Growth Optimal Portfolio Selection Strategies with Transaction Cost

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Notation

investment in the stock market

d assets

$s_n^{(j)}$ price of asset $j$ at the end of trading period (day) $n$

initial price $s_0^{(j)} = 1, j = 1, \ldots, d$
investment in the stock market

d assets

$s^{(j)}_n$ price of asset $j$ at the end of trading period (day) $n$

initial price $s^{(j)}_0 = 1$, $j = 1, \ldots, d$

\[ x^{(j)}_n = \frac{s^{(j)}_n}{s^{(j)}_{n-1}} \]

\[ x_n = (x^{(1)}_n, \ldots, x^{(d)}_n) \] the return vector on trading period $n$
nth trading period a portfolio strategy

\[ b_n = (b_n^{(1)}, \ldots, b_n^{(d)}) = b(x_1, \ldots, x_{n-1}) = b(x_1^{n-1}) \]

\( b_n^{(j)} \geq 0 \) gives the proportion of the investor’s capital invested in stock \( j \) for trading period \( n \) \( (\sum_{j=1}^d b_n^{(j)} = 1) \)
Portfolio selection without transaction cost

In the $n$th trading period a portfolio strategy

$$b_n = (b_n^{(1)}, \ldots, b_n^{(d)}) = b(x_1, \ldots, x_{n-1}) = b(x_1^{n-1})$$

$b_n^{(j)} \geq 0$ gives the proportion of the investor’s capital invested in stock $j$ for trading period $n$ ($\sum_{j=1}^{d} b_n^{(j)} = 1$)

for the $n$th trading period, $S_{n-1}$ is the initial capital (it is invested).

$$S_n = S_{n-1} \sum_{j=1}^{d} b_n^{(j)} x_1^{(j)}$$
nth trading period a portfolio strategy

\[ b_n = (b_n^{(1)}, \ldots, b_n^{(d)}) = \mathbf{b}(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) = \mathbf{b}(\mathbf{x}_1^{n-1}) \]

\( b_n^{(j)} \geq 0 \) gives the proportion of the investor’s capital invested in stock \( j \) for trading period \( n \) (\( \sum_{j=1}^{d} b_n^{(j)} = 1 \))
for the nth trading period, \( S_{n-1} \) is the initial capital (it is invested).

\[ S_n = S_{n-1} \sum_{j=1}^{d} b_n^{(j)} x_1^{(j)} = S_{n-1} \langle \mathbf{b}_n, \mathbf{x}_n \rangle \]
nth trading period a portfolio strategy

\[ b_n = (b_n^{(1)}, \ldots, b_n^{(d)}) = b(x_1, \ldots, x_{n-1}) = b(x_1^{n-1}) \]

\( b_n^{(j)} \geq 0 \) gives the proportion of the investor’s capital invested in stock \( j \) for trading period \( n \) \( (\sum_{j=1}^{d} b_n^{(j)} = 1) \)

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\[ S_n = S_{n-1} \sum_{j=1}^{d} b_n^{(j)} x_1^{(j)} = S_{n-1} \langle b_n, x_n \rangle = S_0 \prod_{i=1}^{n} \langle b_i, x_i \rangle \]
nth trading period a portfolio strategy

$$b_n = (b_n^{(1)}, \ldots, b_n^{(d)}) = b(x_1, \ldots, x_{n-1}) = b(x_1^{n-1})$$

$$b_n^{(j)} \geq 0$$ gives the proportion of the investor’s capital invested in stock $j$ for trading period $n$ ($\sum_{j=1}^{d} b_n^{(j)} = 1$)

for the $n$th trading period, $S_{n-1}$ is the initial capital (it is invested).

$$S_n = S_{n-1} \sum_{j=1}^{d} b_n^{(j)} x_1^{(j)} = S_{n-1} \langle b_n, x_n \rangle = S_0 \prod_{i=1}^{n} \langle b_i, x_i \rangle = S_0 e^{nW_n(b)}$$

with the average growth rate

$$W_n(B) = \frac{1}{n} \sum_{i=1}^{n} \log \langle b_i, x_i \rangle.$$
\[
\frac{1}{n} \log S_n \approx \frac{1}{n} \sum_{i=1}^{n} E\{\log \langle b(X_1^{i-1}), X_i \rangle | X_1^{i-1} \}
\]
\[
\frac{1}{n} \log S_n \approx \frac{1}{n} \sum_{i=1}^{n} E\{\log \langle b(X_{1}^{i-1}), X_i \rangle \mid X_1^{i-1}\}
\]

and

\[
\frac{1}{n} \log S_n^* \approx \frac{1}{n} \sum_{i=1}^{n} E\{\log \langle b^*(X_{1}^{i-1}), X_i \rangle \mid X_1^{i-1}\}
\]
$S_0 = 1$, gross wealth $S_n$, net wealth $N_n$ for the $n$th trading period, $N_{n-1}$ is the initial capital

\[ S_n = N_{n-1} \langle b_n, x_n \rangle \]
Portfolio selection with transaction cost

$S_0 = 1$, gross wealth $S_n$, net wealth $N_n$
for the $n$th trading period, $N_{n-1}$ is the initial capital

$$S_n = N_{n-1} \langle b_n, x_n \rangle$$

Calculate the transaction cost for selecting the portfolio $b_{n+1}$. 
$S_0 = 1$, gross wealth $S_n$, net wealth $N_n$
for the $n$th trading period, $N_{n-1}$ is the initial capital

$$S_n = N_{n-1} \langle b_n, x_n \rangle$$

Calculate the transaction cost for selecting the portfolio $b_{n+1}$.
Before rearranging, at the $j$-th asset there is $b_n^{(j)} x_n^{(j)} N_{n-1}$ dollars.
Portfolio selection with transaction cost

\[ S_0 = 1, \text{ gross wealth } S_n, \text{ net wealth } N_n \]
for the \( n \)th trading period, \( N_{n-1} \) is the initial capital

\[ S_n = N_{n-1} \langle b_n, x_n \rangle \]

Calculate the transaction cost for selecting the portfolio \( b_{n+1} \).
Before rearranging, at the \( j \)-th asset there is \( b_n^{(j)} x_n^{(j)} N_{n-1} \) dollars.
After rearranging, we need \( b_{n+1}^{(j)} N_n \) dollars.
\( S_0 = 1 \), gross wealth \( S_n \), net wealth \( N_n \) 
for the \( n \)th trading period, \( N_{n-1} \) is the initial capital

\[
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\]

Calculate the transaction cost for selecting the portfolio \( b_{n+1} \).
Before rearranging, at the \( j \)-th asset there is \( b_n^{(j)} x_n^{(j)} N_{n-1} \) dollars.
After rearranging, we need \( b_{n+1}^{(j)} N_n \) dollars.
If \( b_n^{(j)} x_n^{(j)} N_{n-1} \geq b_{n+1}^{(j)} N_n \) then we have to sell and the transaction cost at the \( j \)-th asset is

\[
c \left( b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n \right),
\]
Portfolio selection with transaction cost

\( S_0 = 1 \), gross wealth \( S_n \), net wealth \( N_n \) for the \( n \)th trading period, \( N_{n-1} \) is the initial capital

\[
S_n = N_{n-1} \langle b_n, x_n \rangle
\]

Calculate the transaction cost for selecting the portfolio \( b_{n+1} \).

Before rearranging, at the \( j \)-th asset there is \( b_n^{(j)} x_n^{(j)} N_{n-1} \) dollars.

After rearranging, we need \( b_{n+1}^{(j)} N_n \) dollars.

If \( b_n^{(j)} x_n^{(j)} N_{n-1} \geq b_{n+1}^{(j)} N_n \) then we have to sell and the transaction cost at the \( j \)-th asset is

\[
c \left( b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n \right),
\]

otherwise we have to buy and the transaction cost at the \( j \)-th asset is

\[
c \left( b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1} \right).
\]
Let $x^+$ denote the positive part of $x$. 
Let \( x^+ \) denote the positive part of \( x \).
Thus,

\[
N_n = S_n - \sum_{j=1}^{d} c \left( b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n \right)^+ \\
- \sum_{j=1}^{d} c \left( b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1} \right)^+ ,
\]
Let $x^+$ denote the positive part of $x$. Thus,

$$N_n = S_n - \sum_{j=1}^{d} c \left( b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n \right)^+$$

$$- \sum_{j=1}^{d} c \left( b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1} \right)^+, $$

or equivalently

$$S_n = N_n + c \sum_{j=1}^{d} \left| b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n \right|. $$
Dividing both sides by $S_n$ and introducing ratio

$$w_n = \frac{N_n}{S_n},$$

$0 < w_n < 1,$
Dividing both sides by $S_n$ and introducing ratio

$$w_n = \frac{N_n}{S_n},$$

$0 < w_n < 1$,

we get

$$1 = w_n + c \sum_{j=1}^{d} \left| \frac{b^{(j)}(j)}{\langle b_n, x_n \rangle} - b^{(j)}_{n+1} w_n \right|.$$
\[ S_n = N_{n-1}\langle b_n, x_n \rangle = S_{n-1}w_{n-1}\langle b_n, x_n \rangle = \prod_{i=1}^{n}[w(b_{i-1}, b_i, x_{i-1}) \langle b_i, x_i \rangle] \]
\[ S_n = N_{n-1} \langle b_n, x_n \rangle = S_{n-1} w_{n-1} \langle b_n, x_n \rangle = \prod_{i=1}^{n} \left[ w(b_{i-1}, b_i, x_{i-1}) \langle b_i, x_i \rangle \right] \]

Introduce the notation

\[ g(b_{i-1}, b_i, x_{i-1}, x_i) = \log \left( w(b_{i-1}, b_i, x_{i-1}) \langle b_i, x_i \rangle \right), \]
\[ S_n = N_{n-1} \langle b_n, x_n \rangle = S_{n-1} \mathcal{W}_{n-1} \langle b_n, x_n \rangle = \prod_{i=1}^{n} [w(b_{i-1}, b_i, x_{i-1}) \langle b_i, x_i \rangle] \]

Introduce the notation
\[
g(b_{i-1}, b_i, x_{i-1}, x_i) = \log(w(b_{i-1}, b_i, x_{i-1}) \langle b_i, x_i \rangle),
\]
then the average growth rate becomes
\[
\frac{1}{n} \log S_n = \frac{1}{n} \sum_{i=1}^{n} \log(w(b_{i-1}, b_i, x_{i-1}) \langle b_i, x_i \rangle)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} g(b_{i-1}, b_i, x_{i-1}, x_i).
\]
In the sequel $x_i$ will be random variable and is denoted by $X_i$. Let’s use the decomposition

$$
\frac{1}{n} \log S_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{g(b_{i-1}, b_i, X_{i-1}, X_i)|X_{i-1}\} \\
+ \frac{1}{n} \sum_{i=1}^{n} (g(b_{i-1}, b_i, X_{i-1}, X_i) - \mathbb{E}\{g(b_{i-1}, b_i, X_{i-1}, X_i)|X_{i-1}\}),
$$
In the sequel \( x_i \) will be random variable and is denoted by \( X_i \). Let's use the decomposition

\[
\frac{1}{n} \log S_n
\]

\[
= \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{g(b_{i-1}, b_i, X_{i-1}, X_i) \mid X_1^{i-1}\} \right) + \left( \frac{1}{n} \sum_{i=1}^{n} (g(b_{i-1}, b_i, X_{i-1}, X_i) - \mathbb{E}\{g(b_{i-1}, b_i, X_{i-1}, X_i) \mid X_1^{i-1}\}) \right),
\]

therefore

\[
\frac{1}{n} \log S_n \approx \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{g(b_{i-1}, b_i, X_{i-1}, X_i) \mid X_1^{i-1}\}
\]
If the market process \( \{X_i\} \) is a homogeneous and first order Markov process then

\[
E\{g(b_{i-1}, b_i, X_{i-1}, X_i) | X_i^{i-1}\} \\
= E\{\log(w(b_{i-1}, b_i, X_{i-1}) \langle b_i, X_i \rangle) | X_i^{i-1}\}
\]
If the market process \( \{X_i\} \) is a \textit{homogeneous and first order Markov process} then

\[
E\{g(b_{i-1}, b_i, X_{i-1}, X_i)|X_1^{i-1}\} = E\{\log(w(b_{i-1}, b_i, X_{i-1}) \langle b_i, X_i \rangle)|X_1^{i-1}\} = \log w(b_{i-1}, b_i, X_{i-1}) + E\{\log \langle b_i, X_i \rangle |X_1^{i-1}\}
\]
If the market process $\{X_i\}$ is a *homogeneous and first order* Markov process then

\[
E \{ g(b_{i-1}, b_i, X_{i-1}, X_i) | X_{i-1} \} \\
= E \{ \log (w(b_{i-1}, b_i, X_{i-1}) \langle b_i, X_i \rangle) | X_{i-1} \} \\
= \log w(b_{i-1}, b_i, X_{i-1}) + E \{ \log \langle b_i, X_i \rangle | X_{i-1} \} \\
= \log w(b_{i-1}, b_i, X_{i-1}) + E \{ \log \langle b_i, X_i \rangle | b_i, X_{i-1} \} \\
\text{def} = v(b_{i-1}, b_i, X_{i-1}),
\]
If the market process \( \{X_i\} \) is a \textit{homogeneous and first order Markov process} then

\[
E\{g(b_{i-1}, b_i, X_{i-1}, X_i) | X_{1}^{i-1}\} = E\{\log(w(b_{i-1}, b_i, X_{i-1}) \langle b_i, X_i \rangle) | X_{1}^{i-1}\} = \log w(b_{i-1}, b_i, X_{i-1}) + E\{\log \langle b_i, X_i \rangle | X_{1}^{i-1}\} = \log w(b_{i-1}, b_i, X_{i-1}) + E\{\log \langle b_i, X_i \rangle | b_i, X_{i-1}\} = v(b_{i-1}, b_i, X_{i-1}),
\]

therefore the maximization of the average growth rate

\[
\frac{1}{n} \log S_n
\]

is asymptotically equivalent to the maximization of

\[
\frac{1}{n} \sum_{i=1}^{n} v(b_{i-1}, b_i, X_{i-1}).
\]

dynamic programming problem
empirical portfolio selection
empirical portfolio selection
Naive approach
Algorithm 1

empirical portfolio selection
Naive approach
For the optimization, neglect the transaction cost
Algorithm 1

empirical portfolio selection
Naive approach
For the optimization, neglect the transaction cost
kernel based log-optimal portfolio selection
empirical portfolio selection
Naive approach
For the optimization, neglect the transaction cost
kernel based log-optimal portfolio selection
Define an infinite array of experts $B^{(\ell)} = \{b^{(\ell)}(\cdot)\}$, where $\ell$ is a positive integer.
empirical portfolio selection

Naive approach
For the optimization, neglect the transaction cost

kernel based log-optimal portfolio selection

Define an infinite array of experts $\mathbf{B}^{(\ell)} = \{\mathbf{b}^{(\ell)}(\cdot)\}$, where $\ell$ is a positive integer.

For fixed positive integer $\ell$, choose the radius $r_\ell > 0$ such that

$$\lim_{\ell \to \infty} r_\ell = 0.$$
put

\[ b_1 = \{1/d, \ldots, 1/d\} \]
put

\[ b_1 = \{1/d, \ldots, 1/d\} \]

for \( n > 1 \), define the expert \( b^{(\ell)} \) by

\[
b_n^{(\ell)} = \arg \max_{b \in \Delta_d} \sum_{\{i < n : \|x_{i-1} - x_{n-1}\| \leq r_\ell\}} \ln \langle b, x_i \rangle ,
\]

if the sum is non-void,
put
\[ b_1 = \{1/d, \ldots, 1/d\} \]

for \( n > 1 \), define the expert \( b^{(\ell)} \) by
\[
b_n^{(\ell)} = \arg \max_{b \in \Delta_d} \sum_{\{i < n : \|x_{i-1} - x_{n-1}\| \leq r_\ell\}} \ln \langle b, x_i \rangle,
\]
if the sum is non-void,
and \( b_1 = (1/d, \ldots, 1/d) \) otherwise, where \( \| \cdot \| \) denotes the Euclidean norm.
let $\{q_\ell\}$ be a probability distribution over the set of all positive integers $\ell$
let \( \{q_\ell \} \) be a probability distribution over the set of all positive integers \( \ell \)

\( S_n(B^{(\ell)}) \) is the capital accumulated by the elementary strategy \( B^{(\ell)} \) after \( n \) periods with an initial capital \( S_0 = 1 \)
let \( \{q_\ell\} \) be a probability distribution over the set of all positive integers \( \ell \)

\( S_n(B^{(\ell)}) \) is the capital accumulated by the elementary strategy \( B^{(\ell)} \) after \( n \) periods with an initial capital \( S_0 = 1 \)

- after period \( n \), aggregations with the wealths:

\[
S_n = \sum_{\ell} q_\ell S_n(B^{(\ell)}). \tag{1}
\]
let \( \{q_\ell\} \) be a probability distribution over the set of all positive integers \( \ell \)

\( S_n(B^{(\ell)}) \) is the capital accumulated by the elementary strategy \( B^{(\ell)} \) after \( n \) periods with an initial capital \( S_0 = 1 \)

- after period \( n \), aggregations with the wealths:

\[
S_n = \sum_\ell q_\ell S_n(B^{(\ell)}).
\] (1)

- after period \( n \), aggregations with the portfolios:

\[
b_n = \frac{\sum_\ell q_\ell S_{n-1}(B^{(\ell)})b_n^{(\ell)}}{\sum_\ell q_\ell S_{n-1}(B^{(\ell)})}.
\] (2)
let \( \{q_\ell\} \) be a probability distribution over the set of all positive integers \( \ell \).

\( S_n(\mathbf{B}(\ell)) \) is the capital accumulated by the elementary strategy \( \mathbf{B}(\ell) \) after \( n \) periods with an initial capital \( S_0 = 1 \).

- After period \( n \), aggregations with the wealths:

\[
S_n = \sum_\ell q_\ell S_n(\mathbf{B}(\ell)).
\]  

(1)

- After period \( n \), aggregations with the portfolios:

\[
b_n = \frac{\sum_\ell q_\ell S_{n-1}(\mathbf{B}(\ell)) b_n^{(\ell)}}{\sum_\ell q_\ell S_{n-1}(\mathbf{B}(\ell))}.
\]

(2)

The investor’s capital is:

\[
S_n = S_{n-1} \langle \mathbf{b}_n, \mathbf{x}_n \rangle w(\mathbf{b}_{n-1}, \mathbf{b}_n, \mathbf{x}_{n-1}).
\]
Algorithm 2

tempirical portfolio selection
empirical portfolio selection
a one-step optimization as follows:

\[ b_1 = \{1/d, \ldots, 1/d\} \]
Algorithm 2

empirical portfolio selection
a one-step optimization as follows:

\[ b_1 = \{1/d, \ldots, 1/d\} \]

for \( n \geq 1 \),

\[ b_{n}^{(\ell)} = \arg \max_{b \in \Delta_d} \sum_{\{i < n : \|x_i - 1 - x_{n-1}\| \leq r_\ell\}} \left( \ln \langle b, x_i \rangle + \ln w(b_{n-1}^{(\ell)}, b, x_{n-1}) \right), \]

if the sum is non-void,
empirical portfolio selection
a one-step optimization as follows:

\[ b_1 = \{1/d, \ldots, 1/d\} \]

for \( n \geq 1 \),

\[ b^{(\ell)}_n = \arg \max_{b \in \Delta_d} \sum_{\{i < n : \|x_{i-1} - x_{n-1}\| \leq r_\ell\}} \left( \ln \langle b, x_i \rangle + \ln w(b^{(\ell)}_{n-1}, b, x_{n-1}) \right), \]

if the sum is non-void,
and \( b_1 = (1/d, \ldots, 1/d) \) otherwise.
Algorithm 2

empirical portfolio selection
a one-step optimization as follows:

\[
\mathbf{b}_1 = \{1/d, \ldots, 1/d\}
\]

for \( n \geq 1 \),

\[
\mathbf{b}_{n}^{(\ell)} = \arg\max_{\mathbf{b} \in \Delta_d} \sum_{\{i < n: \|\mathbf{x}_{i-1} - \mathbf{x}_{n-1}\| \leq r_\ell\}} \left( \ln \langle \mathbf{b}, \mathbf{x}_i \rangle + \ln w(\mathbf{b}_{n-1}^{(\ell)}, \mathbf{b}, \mathbf{x}_{n-1}) \right),
\]

if the sum is non-void,
and \( \mathbf{b}_1 = (1/d, \ldots, 1/d) \) otherwise.
These elementary portfolios are mixed as before (1) or (2).
At www.szit.bme.hu/~oti/portfolio there are two benchmark data set from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years (5651 trading days ending in 1985).
- The second data set contains 23 stocks and has length 44 years (11178 trading days ending in 2006).
At www.szit.bme.hu/~oti/portfolio there are two benchmark data set from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years (5651 trading days ending in 1985).
- The second data set contains 23 stocks and has length 44 years (11178 trading days ending in 2006).

Our experiment is on the second data set.
Experiments on average annual yields (AAY)

Kernel based log-optimal portfolio selection with $\ell = 1, \ldots, 10$

$$r_\ell^2 = 0.0001 \cdot d \cdot \ell,$$
Experiments on average annual yields (AAY)

Kernel based log-optimal portfolio selection with $\ell = 1, \ldots, 10$

$$r_\ell^2 = 0.0001 \cdot d \cdot \ell,$$

MORRIS had the best AAY, 20%
The average annual yields of the individual experts and of the aggregations with $c = 0.0015$.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$c = 0$</th>
<th>Algorithm 1</th>
<th>Algorithm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20%</td>
<td>-18%</td>
<td>-14%</td>
</tr>
<tr>
<td>2</td>
<td>118%</td>
<td>-2%</td>
<td>25%</td>
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<tr>
<td>3</td>
<td>71%</td>
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<td>55%</td>
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<td>4</td>
<td>103%</td>
<td>28%</td>
<td>73%</td>
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<td>77%</td>
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<td>127%</td>
<td>42%</td>
<td>66%</td>
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<tr>
<td>10</td>
<td>123%</td>
<td>44%</td>
<td>62%</td>
</tr>
<tr>
<td>Aggregation with wealth (1)</td>
<td>137%</td>
<td>40%</td>
<td>83%</td>
</tr>
<tr>
<td>Aggregation with portfolio (2)</td>
<td>137%</td>
<td>49%</td>
<td>89%</td>
</tr>
</tbody>
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non-empirical strategy
non-empirical strategy
$0 < \delta < 1$ denotes a discount factor
non-empirical strategy
0 < \delta < 1 denotes a discount factor
discounted Bellman equation:

\[ F_\delta(b, x) = \max_{b'} \left\{ v(b, b', x) + (1 - \delta) E\{F_\delta(b', X_2) | X_1 = x\} \right\} . \]
non-empirical strategy

$0 < \delta < 1$ denotes a discount factor
discounted Bellman equation:

$$F_\delta(b, x) = \max_{b'} \left\{ v(b, b', x) + (1 - \delta) \mathbb{E}\{F_\delta(b', X_2) \mid X_1 = x\} \right\}.$$ 

$$b_1^* = \{1/d, \ldots, 1/d\}$$

and

$$b_{i+1}^* = \arg \max_{b'} \left\{ v(b_i^*, b', X_i) + (1 - \delta_i) \mathbb{E}\{F_{\delta_i}(b', X_{i+1}) \mid X_i\} \right\},$$

for $1 \leq i$,
non-empirical strategy
0 < \delta < 1 denotes a discount factor
discounted Bellman equation:

\[ F_\delta(b, x) = \max_{b'} \left\{ v(b, b', x) + (1 - \delta)\mathbb{E}\{F_\delta(b', X_2) \mid X_1 = x\} \right\}. \]

\[ b_1^* = \{1/d, \ldots, 1/d\} \]

and

\[ b_{i+1}^* = \arg \max_{b'} \left\{ v(b_i^*, b', X_i) + (1 - \delta_i)\mathbb{E}\{F_{\delta_i}(b', X_{i+1}) \mid X_i\} \right\}, \]

for 1 \leq i, where 0 < \delta_i < 1 is a discount factor such that \delta_i \downarrow 0.
non-empirical strategy
0 < \delta < 1 denotes a discount factor
discounted Bellman equation:

\[
F_\delta(b, x) = \max_{b'} \left\{ v(b, b', x) + (1 - \delta) \mathbb{E}\{F_\delta(b', X_2) | X_1 = x\} \right\}.
\]

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Choose the discount factor \( \delta_i \downarrow 0 \) such that

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\frac{(\delta_i - \delta_{i+1})}{\delta_{i+1}^2} \to 0
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Then, for Strategy 1, the portfolio \( \{b^*_i\} \) with capital \( S^*_n \) is optimal in the sense that for any portfolio strategy \( \{b_i\} \) with capital \( S_n \),

\[
\liminf_{n \to \infty} \left( \frac{1}{n} \log S^*_n - \frac{1}{n} \log S_n \right) \geq 0
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a.s.
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$$b_1^{(k)} = \{1/d, \ldots, 1/d\}$$

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Choose an arbitrary probability distribution $q_k > 0$, and introduce the combined portfolio with its capital

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stationary policy
Assume (i) and (ii) of Theorem 1.
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\[
\lim_{n \to \infty} \left( \frac{1}{n} \log S^*_n - \frac{1}{n} \log \tilde{S}_n \right) = 0 \quad \text{a.s.}
\]
Theorem 2

Assume (i) and (ii) of Theorem 1.
Choose the discount factor \( \delta_i \downarrow 0 \) as \( i \to \infty \).
Then, for Strategy 2,

\[
\lim_{n \to \infty} \left( \frac{1}{n} \log S_n^* - \frac{1}{n} \log \tilde{S}_n \right) = 0
\]
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How to construct empirical (data-driven) optimal portfolio selection strategy?