way stable 3-way exchange for three-sided cyclic digraphs is NP-hard.

Theorem 16 *The decision problem related to* MAXCOVER-S$_3$E$_3$ *is NP-complete, even for three-sided cyclic digraphs [R8].*

Summary, open questions

We summarize the presented complexity results in the following table for (weakly) stable exchanges under normal preferences. Here, P denotes that the problem is polynomial time solvable, NPc denotes that the (related) problem is NP-complete, (NPh) denotes that the NP-hardness of the problem is obvious from the presented results. Finally, ??? means that we think that these unsolved problems are relevant, the reasons are explained below.

	$l =$	2-way exchange		3-way exchange		exchange	
$b =$		(strict)	with ties	(strict)	with ties	(strict)	with ties
2-way	Does exist?	P [R2]	NPc [R3]	???		(Yes)	(Yes)
stable	MAXCOVER	P	NPc [R4]			???	
3-way	Does exist?	NPc [R6]	(NPh)	NPc [R7]	(NPh)	(Yes)	(Yes)
stable	MAXCOVER	(NPh)	(NPh)	NPc [R8]	(NPh)		
(cycle)	Does exist?	NPc [R5]	(NPh)	(NPh)	(NPh)	(Yes)	Yes [R1]
stable	MAXCOVER	(NPh)	(NPh)	(NPh)	(NPh)	???	

In the present applications of kidney exchange, the programs tend to allow three-way exchanges, and the pairwise stability may become a natural expectation. That is why the problem of finding a pairwise stable 3-way exchange in an instance of SE is important. However, it is easy to construct an example to show that such a stable solution may not exist, but the complexity of this problem is unclear.

Considering the exchanges without restriction on the cycle-lengths, the TTC algorithm always provides a stable exchange, that is also a pairwise stable exhange in an instance of SE. Here, MAXCOVER-SE and MAXCOVER-S$_2$E are two important problems to solve.

Moreover, the problems of {weakly, strongly, super-} stable exchange with ties, under normal and $\mathscr{L}$-preferences yield various interesting open questions.

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