Machine learning and portfolio selections. I.

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Growth rate

investment in the stock market

d assets

$S_n^{(j)}$ price of asset j at the end of trading period (day) n

initial price $S_0^{(j)} = 1$, $j = 1, \dots, d$

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$$S_n^{(j)} = e^{nW_n^{(j)}} \approx e^{nW^{(j)}}$$

average growth rate

$$W_n^{(j)} = \frac{1}{n} \ln S_n^{(j)}$$

asymptotic average growth rate

$$W^{(j)} = \lim_{n\to\infty} \frac{1}{n} \ln S_n^{(j)}$$

Static portfolio selection: single period investment

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$$S_n = \sum_j b^{(j)} S_n^{(j)}$$

$$\max_j b^{(j)} S_n^{(j)} \leq S_n \leq d \max_j b^{(j)} S_n^{(j)}$$

assume that $b^{(j)} > 0$

$$\frac{1}{n}\ln\max_j\left(b^{(j)}S_n^{(j)}\right) \le \frac{1}{n}\ln S_n \le \frac{1}{n}\ln\left(d\max_j b^{(j)}S_n^{(j)}\right)$$

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$$\begin{aligned}
\max_j \left(\frac{1}{n}\ln b^{(j)} + \frac{1}{n}\ln S_n^{(j)}\right) &\leq \frac{1}{n}\ln S_n \\
&\leq \max_j \left(\frac{1}{n}\ln(d b^{(j)}) + \frac{1}{n}\ln S_n^{(j)}\right)
\end{aligned}$$

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$$\lim_{n\to\infty} \frac{1}{n}\ln S_n = \lim_{n\to\infty} \max_j \frac{1}{n}\ln S_n^{(j)} = \max_j W^{(j)}$$

we can do much better

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$$

$\mathbf{x}_i = (x_i^{(1)}, \dots x_i^{(d)})$ the return vector on day i
multi-period investment
$x_i^{(j)}$ is the factor by which capital invested in stock j grows during the market period i

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Constantly Re-balanced Portfolio (CRP)

a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$
$b^{(j)}$ gives the proportion of the investor's capital invested in stock j
$\mathbf{b}$ is the portfolio vector for each trading day

for the first day S_0 denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b} \, , \, \mathbf{x}_1 \rangle$$

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for the second day, S_1 new initial capital

$$S_2 = S_1 \cdot \langle \mathbf{b}\,,\, \mathbf{x}_2 \rangle = S_0 \cdot \langle \mathbf{b}\,,\, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}\,,\, \mathbf{x}_2 \rangle \,.$$

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for the nth day:

$$S_n = S_{n-1} \langle \mathbf{b}\,,\, \mathbf{x}_n \rangle = S_0 \prod_{i=1}^{n} \langle \mathbf{b}\,,\, \mathbf{x}_i \rangle = S_0 e^{nW_n(\mathbf{b})}$$

with the average growth rate

$$W_n(\mathbf{b}) = \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b}\,,\, \mathbf{x}_i \rangle\,.$$

log-optimum portfolio

Special market process: $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is independent and identically distributed (i.i.d.)

log-optimum portfolio $\mathbf{b}^*$

$$\mathbf{E}\{\ln \langle \mathbf{b}^* , \mathbf{X}_1 \rangle\} = \max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b} , \mathbf{X}_1 \rangle\}$$

Best Constantly Re-balanced Portfolio (BCRP)

Optimality

If $S_n^* = S_n(\mathbf{b}^*)$ denotes the capital after day n achieved by a log-optimum portfolio strategy $\mathbf{b}^*$, then for any portfolio strategy $\mathbf{b}$ with capital $S_n = S_n(\mathbf{b})$ and for any i.i.d. process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n \leq \lim_{n\to\infty} \frac{1}{n} \ln S_n^* \quad \text{almost surely}$$

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$$\lim_{n\to\infty} \frac{1}{n} \ln S_n \leq \lim_{n\to\infty} \frac{1}{n} \ln S_n^* \quad \text{almost surely}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely,}$$

where

$$W^* = \mathbf{E}\{\ln \langle \mathbf{b}^* , \mathbf{X}_1 \rangle\}$$

is the maximal growth rate of any portfolio.

Proof

$$\frac{1}{n}\ln S_n = \frac{1}{n}\sum_{i=1}^{n}\ln\langle\mathbf{b}\,,\,\mathbf{X}_i\rangle$$

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$$= \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle\}$$

$$+ \frac{1}{n}\sum_{i=1}^{n}\left(\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle - \mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle\}\right)$$

Proof

$$
\begin{aligned}
\frac{1}{n}\ln S_n &= \frac{1}{n}\sum_{i=1}^{n}\ln\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle \\
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\frac{1}{n}\ln S_n^* &= \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\ln\langle \mathbf{b}^*\,,\,\mathbf{X}_i\rangle\} \\
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History

gambling, horse racing, information theory

Kelly (1956)
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Chapter 15 of D. G. Luenberger, *Investment Science*. Oxford University Press, 1998.

Example 1: 1 stock + cash

$d = 2, \qquad \mathbf{X} = (X^{(1)}, X^{(2)})$

Stock:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

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What about $S_n^{(1)}$ or $W^{(1)}$?

$$W^{(1)} = \lim_{n\to\infty} \frac{1}{n} \ln S_n^{(1)} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln X_i^{(1)} = \mathbf{E}\{\ln X^{(1)}\}$$

$$= 1/2 \ln 2 + 1/2 \ln(1/2) = 0$$

zero growth rate

Cash:

$$X^{(2)} = 1$$

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portfolio

$$\mathbf{b} = (b, 1 - b)$$

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$$\begin{aligned}\mathbf{E}\{\ln \langle \mathbf{b}\,,\,\mathbf{X}\rangle\} &= 1/2\,(\ln(2b + (1-b)) + \ln(b/2 + (1-b)) \\ &= 1/2\ln[(1+b)(1-b/2)]\end{aligned}$$

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log-optimal portfolio

$$\mathbf{b}^* = (1/2, 1/2)$$

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log-optimal portfolio

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asymptotic average growth rate

$$\mathbf{E}\{\ln \langle \mathbf{b}^*\,,\,\mathbf{X}\rangle\} = 1/2\ln(9/8) = 0.059 = W^*$$

positive growth rate

Example 2: 2 stocks + cash

$d = 3, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)})$

Stocks:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases}$$

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$$\mathbf{b}^* = (0.46, 0.46, 0.08)$$

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log-optimal portfolio

$$\mathbf{b}^* = (0.46, 0.46, 0.08)$$

asymptotic average growth rate

$$\mathbf{E}\{\ln \langle \mathbf{b}^*, \mathbf{X} \rangle\} = 0.112 = W^*$$

Example 3: 3 stocks + cash

$d = 4, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$
log-optimal portfolio

$$\mathbf{b}^* = (1/3, 1/3, 1/3, 0)$$

the cash has zero weight

Example 3: 3 stocks + cash

$d = 4, \qquad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$

log-optimal portfolio

$$\mathbf{b}^* = (1/3, 1/3, 1/3, 0)$$

the cash has zero weight

asymptotic average growth rate

$$\mathbf{E}\{\ln \langle \mathbf{b}^* , \mathbf{X} \rangle\} = 0.152 = W^*$$

Example 4: many stocks

d is large
log-optimal portfolio

$$\mathbf{b}^* = (1/d, \ldots, 1/d)$$

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asymptotic average growth rate

$$\mathbf{E}\{\ln \langle \mathbf{b}^*, \mathbf{X} \rangle\} = 0.223 = W^*$$

Example 5: horse racing

d horses in a race
horse j wins with probability p_j
payoff o_j: investing 1\$ on horse j results in o_j if it wins, otherwise 0\$

$$\mathbf{X} = (0, \ldots, 0, o_j, 0, \ldots, 0)$$

if horse j wins

Example 5: horse racing

d horses in a race
horse j wins with probability p_j
payoff o_j: investing 1\$ on horse j results in o_j if it wins, otherwise 0\$

$$\mathbf{X} = (0, \dots, 0, o_j, 0, \dots, 0)$$

if horse j wins
repeated races

$$\mathbf{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X} \rangle\} = \sum_{j=1}^{d} p_j \ln(b^{(j)} o_j) = \sum_{j=1}^{d} p_j \ln b^{(j)} + \sum_{j=1}^{d} p_j \ln o_j$$

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therefore

$$\arg\max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b}\,,\,\mathbf{X}\rangle\} = \arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)}$$

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Kullback-Leibler divergence:

$$KL(\mathbf{p}, \mathbf{b}) = \sum_{j=1}^{d} p_j \ln \frac{p_j}{b^{(j)}}$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)}$$

Kullback-Leibler divergence:

$$KL(\mathbf{p}, \mathbf{b}) = \sum_{j=1}^{d} p_j \ln \frac{p_j}{b^{(j)}}$$

basic property:

$$KL(\mathbf{p}, \mathbf{b}) \geq 0$$

Proof:

$$\begin{aligned} KL(\mathbf{p}, \mathbf{b}) = -\sum_{j=1}^{d} p_j \ln \frac{b^{(j)}}{p_j} &\geq -\sum_{j=1}^{d} p_j \left(\frac{b^{(j)}}{p_j} - 1 \right) \\ &= -\sum_{j=1}^{d} b^{(j)} + \sum_{j=1}^{d} p_j = 0 \end{aligned}$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)} = \mathbf{p}$$

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independent of the payoffs

$$W^* = \sum_{j=1}^{d} p_j \ln(p_j o_j)$$

usual choice of payoffs:

$$o_j = \frac{1}{p_j}$$

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)} = \mathbf{p}$$

independent of the payoffs

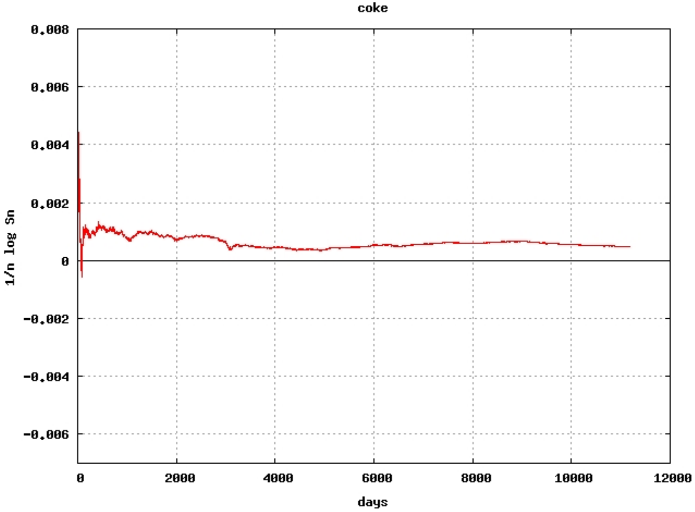
$$W^* = \sum_{j=1}^{d} p_j \ln(p_j o_j)$$

usual choice of payoffs:

$$o_j = \frac{1}{p_j}$$

$$W^* = 0$$

any gambling strategy has negative growth rate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0.004
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-0.006
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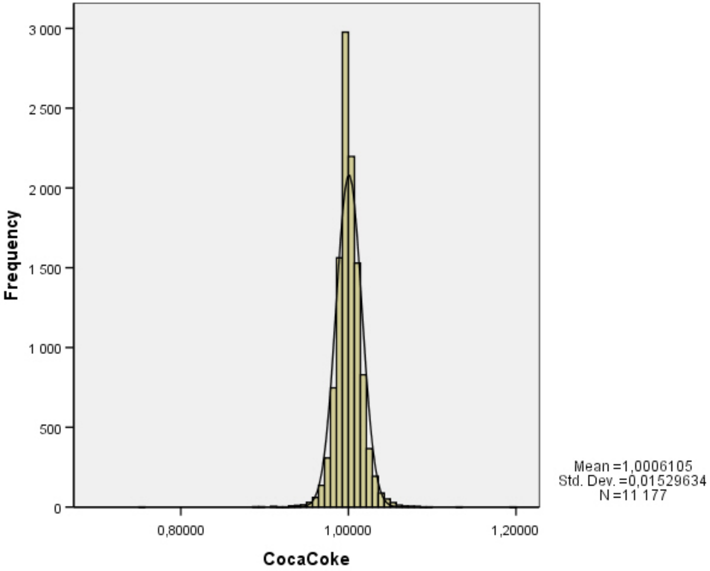
4000
6000
8000
10000
12000
days



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2 000
1 500
1 000
500
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1,00000
1,20000
CocaCoke
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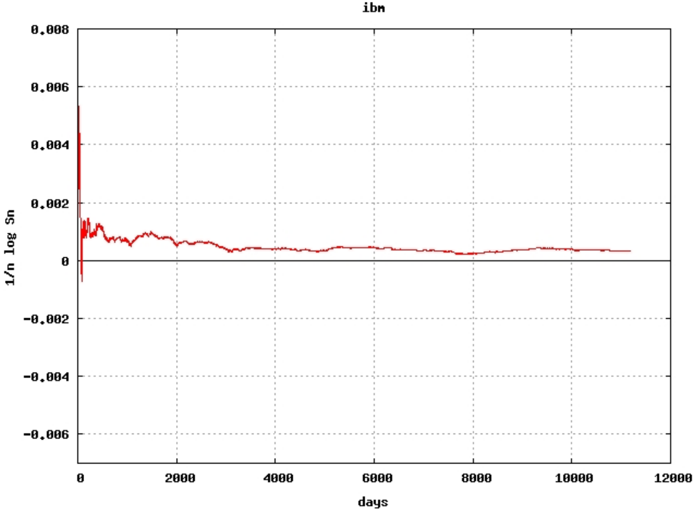
Std. Dev. =0,01529634
N =11 177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0.006
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-0.004
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1/n log Sn
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2000
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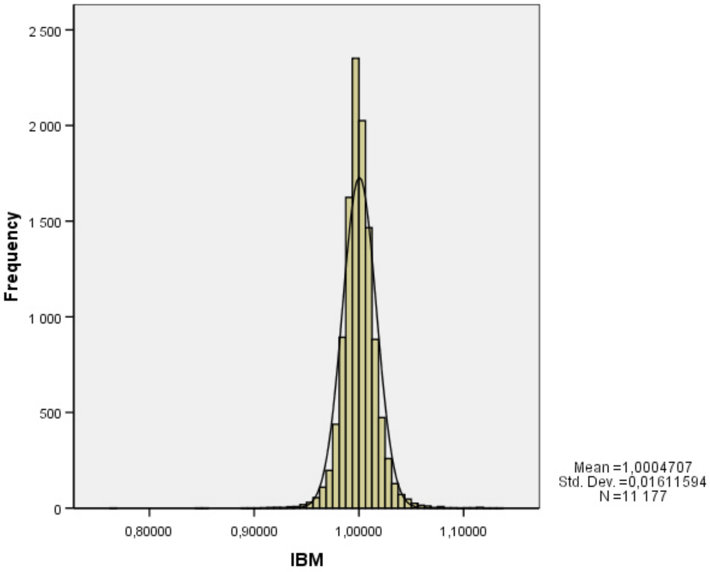
6000
8000
10000
12000
days


2 500
2 000
1 500
1 000
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0
Frequency
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1,00000
1,10000
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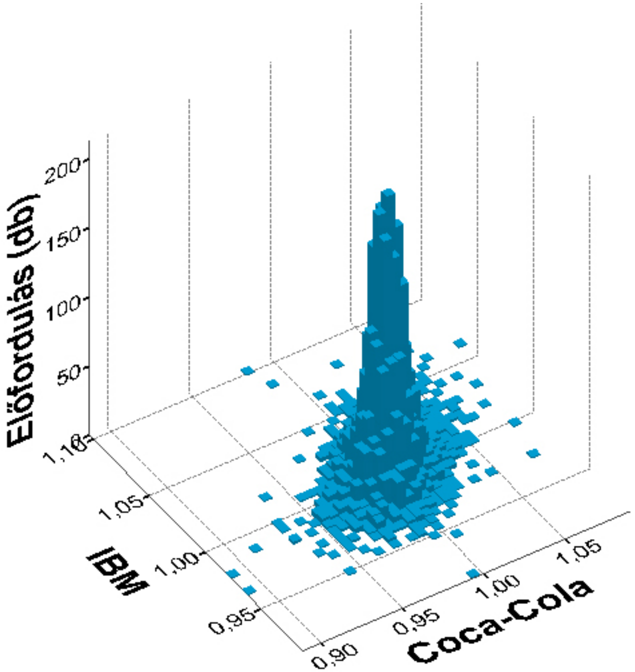
Std. Dev. =0,01611594
N =11 177


Előfordulás (db)
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100
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1,00
0,95
IBM
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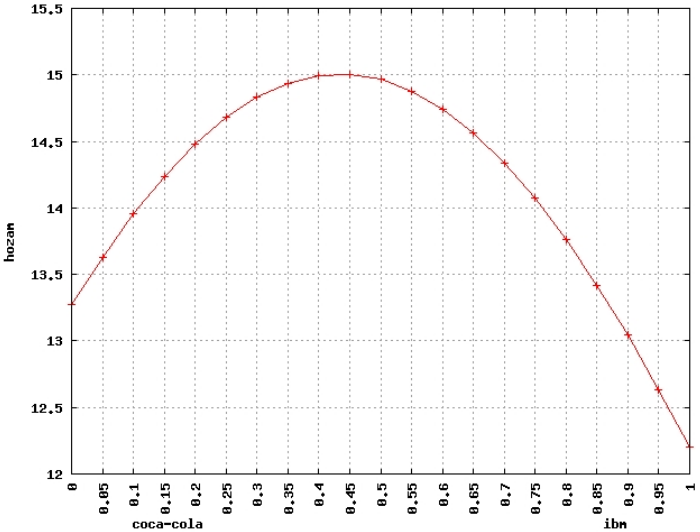
0,95
1,00
1,05
Coca-Cola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14.5
14
13.5
13
12.5
12
hozam
0
0.05
0.1
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0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
coca-cola
ibm

Consequences

$$S_n(\mathbf{b}) \quad \text{is not close to} \quad \mathbf{E}\{S_n(\mathbf{b})\}$$

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$$\left\{-\delta < \frac{1}{n}\ln S_n(\mathbf{b}) - \mathbf{E}\{\ln\langle \mathbf{b}\,,\, \mathbf{X}_1\rangle\} < \delta\right\}$$

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$$\left\{e^{n(-\delta + \mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\})} < S_n(\mathbf{b}) < e^{n(\delta + \mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\})}\right\}$$

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by Jensen inequality

$$\ln\langle \mathbf{b}\,,\,\mathbf{E}\{\mathbf{X}_1\}\rangle > \mathbf{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\}$$

therefore

$$S_n(\mathbf{b}) \quad \text{is much less than} \quad \mathbf{E}\{S_n(\mathbf{b})\}$$

Naive approach

$$\arg\max_{\mathbf{b}} \mathbf{E}\{S_n(\mathbf{b})\}$$

because of

$$\mathbf{E}\{S_n(\mathbf{b})\} = \langle \mathbf{b}\,,\ \mathbf{E}\{\mathbf{X}_1\}\rangle^n$$

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$\arg\max_{\mathbf{b}} \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle$ is a portfolio vector having 1 at the position, where $\mathbf{E}\{\mathbf{X}_1\}$ has the largest component
it is a dangerous portfolio
Markowitz:

$$\arg\max_{\mathbf{b}:\mathbf{Var}(\langle \mathbf{b}\,,\, \mathbf{X}_1\rangle)\le\lambda} \langle \mathbf{b}\,,\, \mathbf{E}\{\mathbf{X}_1\}\rangle$$

Semi-log-optimal portfolio

log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b} \, , \, \mathbf{X}_1 \rangle\}$$

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Connection to the Markowitz theory.

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Connection to the Markowitz theory.
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