Chapter 6

Empirical Pricing American Put Options

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In this note we study the empirical pricing American options. The pricing American option is an optimal stopping problem, which can be derived from a backward recursion such that in each step of the recursion one needs conditional expectations. For empirical pricing, [Longstaff and Schwartz (2001)] suggested to replace the conditional expectations by regression function estimates. We survey the current literature and the main techniques of nonparametric regression estimates, and derive new empirical pricing algorithms.

6.1. Introduction: the valuation of option price

6.1.1. *Notations*

One of the most important problems in option pricing theory is the valuation and optimal exercise of derivatives with American-style exercise features. Such derivatives are, for example, the equity, commodity, foreign exchange, insurance, energy, municipal, mortgage, credit, convertible, swap, emerging markets, etc. Despite recent progresses, the valuation and optimal exercise of American options remains one of the most challenging problems in derivatives finance. In many financial contracts it is allowed to exercise the contract early before expiry. For example, many exchange traded options are of American type and allow the holder any exercise date before expiry, mortgages have often embedded prepayment options such that the mortgage can be amortized or repayed, or life insurance contracts allow often for early surrender. In this paper we consider data driven pricing of options with early exercise features.

Let X_t be the asset price at time t, K the strike price, r the discount rate. For American put option, the payoff function f_t with discount factor e^{-rt} is

$$f_t(X_t) = e^{-rt}\left(K - X_t\right)^+ \ .$$

For maturity time T, let $\mathbb{T}=\{1,\dots,T\}$ be the time frame for American options. Let $\mathcal{F}_t$ denote the σ-algebra generated by $X_0 = 1, X_1, \dots, X_t$ then an integer valued random variable τ is called stopping time if $\{\tau = t\} \in \mathcal{F}_t$, for all $t = 1, \dots, T$. If $\widetilde{\mathfrak{T}}(0,\dots,T)$ stands for the set of stopping times taking values in $(0,\dots,T)$ then the task of pricing the American option is to determine

$$V_0 = \sup_{\tau \in \widetilde{\mathfrak{T}}(0,\dots,T)} \mathbb{E}\left\{f_\tau\left(X_\tau\right)\right\}. \tag{6.1}$$

The main principles of pricing American put option described below can be extended to more general payoffs, for example, the payoffs may depend on many assets' prices (cf. [Tsitsiklis and Roy (2001)]).

Let τ^* be the optimum stopping time, i.e.,

$$\mathbb{E}\left\{f_{\tau^*}\left(X_{\tau^*}\right)\right\} = \sup_{\tau \in \widetilde{\mathfrak{T}}(0,\dots,T)} \mathbb{E}\left\{f_\tau\left(X_\tau\right)\right\}$$

6.1.2. *Optimal stopping*

An alternative formulation of τ^* can be derived as follows. Introduce the notation

$$q_t(x) = \sup_{\tau \in \widetilde{\mathfrak{T}}\{t+1,\dots,T\}} \mathbb{E}\left\{f_\tau(X_\tau) \mid X_t = x\right\} \tag{6.2}$$

continuation value, where $\widetilde{\mathfrak{T}}\{t+1,\dots,T\}$ refers to the possible stopping times taking values in $\{t+1,\dots,T\}$.

Theorem 6.1 (cf. Chow *et. al*, 1971, Shiryayev, 1978, Kohler, 2010). *Put*

$$\tau^q = \min\left\{1 \le s \le T : q_s\left(X_s\right) \le f_s\left(X_s\right)\right\}.$$

If the assets prices $\{X_t\}$ form a Markov process then

$$\tau^* = \tau^q.$$

The intuition behind the optimal stopping rule τ^q is that at any exercise time, the holder of an American option optimally compares the payoff from immediate exercise with the expected payoff from continuation, and then exercises if the immediate payoff is higher. Thus, the optimal exercise strategy is fundamentally determined by the conditional expectation of the payoff from continuing to keep the option alive. The key insight underlying the current approaches is that this conditional expectation can be estimated from data.

As a byproduct of the proof of Theorem 6.1, one may check the the following:

Theorem 6.2 (cf. Tsitsiklis and Roy, 1999, Kohler, 2010). *We get that*

$$q_T(x) = 0,$$

while at any $t < T$

$$q_t(x) = \mathbb{E}\left\{\max\left\{f_{t+1}\left(X_{t+1}\right), q_{t+1}\left(X_{t+1}\right)\right\} \mid X_t = x\right\} \tag{6.3}$$

which means that there is a backward recursive scheme.

(6.3) implies that

$$\begin{aligned}
q_t(x) &= \mathbb{E}\left\{\max\left\{f_{t+1}\left(X_{t+1}\right), q_{t+1}\left(X_{t+1}\right)\right\} \mid X_t = x\right\} \\
&= \mathbb{E}\left\{\max\left\{e^{-r(t+1)}\left(K - X_{t+1}\right)^{+}, q_{t+1}\left(X_{t+1}\right)\right\} \mid X_t = x\right\} \\
&= \mathbb{E}\left\{\max\left\{e^{-r(t+1)}\left(K - \frac{X_{t+1}}{X_t}X_t\right)^{+}, q_{t+1}\left(\frac{X_{t+1}}{X_t}X_t\right)\right\} \mid X_t = x\right\} \\
&= \mathbb{E}\left\{\max\left\{e^{-r(t+1)}\left(K - \frac{X_{t+1}}{X_t}x\right)^{+}, q_{t+1}\left(\frac{X_{t+1}}{X_t}x\right)\right\} \mid X_t = x\right\}.
\end{aligned} \tag{6.4}$$

6.1.3. *Martingale approach: the primal-dual problem*

As we defined in the Introduction, the initial problem is to find the optimal stopping time which provides the price of American option:

$$V_0 = \sup_{\tau \in \widetilde{\mathfrak{T}}(0,\ldots,T)} \mathbb{E}\left\{f_\tau\left(X_\tau\right)\right\},$$

where the sup is taken over the stopping times τ. The dual problem is formulated by [Rogers (2002)], [Haugh and Kogan (2004)] to obtain an alternative valuation method. Let

$$U_0 = \inf_{M \in \mathcal{M}} \mathbb{E} \left\{ \max_{t \in \{0,1 \ldots T\}} \left(f_t \left(X_t \right) - M_t \right) \right\} \tag{6.5}$$

where $\mathcal{M}$ is the set of martingales with $M_0 = 0$ and with the same filtration $\sigma \left(X_t, \ldots, X_1 \right)$. The dual method is based on the next theorem.

Theorem 6.3. (cf. Rogers, 2002, Haugh and Kogan, 2004, Glasserman, 2004, Kohler, 2010) *If X_t is a Markov process then*

$$U_0 = V_0$$

This result is based on the important observation that one can obtain a martingale from the pay-off function and continuation value in a natural way.

Theorem 6.4. (cf. Glasserman, 2004, Tsitsiklis and Roy, 1999, Kohler, 2010) *The optimal martingale is of form*

$$M_t^* = \sum_{s=1}^{t} \left(\max \left\{ f_s \left(X_s \right), q_s \left(X_s \right) \right\} - q_{s-1} \left(X_{s-1} \right) \right)$$

and indeed M_t^* is a martingale.

The valuation task now is converted into an estimate of the martingale M_t^*.

6.1.4. *Lower and upper bounds of $q_t(x)$*

In pricing American option, the continuation values $q_t(x)$ play an important role. For empirical pricing, one has to estimate them, which is possible using the backward recursion (6.3). However, using this recursion the estimation errors are accumulated, therefore there is a need to control the error propagation.

We introduce a lower bound of $q_t(x)$:

$$q_t^{(l)}(x) = \max_{s \in \{t+1, \ldots, T\}} \mathbb{E} \left\{ f_s(X_s) | X_t = x \right\}.$$

Since any constant $\tau = s$ is a stopping time, we have that

$$q_t^{(l)}(x) \leq q_t(x).$$

We shall show that $q_t^{(l)}(x)$ can be estimated easier than that of $q_t(x)$ and the estimate has a fast rate of convergence, so if $q_{t,n}^{(l)}(x)$ and $q_{t,n}(x)$ are the estimates of $q_t^{(l)}(x)$ and $q_t(x)$, resp., then

$$\hat{q}_{t,n}(x) := \max\{q_{t,n}(x), q_{t,n}^{(l)}(x)\}$$

is an (hopefully) improved estimate of $q_t(x)$.

Next we introduce an upper bound. For $\tau \in \widetilde{\mathfrak{T}}\{t+1,\ldots,T\}$, we have that

$$f_\tau(X_\tau) \le \max_{s\in\{t+1,\ldots,T\}} f_s(X_s),$$

therefore

$$q_t(x) = \sup_{\tau\in\widetilde{\mathfrak{T}}\{t+1,\ldots,T\}} \mathbb{E}\left\{f_\tau(X_\tau) \mid X_t = x\right\} \le \mathbb{E}\left\{\max_{s\in\{t+1,\ldots,T\}} f_s(X_s) \mid X_t = x\right\}.$$

Introduce the notation

$$q_t^{(u)}(x) := \mathbb{E}\left\{\max_{s\in\{t+1,\ldots,T\}} f_s(X_s) \mid X_t = x\right\},$$

then we get an upper bound

$$q_t(x) \le q_t^{(u)}(x).$$

Again, $q_t^{(u)}(x)$ can be estimated easier than that of $q_t(x)$ and the estimate has a fast rate of convergence, so if $q_{t,n}^{(u)}(x)$ and $q_{t,n}(x)$ are the estimates of $q_t^{(u)}(x)$ and $q_t(x)$, resp., then

$$\hat{q}_{t,n}(x) := \min\{q_{t,n}(x), q_{t,n}^{(u)}(x)\}$$

is an improved estimate of $q_t(x)$.

The combination of the lower an upper bounds reads as follows:

$$\max_{s\in\{t+1,\ldots,T\}} \mathbb{E}\left\{f_s(X_s) | X_t = x\right\} \le q_t(x) \le \mathbb{E}\left\{\max_{s\in\{t+1,\ldots,T\}} f_s(X_s) \mid X_t = x\right\},$$

while the improved estimate has the form

$$\hat{q}_{t,n}(x) = \begin{cases} q_{t,n}^{(u)}(x) \text{ if } & q_{t,n}^{(u)}(x) < q_{t,n}(x), \\ q_{t,n}(x) \text{ if } & q_{t,n}^{(u)}(x) \ge q_{t,n}(x) \ge q_{t,n}^{(l)}(x), \\ q_{t,n}^{(l)}(x) \text{ if } & q_{t,n}(x) < q_{t,n}^{(l)}(x). \end{cases}$$

6.1.5. *Sampling*

In a real life problem we have a single historical data sequence $X_1, \ldots, X_N$.

Definition 6.1. The process $\{X_t\}$ is called of memoryless multiplicative increments, if $X_1/X_0, X_2/X_1, \ldots$ are independent random variables.

Definition 6.2. The process $\{X_t\}$ is called of stationary multiplicative increments, if the sequence $X_1/X_0 = X_1, X_2/X_1, \ldots$ is strictly stationary.

As mentioned earlier, the continuation value $q_t(x)$ plays an important role in the optimum pricing, which is the supremum of conditional expectations. Conditional expectations can be considered as regression functions, and in the empirical pricing the regression function is replaced by its estimate. For regression function estimation, we are given independent and identically distributed (i.i.d) copies of $X_1, \ldots, X_T$, i.e., one generates i.i.d. sample path prices:

$$X_{i,1}, \ldots, X_{i,T}, \tag{6.6}$$

$i = 1, ...n$.

Based on the historical data sequence $X_1, \ldots, X_N$, one can construct samples for (6.6) as follows:

(i) For the Monte Carlo sampling, one assumes that the data generating process is completely known, i.e., that there is perfect parametric model and all parameters of this process are already estimated from historical data $X_1, \ldots, X_N$ (cf. Longstaff, Schwartz [Longstaff and Schwartz (2001)]). Thus, one can artificially generate independent sample paths (6.6). The weakness of this approach is that usually the size N of the historical data is not large enough in order to have a good model and reliable parameter estimates.

(ii) For disjoint sampling, $N = nT$ and2$i = 1, \ldots, n = N/T$. However, we haven't the required i.i.d. property unless the process $X_1, \ldots, X_{nT}$ have memoryless and stationary multiplicative increments, which means that $X_1/X_0, X_2/X_1, \ldots, X_{nT}/X_{nT-1}$ are i.i.d.

(iii) For sliding sampling,

$$X_{i,t} := \frac{X_{i+t}}{X_i}, \tag{6.7}$$

$i = 1, \ldots, n = N - T$. In this way we get a large sample, however, there is no i.i.d. property.

(iv) For bootstrap sampling, we generate i.i.d. random variables $T_1, \ldots, T_n$ uniformly distributed on $1, \ldots, N-T$ and

$$X_{i,t} := \frac{X_{T_i+t}}{X_{T_i}}, \tag{6.8}$$

$i = 1, \ldots, n$.

6.1.6. *Empirical pricing and optimal exercising of American option*

If the continuation values $q_t(x)$, $t = 1, \ldots T$ were known, then the **optimal stopping time** τ_i for path $X_{i,1}, \ldots, X_{i,T}$ can be calculated:

$$\tau_i = \min \left\{1 \leq s \leq T : q_s\left(X_{i,s}\right) \leq f_s\left(X_{i,s}\right)\right\}.$$

Then the price V_0 can be estimated by the average

$$\frac{1}{n} \sum_{i=1}^{n} f_{\tau_i}\left(X_{\tau_i}\right). \tag{6.9}$$

The continuation values $q_t(x)$, $t = 1, \ldots T$ are unknown, therefore one has to generate some estimates $q_{t,n}(x)$, $t = 1, \ldots T$. [Kohler *et al.* (2008)] suggested a splitting approach as follows. Split the sample $\{X_{i,1}, \ldots, X_{i,T}\}_{i=1}^{n}$ into two samples: $\{X_{i,1}, \ldots, X_{i,T}\}_{i=1}^{m}$ and $\{X_{i,1}, \ldots, X_{i,T}\}_{i=m+1}^{n}$. We estimate $q_t(x)$ by $q_{t,m}(x)$, $(t = 1, \ldots T)$ from $\{X_{i,1}, \ldots, X_{i,T}\}_{i=1}^{m}$, and construct some approximations of the optimal stopping time τ_i for path $X_{i,1}, \ldots, X_{i,T}$

$$\tau_{i,m} = \min \left\{1 \leq s \leq T : q_{s,m}\left(X_{i,s}\right) \leq f_s\left(X_{i,s}\right)\right\},$$

and then the price V_0 can be estimated by the average

$$\frac{1}{n-m} \sum_{i=m+1}^{n} f_{\tau_{i,m}}\left(X_{\tau_{i,m}}\right).$$

For empirical exercising at the time frame $[N+1, N+T]$, we are given the past data $X_1, \ldots, X_N$ based on which generate some estimates $q_{t,N}(x)$, $t = 1, \ldots T$. Then the **empirical exercising of American option** can be defined by the stopping time

$$\tau_N = \min \left\{1 \leq s \leq T : q_{s,N}\left(X_{N+s}/X_N\right) \leq f_s\left(X_{N+s}/X_N\right)\right\}.$$

If the continuation values $q_t(x)$, $t = 1, \ldots T$ were known, then the **optimal martingale** $M^*_{i,t}$ for path $X_{i,1}, \ldots, X_{i,T}$ can be calculated:

$$M^*_{i,t} = \sum_{s=1}^{t} \left(\max \left\{ f_s \left(X_{i,s} \right), q_s \left(X_{i,s} \right) \right\} - q_{s-1} \left(X_{i,s-1} \right) \right).$$

Then the price V_0 can be estimated by the average

$$\frac{1}{n} \sum_{i=1}^{n} \max_{t \in \{0,1 \ldots T\}} \left(f_t \left(X_{i,t} \right) - M^*_{i,t} \right). \tag{6.10}$$

The continuation values $q_t(x)$, $t = 1, \ldots T$ are unknown, then using the splitting approach described above generate some estimates $q_{t,m}(x)$, $t = 1, \ldots T$ are available and the approximations of the optimal martingale $M^*_{i,t}$ for path $X_{i,1}, \ldots, X_{i,T}$:

$$M^*_{i,t,m} = \sum_{s=1}^{t} \left(\max \left\{ f_s \left(X_{i,s} \right), q_{s,m} \left(X_{i,s} \right) \right\} - q_{s-1,m} \left(X_{i,s-1} \right) \right).$$

Then the price V_0 can be estimated by the average

$$V_{0,n} = \frac{1}{n-m} \sum_{i=m+1}^{n} \max_{t \in \{0,1 \ldots T\}} \left(f_t \left(X_{i,t} \right) - M^*_{i,t,m} \right).$$

For option pricing, a nonparametric estimation scheme was firstly proposed by [Carrier (1996)], while [Tsitsiklis and Roy (1999)] and [Longstaff and Schwartz (2001)] estimated the continuation value.

6.2. Special case: pricing for process with memoryless and stationary multiplicative increments

In this section we assume that the assets prices $\{X_t\}$ have memoryless and stationary multiplicative increments. This properties imply that, for $s > t$, $\frac{X_s}{X_t}$ and X_t are independent, and $\frac{X_s}{X_t}$ and $\frac{X_{s-t}}{X_0} = X_{s-t}$ have the same distribution.

6.2.1. *Estimating q_t*

For $t < T$, the recursion (6.4) implies that

$$\begin{aligned}
q_t(x) &= \mathbb{E}\left\{\max\left\{f_{t+1}\left(X_{t+1}\right), q_{t+1}\left(X_{t+1}\right)\right\} \mid X_t = x\right\} \\
&= \mathbb{E}\left\{\max\left\{e^{-r(t+1)}\left(K - \frac{X_{t+1}}{X_t}x\right)^+, q_{t+1}\left(\frac{X_{t+1}}{X_t}x\right)\right\} \mid X_t = x\right\} \\
&= \mathbb{E}\left\{\max\left\{e^{-r(t+1)}\left(K - \frac{X_{t+1}}{X_t}x\right)^+, q_{t+1}\left(\frac{X_{t+1}}{X_t}x\right)\right\}\right\} \\
&= \mathbb{E}\left\{\max\left\{e^{-r(t+1)}\left(K - X_1 x\right)^+, q_{t+1}\left(X_1 x\right)\right\}\right\},
\end{aligned} \tag{6.11}$$

where in the last two steps we assumed independent and stationary multiplicative increments. By a backward induction we get that, for fixed t, $q_t(x)$ is a monotonically decreasing and convex function of x.

If we are given data $X_1, \dots, X_N$, $i = 1, \dots, N$ then, for any fixed t, let $q_{t+1,N}(x)$ be an estimate of $q_{t+1}(x)$. Thus, introduce the estimate of $q_t(x)$ in a backward recursive way as follows:

$$q_{t,N}(x) = \frac{1}{N}\sum_{i=1}^{N} \max\left\{e^{-r(t+1)}\left(K - xX_i/X_{i-1}\right)^+, q_{t+1,N}(xX_i/X_{i-1})\right\}. \tag{6.12}$$

From (6.12) we can derive a numerical procedure such that consider a grid

$$\mathbf{G} := \{j \cdot h\},$$

$j = 1, 2, \dots$, where the step size of the grid $h > 0$, for example $h = 0.01$. In each step of (6.12) we make the recursion for $x \in \mathbf{G}$, and then linearly interpolate for $x \notin \mathbf{G}$.

The weakness of this estimate can be that maybe the estimation errors are cumulated, therefore we consider the estimates of the lower and upper bounds, too.

6.2.2. *Estimating the lower and upper bounds of $q_t(x)$*

For memoryless process, the lower bound of $q_t(x)$ has a simple form:

$$
\begin{aligned}
q_t^{(l)}(x) &= \max_{s\in\{t+1,\ldots,T\}} \mathbb{E}\left\{f_s(X_s)|X_t = x\right\} \\
&= \max_{s\in\{t+1,\ldots,T\}} e^{-rs}\mathbb{E}\left\{\left(K - \frac{X_s}{X_t}X_t\right)^+ \mid X_t = x\right\} \\
&= \max_{s\in\{t+1,\ldots,T\}} e^{-rs}\mathbb{E}\left\{\left(K - \frac{X_s}{X_t}x\right)^+ \mid X_t = x\right\} \\
&= \max_{s\in\{t+1,\ldots,T\}} e^{-rs}\mathbb{E}\left\{\left(K - \frac{X_s}{X_t}x\right)^+\right\} \\
&= \max_{s\in\{t+1,\ldots,T\}} e^{-rs}\mathbb{E}\left\{(K - X_{s-t}x)^+\right\},
\end{aligned}
$$

where in the last two steps we assumed memoryless and stationary multiplicative increments.

Thus

$$
q_t^{(l)}(x) = \sup_{s\in\{t+1,\ldots,T\}} e^{-rs}\mathbb{E}\left\{(K - X_{s-t}x)^+\right\}.
$$

If we are given data $X_{i,1},\ldots,X_{i,T}$, $i = 1,\ldots n$ then the estimate of $q_t^{(l)}(x)$ would be

$$
q_{t,n}^{(l)}(x) = \max_{s\in\{t+1,\ldots,T\}} e^{-rs}\frac{1}{n}\sum_{i=1}^{n}(K - X_{i,s-t}x)^+ .
$$

Concerning the upper bound, the previous arguments imply that

$$\begin{aligned}
q_t^{(u)}(x) &= \mathbb{E}\left\{\max_{s\in\{t+1,\dots,T\}} f_s(X_s) \mid X_t = x\right\} \\
&= \mathbb{E}\left\{\max_{s\in\{t+1,\dots,T\}} e^{-rs}(K - X_s)^+ \mid X_t = x\right\} \\
&= \mathbb{E}\left\{\max_{s\in\{t+1,\dots,T\}} e^{-rs}\left(K - \frac{X_s}{X_t}X_t\right)^+ \mid X_t = x\right\} \\
&= \mathbb{E}\left\{\max_{s\in\{t+1,\dots,T\}} e^{-rs}\left(K - \frac{X_s}{X_t}x\right)^+ \mid X_t = x\right\} \\
&= \mathbb{E}\left\{\max_{s\in\{t+1,\dots,T\}} e^{-rs}\left(K - \frac{X_s}{X_t}x\right)^+ \right\} \\
&= \mathbb{E}\left\{\max_{s\in\{t+1,\dots,T\}} e^{-rs}\left(K - X_{s-t}x\right)^+ \right\}.
\end{aligned}$$

If we are given data $X_{i,1}, \dots, X_{i,T}$, $i = 1, ...n$, then the estimate of $q_t^{(u)}(x)$ would be

$$q_{t,n}^{(u)}(x) = \frac{1}{n}\sum_{i=1}^{n} \max_{s\in\{t+1,\dots,T\}} e^{-rs}\left(K - X_{i,s-t}x\right)^+ .$$

The combination of the lower an upper bounds reads as follows:

$$\max_{s\in\{t+1,\dots,T\}} \mathbb{E}\left\{e^{-rs}\left(K - X_{s-t}x\right)^+\right\} \le q_t(x) \le \mathbb{E}\left\{\max_{s\in\{t+1,\dots,T\}} e^{-rs}\left(K - X_{s-t}x\right)^+\right\}.$$

Again, using the estimates of the lower and upper bound, we suggest a truncation of the estimates of the continuation value:

$$\hat{q}_{t,N}(x) = \begin{cases} q_{t,n}^{(u)}(x) & \text{if} \quad q_{t,n}^{(u)}(x) < q_{t,N}(x), \\ q_{t,N}(x) & \text{if} \quad q_{t,n}^{(u)}(x) \ge q_{t,N}(x) \ge q_{t,n}^{(l)}(x), \\ q_{t,n}^{(l)}(x) & \text{if} \quad q_{t,N}(x) < q_{t,n}^{(l)}(x). \end{cases}$$

6.2.3. *The growth rate of an asset and the Black-Scholes model*

In this section we still assume that the assets prices $\{X_t\}$ have memoryless and stationary multiplicative increments, and in discrete time show that the Black-Scholes formula results in a good approximation of the lower bound $q_t^{(l)}(x)$. Consider an asset, the evolution of which characterized by its price

X_t at trading period (let's say trading day) t. In order to normalize, put $X_0 = 1$. X_t has exponential trend:

$$X_t = e^{tW_t} \approx e^{tW},$$

with average growth rate (average daily yield)

$$W_t := \frac{1}{t} \ln X_t$$

and with asymptotic average growth rate

$$W := \lim_{t \to \infty} \frac{1}{t} \ln X_t.$$

Introduce the returns Z_t as follows:

$$Z_t = \frac{X_t}{X_{t-1}}.$$

Thus, the return Z_t denotes the amount obtained after investing a unit capital in the asset on the t-th trading period. Because $\{X_t\}$ is of independent and stationary multiplicative increments, the sequence $\{Z_t\}$ is i.i.d. Then the strong law of large numbers (cf. [Stout (1974)]) implies that

$$\begin{aligned}
W_t &= \frac{1}{t} \ln X_t \\
&= \frac{1}{t} \ln \prod_{i=1}^{t} \frac{X_i}{X_{i-1}} \\
&= \frac{1}{n} \ln \prod_{i=1}^{n} Z_i \\
&= \frac{1}{n} \sum_{i=1}^{n} \ln Z_i \\
&\to \mathbb{E}\{\ln Z_1\} = \mathbb{E}\{\ln X_1\}
\end{aligned}$$

almost surely (a.s.), therefore

$$W = \mathbb{E}\{\ln X_1\}.$$

The problem is how to calculate $\mathbb{E}\{\ln X_1\}$. It is not an easy task, one should know the distribution of X_1. For the approximate calculation of log-optimal portfolio, [Vajda (2006)] suggested to use the second order Taylor expansion of the function $\ln z$ at $z = 1$:

$$h(z) := z - 1 - \frac{1}{2}(z-1)^2.$$

Table 6.1. The average empirical daily yield, variance, growth rate and estimated growth rate for the 19 stocks from [Gelencsér and Ottucsák (2006)].

STOCK	r_a	σ	W	$\widetilde{W}$
AHP	0.000602	0.0160	0.000473	0.000474
ALCOA	0.000516	0.0185	0.000343	0.000343
AMERB	0.000616	0.0145	0.000511	0.000510
COKE	0.000645	0.0152	0.000528	0.000528
DOW	0.000576	0.0167	0.000436	0.000436
DUPONT	0.000442	0.0153	0.000325	0.000324
FORD	0.000526	0.0184	0.000356	0.000356
GE	0.000591	0.0151	0.000476	0.000476
GM	0.000408	0.0171	0.000261	0.000261
HP	0.000807	0.0227	0.000548	0.000548
IBM	0.000495	0.0161	0.000365	0.000365
INGER	0.000571	0.0177	0.000413	0.000413
JNJ	0.000712	0.0153	0.000593	0.000593
KIMBC	0.000599	0.0154	0.000479	0.000480
MERCK	0.000669	0.0156	0.000546	0.000546
MMM	0.000513	0.0144	0.000408	0.000408
MORRIS	0.000874	0.0169	0.000729	0.000730
PANDG	0.000579	0.0140	0.000478	0.000479
SCHLUM	0.000741	0.0191	0.000557	0.000557

For daily returns, this is a very good approximation, so it is a natural idea to introduce the semi-log approximation of the asymptotic growth rate:

$$\tilde{W} = \mathbb{E}\{h(X_1)\}.$$

$\tilde{W}$ has the advantage that it can be calculated just knowing the first and second moments of X_1. Put

$$\mathbb{E}\{X_1\} = 1 + r_a$$

and

$$\mathbf{Var}(X_1) = \sigma^2,$$

then

$$\tilde{W} = \mathbb{E}\{h(X_1)\} = \mathbb{E}\{X_1 - 1 - \frac{1}{2}(X_1 - 1)^2\} = r_a - \frac{\sigma^2 + r_a^2}{2} \approx r_a - \frac{\sigma^2}{2}.$$

Table 6.1 summarizes the growth rate of some big stocks on New York Stock Exchange (NYSE). The used database contains daily relative closing prices of several stocks and it is normalized by divident and splits for all trading days. For more information about the database see the homepage

[Gelencsér and Ottucsák (2006)]. One can see that $\widetilde{W}$ is really a good approximation of W.

If the expiration time T is much larger than 1 day then for $\ln X_T$ we cannot apply the semi-log approximation, we should approximate the distribution of $\ln X_T$.

As for the binomial model or for the Cox-Ross-Rubinstein model or for the construction of geometric Brownian motion (cf. [Luenberger (1998)]), in addition, we assumed that $\{Z_t\}$ are i.i.d. Then

$$\begin{aligned}
&\mathbf{Var}\left(\sum_{i=1}^{t} \ln Z_i\right) \\
&\approx \mathbf{Var}\left(\sum_{i=1}^{t} h(Z_i)\right) \\
&= t\mathbf{Var}\left(h(Z_1)\right) \\
&= t\mathbf{Var}\left(X_1 - 1 - \frac{1}{2}(X_1-1)^2\right) \\
&= t\left(\mathbb{E}\{(X_1-1)^2\} - \mathbb{E}\{(X_1-1)^3\} + \frac{1}{4}\mathbb{E}\{(X_1-1)^4\} - (r_a - \frac{1}{2}(\sigma^2 + r_a^2))^2\right) \\
&\approx t\sigma^2.
\end{aligned}$$

Thus, by the central limit theorem we get that $\ln X_t$ is approximately Gaussian distributed with mean $t(r_a - (\sigma^2 + r_a^2)/2) \approx t(r_a - \sigma^2/2)$ and variance $t\sigma^2$:

$$\ln X_t \overset{\mathcal{D}}{\approx} \mathcal{N}\left(t(r_a - \sigma^2/2), t\sigma^2\right),$$

so we derived the discrete time version of the Black-Scholes model.

We have that

$$\ln X_t \overset{\mathcal{D}}{\approx} \mathcal{N}\left(tv_0, t\sigma^2\right)$$

where

$$v_0 = r_a - \sigma^2/2.$$

Let $Z \overset{\mathcal{D}}{=} \mathcal{N}(0,1)$ then

$$\begin{aligned}
\mathbb{E}\left\{(K - xX_t)^+\right\} &= \mathbb{E}\left\{\left(K - xe^{\ln X_t}\right)^+\right\} \\
&= \int_{-\infty}^{\infty} \left(K - xe^{tv_0 + \sqrt{t}\sigma z}\right)^+ \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}} dz.
\end{aligned}$$

We have

$$K - xe^{tv_0+\sqrt{t}\sigma z} > 0$$

if and only if

$$\log \frac{K}{x} > tv_0 + \sqrt{t}\sigma z,$$

equivalently

$$z_0 := \frac{\log \frac{K}{x} - tv_0}{\sqrt{t}\sigma} > z.$$

Thus

$$\begin{aligned}
\mathbb{E}\left\{(K - xX_t)^+\right\} &= \int_{-\infty}^{z_0} \left(K - xe^{tv_0+\sqrt{t}\sigma z}\right)^+ \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}} dz \\
&= K\Phi(z_0) - \frac{xe^{tv_0}}{\sqrt{2\pi}} \int_{-\infty}^{z_0} e^{\sqrt{t}\sigma z - z_0^2/2} dz \\
&= K\Phi(z_0) - \frac{xe^{t\left(v_0+\sigma^2/2\right)}}{\sqrt{2\pi}} \int_{-\infty}^{z_0} e^{\frac{(z-\sqrt{t}\sigma)^2}{2}} dz \\
&= K\Phi(z_0) - xe^{t\left(v_0+\sigma^2/2\right)} \Phi\left(z_0 - \sqrt{t}\sigma\right).
\end{aligned}$$

Consequently

$$\begin{aligned}
&e^{-rt}\mathbb{E}\left\{(K - xX_t)^+\right\} \\
&= e^{-rt}\left(K\Phi\left(\frac{\log \frac{K}{x} - tv_0}{\sqrt{t}\sigma}\right) - xe^{t\left(v_0+\sigma^2/2\right)}\Phi\left(\frac{\log \frac{K}{x} - tv_0}{\sqrt{t}\sigma} - \sqrt{t}\sigma\right)\right),
\end{aligned}$$

therefore we get that

$$\begin{aligned}
&q_t^{(l)}(x) \\
&= \sup_{s\in\{t+1,\dots,T\}} e^{-rs}\mathbb{E}\left\{(K - X_{s-t}x)^+\right\} \\
&= e^{-rt} \\
&\cdot \sup_{s\in\{1,\dots,T-t\}} e^{-rs}\left(K\Phi\left(\frac{\log \frac{K}{x} - sv_0}{\sqrt{s}\sigma}\right) - xe^{s\left(v_0+\sigma^2/2\right)}\Phi\left(\frac{\log \frac{K}{x} - sv_0 - s\sigma^2}{\sqrt{s}\sigma}\right)\right).
\end{aligned}$$

6.3. Nonparametric regression estimation

In order to introduce efficient estimates of $q_t(x)$, for general Markov process, we briefly summarize the basics of nonparametric regression estimation. In regression analysis one considers a random vector (X, Y), where X and

Y are $\mathbb{R}$-valued, and one is interested how the value of the so-called response variable Y depends on the value of the observation X. This means that one wants to find a function $f : \mathbb{R} \to \mathbb{R}$, such that $f(X)$ is a "good approximation of Y," that is, $f(X)$ should be close to Y in some sense, which is equivalent to making $|f(X) - Y|$ "small." Since X and Y are random, $|f(X) - Y|$ is random as well, therefore it is not clear what "small $|f(X) - Y|$" means. We can resolve this problem by introducing the so-called *mean squared error* of f,

$$\mathbb{E}|f(X) - Y|^2,$$

and requiring it to be as small as possible. So we are interested in a function $m : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}|m(X) - Y|^2 = \min_{f:\mathbb{R}\to\mathbb{R}} \mathbb{E}|f(X) - Y|^2.$$

According to Chapter 5 of this volume, such a function can be obtained explicitly by the *regression function*:

$$m(x) = \mathbb{E}\{Y|X = x\}.$$

In applications the distribution of (X, Y) (and hence also the regression function) is usually unknown. Therefore it is impossible to predict Y using $m(X)$. But it is often possible to observe data according to the distribution of (X, Y) and to estimate the regression function from these data.

To be more precise, denote by (X, Y), (X_1, Y_1), $(X_2, Y_2), \ldots$ i.i.d. random variables with $\mathbb{E}Y^2 < \infty$. Let $\mathcal{D}_n$ be the set of *data* defined by

$$\mathcal{D}_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}.$$

In the regression function estimation problem one wants to use the data $\mathcal{D}_n$ in order to construct an estimate $m_n : \mathbb{R} \to \mathbb{R}$ of the regression function m. Here $m_n(x) = m_n(x, \mathcal{D}_n)$ is a measurable function of x and the data. For simplicity, we will suppress $\mathcal{D}_n$ in the notation and write $m_n(x)$ instead of $m_n(x, \mathcal{D}_n)$.

In this section we describe the basic principles of nonparametric regression estimation: *local averaging*, or *least squares estimation*). (Concerning the details see Chapter 5 of this volume and [Györfi *et al.* (2002)].)

The local averaging estimates of $m(x)$ can be written as

$$m_n(x) = \sum_{i=1}^{n} W_{n,i}(x) \cdot Y_i,$$

where the weights $W_{n,i}(x) = W_{n,i}(x, X_1, \ldots, X_n) \in \mathbb{R}$ depend on $X_1, \ldots, X_n$. Usually the weights are nonnegative and $W_{n,i}(x)$ is "small" if X_i is "far" from x.

An example of such an estimate is the *partitioning estimate*. Here one chooses a finite or countably infinite partition $\mathcal{P}_n = \{A_{n,1}, A_{n,2}, \ldots\}$ of $\mathbb{R}$ consisting of cells $A_{n,j} \subseteq \mathbb{R}$ and defines, for $x \in A_{n,j}$, the estimate by averaging Y_i's with the corresponding X_i's in $A_{n,j}$, i.e.,

$$m_n(x) = \frac{\sum_{i=1}^n I_{\{X_i \in A_{n,j}\}} Y_i}{\sum_{i=1}^n I_{\{X_i \in A_{n,j}\}}} \quad \text{for } x \in A_{n,j},$$

where I_A denotes the indicator function of set A. Here and in the following we use the convention $\frac{0}{0} = 0$. For the partition $\mathcal{P}_n$, the most important example is when the cells $A_{n,j}$ are intervals of length h_n. For interval partition, the consistency conditions mean that

$$\lim_{n\to\infty} h_n = 0 \quad \text{and} \quad \lim_{n\to\infty} nh_n = \infty. \tag{6.13}$$

The second example of a local averaging estimate is the *Nadaraya–Watson kernel estimate*. Let $K : \mathbb{R} \to \mathbb{R}_+$ be a function called the kernel function, and let $h > 0$ be a bandwidth. The kernel estimate is defined by

$$m_n(x) = \frac{\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right) Y_i}{\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right)}.$$

Here the estimate is a weighted average of the Y_i, where the weight of Y_i (i.e., the influence of Y_i on the value of the estimate at x) depends on the distance between X_i and x. For the bandwidth $h = h_n$, the consistency conditions are (6.13). If one uses the so-called naive kernel (or window kernel) $K(x) = I_{\{\|x\| \le 1\}}$, then

$$m_n(x) = \frac{\sum_{i=1}^n I_{\{\|x - X_i\| \le h\}} Y_i}{\sum_{i=1}^n I_{\{\|x - X_i\| \le h\}}},$$

i.e., one estimates $m(x)$ by averaging Y_i's such that the distance between X_i and x is not greater than h.

Our final example of local averaging estimates is the *k-nearest neighbor* (k-NN) *estimate*. Here one determines the k nearest X_i's to x in terms of distance $\|x - X_i\|$ and estimates $m(x)$ by the average of the corresponding Y_i's. More precisely, for $x \in \mathbb{R}$, let

$$(X_{(1)}(x), Y_{(1)}(x)), \ldots, (X_{(n)}(x), Y_{(n)}(x))$$

be a permutation of

$$(X_1, Y_1), \ldots, (X_n, Y_n)$$

such that

$$|x - X_{(1)}(x)| \leq \cdots \leq |x - X_{(n)}(x)|.$$

The k-NN estimate is defined by

$$m_n(x) = \frac{1}{k} \sum_{i=1}^{k} Y_{(i)}(x).$$

If $k = k_n \to \infty$ such that $k_n/n \to 0$ then the k-nearest-neighbor regression estimate is consistent.

Least squares estimates are defined by minimizing the empirical L_2 risk

$$\frac{1}{n} \sum_{i=1}^{n} |f(X_i) - Y_i|^2$$

over a general set of functions $\mathcal{F}_n$. Observe that it doesn't make sense to minimize the empirical L_2 risk over all functions f, because this may lead to a function which interpolates the data and hence is not a reasonable estimate. Thus one has to restrict the set of functions over which one minimizes the empirical L_2 risk. Examples of possible choices of the set $\mathcal{F}_n$ are sets of piecewise polynomials with respect to a partition $\mathcal{P}_n$, or sets of smooth piecewise polynomials (splines). The use of spline spaces ensures that the estimate is a smooth function. An important member of least squares estimates is the generalized linear estimates. Let $\{\phi_j\}_{j=1}^{\infty}$ be real-valued functions defined on $\mathbb{R}$ and let $\mathcal{F}_n$ be defined by

$$\mathcal{F}_n = \left\{ f;\ f = \sum_{j=1}^{\ell_n} c_j \phi_j \right\}.$$

Then the generalized linear estimate is defined by

$$\begin{aligned} m_n(\cdot) &= \arg\min_{f \in \mathcal{F}_n} \left\{ \frac{1}{n} \sum_{i=1}^{n} (f(X_i) - Y_i)^2 \right\} \\ &= \arg\min_{c_1, \ldots, c_{\ell_n}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{\ell_n} c_j \phi_j(X_i) - Y_i \right)^2 \right\}. \end{aligned}$$

If the set

$$\left\{ \sum_{j=1}^{\ell} c_j \phi_j;\quad (c_1, \ldots, c_\ell),\ \ell = 1, 2, \ldots \right\}$$

is dense in the set of continuous functions, $\ell_n \to \infty$ and $\ell_n/n \to 0$ then the generalized linear regression estimate defined above is consistent. For least squares estimates, other example can be the neural networks or radial basis functions or orthogonal series estimates or splines.

6.4. General case: pricing for process with stationary multiplicative increments

6.4.1. *The backward recursive estimation scheme*

Using the recursion (6.3), if the function $q_{t+1}(x)$ were known, then $q_t(x)$ would be a regression function, which can be estimated from data

$$\mathcal{D}_t = \{(X_{i,t}, Y_{i,t})\}_{i=1}^n ,$$

with

$$Y_{i,t} = \max\{f_{t+1}(X_{i,t+1}), q_{t+1}(X_{i,t+1})\} .$$

However, the function $q_{t+1}(x)$ is unknown. Once we have an estimate $q_{t+1,n}$ of q_{t+1} we can get an estimate of the next q_t by generating samples $\mathcal{D}_t$ with

$$Y_{i,t}^{(n)} = \max\{f_{t+1}(X_{i,t+1}), q_{t+1,n}(X_{i,t+1})\} .$$

6.4.2. *The Longstaff-Schwartz (LS) method*

In this section we briefly survey on recent papers which generalized or improved the Markov chain Monte Carlo and/or LS method.

First we recall the original method developed by [Longstaff and Schwartz (2001)] then we elaborate on some refinements and variations. All these methods have the following basic characteristics. They assume that the price process of the underlying asset very well described by a theoretical model, by the Black-Scholes (BS) model or a Markov chain model. In both cases it is also assumed that we have from historical data a perfect estimate of the model parameters hence Monte Carlo (MC) generation of arbitrary large number of sample paths of the price process provide arbitrarily good approximation of the real situation, i.e., one applies a Monte Carlo sampling.

[Longstaff and Schwartz (2001)] suggested a quadratic regression as follows. Given that q_t is expressed by a conditional expectation (6.2), we can seek for a regression function which determine the value of q_t. Let us consider a function space e.g. L_2 and an orthonormal basis, the weighted Laguerre polynomials

$$\begin{aligned} L_0(x) &= \exp(-x/2) \\ L_1(x) &= (1-x)\,L_0(x) \\ L_2(x) &= \left(1-2x+x^2/2\right)L_0(x) \\ L_n(x) &= \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right). \end{aligned}$$

we determine the coefficients: in case of $k = 2, a_1, a_2, a_3$:

$$(a_{0,t}, a_{1,t}, a_{2,t}) = \underset{(a_0,a_1,a_2)}{\arg\min} \sum_{i=1}^{n} \left(a_0 L_0(X_{i,t}) + a_1 L_1(X_{i,t}) + a_2 L_2(X_{i,t}) - Y_{i,t}\right)^2$$

and obtain the estimate of q_t

$$q_{t,n}(x) = \sum_{i=0}^{2} a_{i,t} L_i(x).$$

Other choices might be, Hermite, Legendre, Chebysev, Gegenbauer, Jacoby, trigonometric or even power functions do the job.

[Egloff (2005)] suggested to replace the parametric regression in the LS method by nonparametric estimates. For example, in the possession of the generated variables one can get the least square estimate of q_t by

$$q_{t,n} = \arg\min_{f\in\mathcal{F}} \left\{ \frac{1}{n}\sum_{i=1}^{n} \left(f(X_{i,t}) - Y_{i,t}\right)^2 \right\},$$

where $\mathcal{F}$ is a function space.

[Kohler (2008)] studied the possible refinement, improvement of the LS method in several papers. One significant extension is the computational adaptation of the original LS method to options based on d underlying assets, which lifts up the problem. This amounts to analyze d-dimensional time-series such that [Kohler (2008)] suggested a penalized spline estimate over a Sobolev space.

[Kohler *et al.* (2010)] investigated a least squares method for empirical pricing compound American option if the corresponding space of functions $\mathcal{F}$ is defined by neural networks (NN).

[Egloff *et al.* (2007)] reduced the error propagation with the rule such that the non-in the money paths are sorted out, and for $(X_{i,s}, Y_{i,s})$ generate new path working on $t, ...T$ (not the already used for $t+1...T$) reducing error propagation. They studied an empirical error minimization estimate for a function space of polynomial splines.

6.4.3. *A new estimator*

Let's introduce a partitioning like estimate, i.e., for the grid $\mathbf{G}$ and for $x \in \mathbf{G}$ put

$$q_{t,n}(x) = \frac{\sum_{i=1}^{n} \max\left\{f_{t+1}(X_{i,t+1}), q_{t+1,n}(X_{i,t+1})\right\} I_{\{|X_{i,t}-x|\leq h/2\}}}{\sum_{i=1}^{n} I_{\{|X_{i,t}-x|\leq h/2\}}}, \quad (6.14)$$

where I denotes the indicator, and $0/0 = 0$ by definition. Obviously, this estimate should be slightly modified if the denominator of the estimate is not large enough. Then linearly interpolate for $x \notin \mathbf{G}$.

We have that

$$\max_{s\in\{t+1,\ldots,T\}} \mathbb{E}\left\{f_s(X_s) \mid X_t = x\right\} \leq q_t(x) \leq \mathbb{E}\left\{\max_{s\in\{t+1,\ldots,T\}} f_s(X_s) \mid X_t = x\right\},$$

where both the lower and the upper bounds are true regression function.

For $x \in \mathbf{G}$, the lower bound can be estimated by

$$q_{t,n}^{(l)}(x) = \max_{s\in\{t+1,\ldots,T\}} \frac{\sum_{i=1}^{n} f_s(X_{i,s}) I_{\{|X_{i,t}-x|\leq h/2\}}}{\sum_{i=1}^{n} I_{\{|X_{i,t}-x|\leq h/2\}}},$$

while an estimate of the upper bound can be

$$q_{t,n}^{(u)}(x) = \frac{\sum_{i=1}^{n} \max_{s\in\{t+1,\ldots,T\}} f_s(X_{i,s}) I_{\{|X_{i,t}-x|\leq h/2\}}}{\sum_{i=1}^{n} I_{\{|X_{i,t}-x|\leq h/2\}}}.$$

Again, a truncation is proposed:

$$\hat{q}_{t,n}(x) = \begin{cases} q_{t,n}^{(u)}(x) \text{ if } & q_{t,n}^{(u)}(x) < q_{t,n}(x), \\ q_{t,n}(x) \text{ if } & q_{t,n}^{(u)}(x) \geq q_{t,n}(x) \geq q_{t,n}^{(l)}(x), \\ q_{t,n}^{(l)}(x) \text{ if } & q_{t,n}(x) < q_{t,n}^{(l)}(x). \end{cases}$$

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