Chapter 4

Growth Optimal Portfolio Selection with Short Selling and Leverage

Márk Horváth and András Urbán

Department of Computer Science and Information Theory, Budapest University of Technology and Economics. H-1117, Magyar tudósok körútja 2., Budapest, Hungary, mhorvath@math.bme.hu, urbi@shannon.szit.bme.hu

The growth optimal strategy on non-leveraged, long only memoryless markets is the best constantly rebalanced portfolio (BCRP), also called log-optimal strategy. Optimality conditions are derived to frameworks on leverage and short selling, and generalizing BCRP by establishing no-ruin conditions. Moreover the strategy and its asymptotic growth rate are investigated under memoryless assumption, both from theoretical and empirical points of view. The empirical performance of the methods was tested for NYSE data, demonstrating spectacular gains for leveraged portfolios and showing unimportance of short selling in the growth-rate sense both in case of BCRP and dynamic portfolios.

4.1. Introduction

Earlier results in the non-parametric statistics, information theory and economics literature (such as [Kelly (1956)], [Latané (1959)], [Breiman (1961)], [Markowitz (1952)], [Markowitz (1976)], [Finkelstein and Whitley (1981)]) established optimality criterion for long-only, non-leveraged investment. These results have shown that the market is inefficient, i.e. substantial gain is achievable by rebalancing and predicting market returns based on market's history. Our aim is to show that using leverage through margin buying (the act of borrowing money and increasing market exposure) yields substantially higher growth rate in the case of memoryless (independent identically distributed, i.i.d.) assumption on returns. Besides a framework for leveraged investment, we also establish mathematical basis for short selling, i.e. creating negative exposure to asset prices. Short selling means the process of borrowing assets and selling them immediately, with the obligation to rebuy them later.

It can be shown that the optimal asymptotic growth rate on a memoryless market coincides with that of the best constantly rebalanced portfolio (BCRP). The idea is that on a frictionless market the investor can rebalance his portfolio for free at each trading period. Hence asymptotic optimization on a memoryless market means that the growth optimal strategy will pick the same portfolio vector at each trading period. Strategies based on this observation are called constantly rebalanced portfolios (CRP), while the one with the highest asymptotic average growth rate is referred to as BCRP. Our results include the generalization of BCRP for margin buying and short selling frameworks.

To allow short and leverage our formulation weakens the constraints on feasible set of possible portfolio vectors, thus they are expected to improve performance. Leverage is anticipated to have substantial merit in terms of growth rate, while short selling is not expected to yield much better results. We do not expect increased profits on short CRP strategy, since companies worth to short in our test period should have already defaulted by now. Nonetheless short selling might yield increased profits in case of markets with memory, since earlier results have shown that the market was inefficient (cf. [Györfi *et al.* (2006)]). In case of i.i.d. returns with known distribution, [Cover (1984)] has introduced a gradient based method for optimization of long-only log-optimal portfolios, and gave necessary and sufficient conditions on growth optimal investment in [Bell and Cover (1980)]. We extend these results to short selling and leverage.

Contrary to non-leveraged long only investment in earlier literature, in case of margin buying and short selling it is easy to default on total initial investment. In this case asymptotic growth rate is minus infinity. By bounding possible market returns, we establish circumstances such that default is impossible. We do this in such a way that debt and positions are limited and the investor is always able to satisfy his liabilities selling assets. Restriction of market exposure and amount of debt is in line with the practice of brokerages and regulators.

Our notation for asset prices and returns are as follows. Consider a market consisting of d assets. Evolution of prices is represented by a sequence of price vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots \in \mathbb{R}_+^d$, where

$$\mathbf{s}_n = (s_n^{(1)}, \ldots, s_n^{(d)}). \tag{4.1}$$

$s_n^{(j)}$ denotes the price of the j-th asset at the end of the n-th trading period.

In order to apply the usual techniques for time series analysis, we transform the sequence of price vectors $\{\mathbf{s}_n\}$ into return vectors:

$$\mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(d)}),$$

where

$$x_n^{(j)} = \frac{s_{n+1}^{(j)}}{s_n^{(j)}}.$$

Here the j-th component $x_n^{(j)}$ of the return vector $\mathbf{x}_n$ denotes the amount obtained by investing unit capital in the j-th asset during the n-th trading period.

4.2. Non-leveraged, long only investment

A representative example of the dynamic portfolio selection in the long only case is the constantly rebalanced portfolio (CRP), introduced and studied by [Kelly (1956)], [Latané (1959)], [Breiman (1961)], [Markowitz (1976)], [Finkelstein and Whitley (1981)], [Móri (1982)], [Móri and Székely (1982)] and [Cover (1984)]. For a comprehensive survey, see also Chapters 6 and 15 in [Cover and Thomas (1991)], and Chapter 15 in [Luenberger (1998)].

CRP is a self-financing portfolio strategy, rebalancing to the same proportional portfolio in each investment period. This means that the investor neither consumes from, nor deposits new cash into his account, but reinvests his capital in each trading period. Using this strategy the investor chooses a proportional portfolio vector $\mathbf{b} = (b^{(1)}, \ldots, b^{(d)})$, and rebalances his portfolio after each period to correct the price shifts in the market. This way the proportion of his wealth invested in each asset at the beginning of trading periods is constant.

The j-th component $b^{(j)}$ of $\mathbf{b}$ denotes the proportion of the investor's capital invested in asset j. Thus the portfolio vector has nonnegative components that sum up to 1. The set of portfolio vectors is denoted by

$$\Delta_d = \left\{ \mathbf{b} = (b^{(1)}, \ldots, b^{(d)});\, b^{(j)} \geq 0,\, \sum_{j=1}^{d} b^{(j)} = 1 \right\}. \tag{4.2}$$

Let S_0 denote the investor's initial capital. At the beginning of the first trading period $S_0 b^{(j)}$ is invested into asset j, and it results in position size $S_0 b^{(j)} x_1^{(j)}$ after changes in market prices. Therefore, at the end of the first trading period the investor's wealth becomes

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b} , \mathbf{x}_1 \rangle ,$$

where $\langle \cdot , \cdot \rangle$ denotes inner product. For the second trading period S_1 is the new initial capital, hence

$$S_2 = S_1 \langle \mathbf{b} , \mathbf{x}_2 \rangle = S_0 \langle \mathbf{b} , \mathbf{x}_1 \rangle \langle \mathbf{b} , \mathbf{x}_2 \rangle .$$

By induction for trading period n,

$$S_n = S_{n-1} \langle \mathbf{b} , \mathbf{x}_n \rangle = S_0 \prod_{i=1}^{n} \langle \mathbf{b} , \mathbf{x}_i \rangle . \tag{4.3}$$

Including cash account into the framework is straight forward by assuming

$$x_n^{(j)} = 1$$

for some j and for all n. The asymptotic average growth rate of this portfolio selection is

$$\begin{aligned} W(\mathbf{b}) &= \lim_{n\to\infty} \ln \sqrt[n]{S_n} = \lim_{n\to\infty} \frac{1}{n} \ln S_n \\ &= \lim_{n\to\infty} \left(\frac{1}{n} \ln S_0 + \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b} , \mathbf{x}_i \rangle \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b} , \mathbf{x}_i \rangle , \end{aligned}$$

if the limit exists. This also means that without loss of generality we can assume that the initial capital $S_0 = 1$.

If the market process $\{\mathbf{X}_i\}$ is memoryless, i.e., is a sequence of independent and identically distributed (i.i.d.) random return vectors, then the asymptotic rate of growth exists almost surely (a.s.), where, with random vector X being distributed as X_i,

$$W(\mathbf{b}) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b} , \mathbf{X}_i \rangle = \mathbb{E} \ln \langle \mathbf{b} , \mathbf{X} \rangle \ \text{ a.s.}, \tag{4.4}$$

given that $\mathbb{E} \ln \langle \mathbf{b} , \mathbf{X} \rangle$ is finite, due to strong law of large numbers. We can ensure this property by assuming finiteness of $\mathbb{E} \ln X^{(j)}$, i.e., $\mathbb{E} | \ln X^{(j)} | < \infty$ for each $j \in \{1, \ldots , d\}$.

In fact, because of $b^{(i)} > 0$ for some i, we have

$$\begin{aligned}\mathbb{E}\ln\langle\mathbf{b}\,,\,\mathbf{X}\rangle &\geq \mathbb{E}\ln\left(b^{(i)}X^{(j)}\right)\\ &= \ln b^{(i)} + \mathbb{E}\ln X^{(i)} > -\infty\ ,\end{aligned}$$

and because of $b^{(j)} \leq 1$ for all j, we have

$$\begin{aligned}\mathbb{E}\ln\langle\mathbf{b}\,,\,\mathbf{X}\rangle &\leq \mathbb{E}\ln\left(d\max_j X^{(j)}\right)\\ &= \ln d + \mathbb{E}\max_j \ln X^{(j)}\\ &\leq \ln d + \mathbb{E}\max_j \ln|X^{(j)}|\\ &\leq \ln d + \sum_j \mathbb{E}\ln|X^{(j)}| < \infty\ .\end{aligned}$$

From (4.4) it follows that rebalancing according to the best log-optimal strategy

$$\mathbf{b}^* \in \arg\max_{\mathbf{b}\in\Delta_d} \mathbb{E}\ln\langle\mathbf{b}\,,\,\mathbf{X}\rangle\ \ ,$$

is also an asymptotically optimal trading strategy, i.e., a strategy with a.s. optimum asymptotic growth

$$W(\mathbf{b}^*) \geq W(\mathbf{b}),$$

for any $\mathbf{b} \in\Delta_d$. The strategy of rebalancing according to $\mathbf{b}^*$ at the beginning of each trading period, is called best constantly rebalanced portfolio (BCRP).

In the following we repeat calculations of [Bell and Cover (1980)]. Our aim is to maximize asymptotic average rate of growth. $W(\mathbf{b})$ being concave, we minimize the convex objective function

$$f_{\mathbf{X}}(\mathbf{b}) = -W(\mathbf{b}) = -\mathbb{E}\ln\langle\mathbf{b},\mathbf{X}\rangle\,. \tag{4.5}$$

To use Kuhn-Tucker theorem we establish linear, inequality type constraints over the search space Δ_d in (4.2):

$$-b^{(i)} \leq 0,$$

for $i = 1,\ldots,d$, i.e.

$$\langle\mathbf{b},\,\mathbf{a}_i\rangle \leq 0, \tag{4.6}$$

where $\mathbf{a}_i \in \mathbb{R}^d$ denotes the i-th unit vector, having -1 at position i.

Our only equality type constraint is

$$\sum_{j=1}^{d} b^{(j)} - 1 = 0,$$

i.e.

$$\langle \mathbf{b}, \mathbf{e} \rangle - 1 = 0, \tag{4.7}$$

where $\mathbf{e} \in \mathbb{R}^d, \mathbf{e} = (1, 1, \ldots, 1)$.

The partial derivatives of the objective function are

$$\frac{\partial f_{\mathbf{X}}(\mathbf{b})}{\partial b^{(i)}} = -\mathbb{E}\frac{X^{(i)}}{\langle \mathbf{b}, \mathbf{X} \rangle},$$

for $i = 1, \ldots, d$.

According to Kuhn-Tucker theorem ([Kuhn and Tucker (1951)]), the portfolio vector $\mathbf{b}^*$ is optimal if and only if, there are constants $\mu_i \geq 0$ ($i = 1, \ldots, d$) and $\vartheta \in \mathbb{R}$, such that

$$f'_{\mathbf{X}}(\mathbf{b}^*) + \sum_{i=1}^{d} \mu_i \mathbf{a}_i + \vartheta \mathbf{e} = 0$$

and

$$\mu_j \langle \mathbf{b}^*, \mathbf{a}_j \rangle = 0,$$

for $j = 1, \ldots, d$.

This means that

$$-\mathbb{E}\frac{X^{(j)}}{\langle \mathbf{b}^*, \mathbf{X} \rangle} - \mu_j + \vartheta = 0 \tag{4.8}$$

and

$$\mu_j b^{*(j)} = 0,$$

for $j = 1, \ldots, d$. Summing up (4.8) weighted by $b^{*(j)}$, we obtain:

$$-\mathbb{E}\frac{\langle \mathbf{b}^*, \mathbf{X} \rangle}{\langle \mathbf{b}^*, \mathbf{X} \rangle} - \sum_{j=1}^{d} \mu_j b^{*(j)} + \sum_{j=1}^{d} \vartheta b^{*(j)} = 0,$$

hence

$$\vartheta = 1.$$

We can state the following necessary condition for optimality of $\mathbf{b}^*$. If

$$\mathbf{b}^* \in \arg\max_{\mathbf{b} \in \Delta_d} W(\mathbf{b}),$$

then

$$b^{*(j)} > 0 \Longrightarrow \mu_j = 0 \Longrightarrow \mathbb{E}\frac{X^{(j)}}{\langle \mathbf{b}^*, \mathbf{X}\rangle} = 1, \tag{4.9}$$

and

$$b^{*(j)} = 0 \Longrightarrow \mu_j \geq 0 \Longrightarrow \mathbb{E}\frac{X^{(j)}}{\langle \mathbf{b}^*, \mathbf{X}\rangle} \leq 1. \tag{4.10}$$

Because of convexity of $f_{\mathbf{X}}(\mathbf{b})$ the former conditions are sufficient, too. Assume $\mathbf{b}^*\in\Delta_d$. If for any fixed $j = 1, \dots, d$ either

$$\mathbb{E}\frac{X^{(j)}}{\langle \mathbf{b}^*, \mathbf{X}\rangle} = 1 \text{ and } b^{*(j)} > 0,$$

or

$$\mathbb{E}\frac{X^{(j)}}{\langle \mathbf{b}^*, \mathbf{X}\rangle} \leq 1 \text{ and } b^{*(j)} = 0,$$

then $\mathbf{b}^*$ is optimal. The latter two conditions pose a necessary and sufficient condition on optimality of b^*.

Remark 4.1. In case of an *independent* asset, i.e. for some $j \in 1, \dots, d$, $X^{(j)}$ being independent from the rest of the assets,

$$b^{*(j)} = 0 \Longrightarrow \mathbb{E}\frac{X^{(j)}}{\langle \mathbf{b}^*, \mathbf{X}\rangle} \leq 1$$

implies by $b^{*(j)} = 0$ that $X^{(j)}$ is independent of $\langle \mathbf{b}^*, \mathbf{X}\rangle$. This means that

$$b^{*(j)} = 0 \Longrightarrow \mathbb{E}X^{(j)}\mathbb{E}\frac{1}{\langle \mathbf{b}^*, \mathbf{X}\rangle} \leq 1,$$

therefore

$$b^{*(j)} = 0 \Longrightarrow \mathbb{E}X^{(j)} \leq \frac{1}{\mathbb{E}\frac{1}{\langle \mathbf{b}^*, \mathbf{X}\rangle}}.$$

According to Kuhn-Tucker theorem, for any fixed $j = 1, \dots, d$ either

$$\mathbb{E}\frac{X^{(j)}}{\langle \mathbf{b}^*, \mathbf{X}\rangle} = 1 \text{ and } b^{*(j)} > 0,$$

or

$$\mathbb{E}X^{(j)} \leq \frac{1}{\mathbb{E}\frac{1}{\langle \mathbf{b}^*, \mathbf{X}\rangle}} \text{ and } b^{*(j)} = 0,$$

if and only if $\mathbf{b}^*$ is optimal, i.e.,

$$\mathbf{b}^* \in \arg\max_{\mathbf{b}\in\Delta_d} W(\mathbf{b}). \tag{4.11}$$

Remark 4.2. Assume optimal portfolio $\mathbf{b}^*$ for d assets

$$\mathbf{X} = (X^{(1)}, X^{(2)}, \ldots, X^{(d)})$$

is already established. Given a new asset – being independent of our previous d assets – we can formulate a condition on its inclusion in the new optimal portfolio $\mathbf{b}^{**}$. If

$$\mathbb{E}X^{(d+1)} < \frac{1}{\mathbb{E}\frac{1}{\langle \mathbf{b}^*, \mathbf{X} \rangle}}$$

then

$$b^{**(d+1)} = 0.$$

This means that, for a new independent asset like the cash, we do not have to do the *optimization* for each asset in the portfolio, and we can reach substantial reduction in dimension of the search for an optimal portfolio.

Remark 4.3. The same trick can be applied in case of *dependent* returns as well. If

$$\mathbb{E}\frac{X^{(d+1)}}{\langle \mathbf{b}^*, \mathbf{X} \rangle} < 1 \text{ then } b^{**(d+1)} = 0,$$

which is much simpler to verify then performing *optimization* of asymptotic average growth. This condition can be formulated as

$$\mathbb{E}X^{(d+1)}\mathbb{E}\frac{1}{\langle \mathbf{b}^*, \mathbf{X} \rangle} + \mathbf{Cov}\left(X^{(d+1)}, \frac{1}{\langle \mathbf{b}^*, \mathbf{X} \rangle}\right) \leq 1,$$

which poses a condition on covariance and expected value of the new asset.

Remark 4.4. [Roll (1973)], [Pulley (1994)] and [Vajda (2006)] suggested an approximation of $\mathbf{b}^*$ using

$$\ln z \approx h(z) = z - 1 - \frac{1}{2}(z-1)^2,$$

which is the second order Taylor approximation of the function $\ln z$ at $z = 1$. Then the *semi-log-optimal* portfolio selection is

$$\bar{\mathbf{b}} \in \arg\max_{\mathbf{b} \in \Delta_d} \mathbb{E}\{h\,\langle \mathbf{b}\,,\,\mathbf{X} \rangle\}.$$

Our new objective function is convex:

$$\begin{aligned}
\overline{f}_{\mathbf{X}}(\mathbf{b}) &= -\mathbb{E}\{h\left<\mathbf{b},\mathbf{X}\right>\} \\
&= -\mathbb{E}\{\left<\mathbf{b},\mathbf{X}\right> - 1 - \frac{1}{2}(\left<\mathbf{b},\mathbf{X}\right> - 1)^2\} \\
&= \mathbb{E}\{-\left<\mathbf{b},\mathbf{X}\right> + 1 + \frac{1}{2}(\left<\mathbf{b},\mathbf{X}\right> - 1)^2\} \\
&= \mathbb{E}\{-\left<\mathbf{b},\mathbf{X}\right> + 1 + \frac{\mathbf{1}}{2}\left<\mathbf{b},\mathbf{X}\right>^2 - \left<\mathbf{b},\mathbf{X}\right> + \frac{1}{2}\} \\
&= \mathbb{E}\{\frac{\mathbf{1}}{2}\left<\mathbf{b},\mathbf{X}\right>^2 - 2\left<\mathbf{b},\mathbf{X}\right> + \frac{3}{2}\} \\
&= \frac{\mathbf{1}}{2}\langle\mathbf{b},\mathbb{E}(\mathbf{X}\mathbf{X}^T)\mathbf{b}\rangle - \langle\mathbf{b},2\mathbb{E}\mathbf{X}\rangle + \frac{3}{2} \\
&= \mathbb{E}\left\{\left(\frac{\mathbf{1}}{\sqrt{2}}\left<\mathbf{b},\mathbf{X}\right> - \sqrt{2}\right)^2 - \frac{1}{2}\right\}\ ,
\end{aligned}$$

where $\mathbf{X}$ is column vector, $\mathbf{X}\mathbf{X}^T$ denotes outer product. This is equivalent to minimizing

$$\overline{\overline{f}}_{\mathbf{X}}(\mathbf{b}) = \mathbb{E}(\left<\mathbf{b},\mathbf{X}\right> - 2)^2.$$

Thus $\overline{\mathbf{b}}$ can be simply calculated to minimize the squared distance from 2. In case of data driven algorithms, the solution is using linear regression, under the constraint $\mathbf{b}^*\in\Delta_d$.

$$\begin{aligned}
\overline{\overline{f}}_{\mathbf{X}}(\mathbf{b}) &= \mathbf{Var}\left<\mathbf{b},\mathbf{X}\right> + (2 - \mathbb{E}\left<\mathbf{b},\mathbf{X}\right>)^2 \\
&= \mathbf{Var}\left<\mathbf{b},\mathbf{X}\right> + (2 - \left<\mathbf{b},\mathbb{E}\mathbf{X}\right>)^2.
\end{aligned}$$

This means we minimize variance of returns while maximizing expected return. This is in close resemblance with Markowitz type portfolio selection. For a discussion of the relationship between Markowitz type portfolio selection and the semi-log-optimal strategy (see [Ottucsák and Vajda (2007)] and Chapter 2 of this volume). The problem can be formulated as a quadratic optimization problem as well,

$$\overline{\overline{f}}_{\mathbf{X}}(\mathbf{b}) = \left<\mathbf{b},\mathbf{R}\mathbf{b}\right> + 4 - 4\left<\mathbf{b},\mathbf{m}\right>$$

where

$$\mathbf{R} = \mathbb{E}(\mathbf{X}\mathbf{X}^T),$$

and

$$\mathbf{m} = \mathbb{E}(\mathbf{X}).$$

Note that $\mathbf{R}$ is symmetric, and positive semidefinite, since for any $\mathbf{z} \in \mathbb{R}^d$

$$\mathbf{z}^T \mathbf{R} \mathbf{z} = \mathbf{z}^\mathbf{T} \mathbb{E}(\mathbf{X}\mathbf{X}^T)\mathbf{z} = \mathbb{E}(\mathbf{z}^\mathbf{T}\mathbf{X})^2 \geq \mathbf{0}.$$

This means we face a convex programming problem again.

4.3. Short selling

4.3.1. *No-ruin constraints*

Short selling an asset is usually done by borrowing the asset under consideration and selling it. As collateral the investor has to provide securities of the same value to the lender of the shorted asset. This ensures that if anything goes wrong, the lender still has high recovery rate.

While the investor has to provide collateral, after selling the assets having been borrowed, he obtains the price of the shorted asset again. This means that short selling is virtually for free.

$$S' = S - C + P,$$

where S' is wealth after opening the short position, S is wealth before, C is collateral for borrowing and P is price income of selling the asset being shorted. For simplicity we assume

$$C = P,$$

hence

$$S' = S,$$

and short selling is free. In practice the act of short selling is more complicated. For institutional investors the size of collateral depends on supply and demand on the short market, and the receiver of the more liquid asset usually pays interest. For simplicity we ignore these problems.

Let us elaborate this process on a real life example. Assume the investor wants to short sell 10 shares of IBM at \$100, and he has \$1000 in cash. First he has to find a lender – the short provider – who is willing to lend the shares. After exchanging the shares and the \$1000 collateral, the investor sells the borrowed shares. After selling the investor has \$1000 in cash again, and the obligation to cover the shorted assets later.

In contrast with our modelling approach where short selling is free, it is also modelled in literature such that selling an asset short yields immediate cash – this is called naked short transaction. This is the case in the Chapter on Mean-Variance Portfolio Theory of [Luenberger (1998)] and in [Cover and Ordentlich (1998)].

Assume our only investment is in asset j and our initial wealth is S_0. We invest a proportion of $b \in (-1, 1)$ of our wealth. If the position is long $(b > 0)$ it results in wealth

$$S_0(1-b) + S_0 b x_1^{(j)} = S_0 + S_0 b(x_1^{(j)} - 1),$$

while if the position is short $(b < 0)$, we win as much money, as price drop of the asset:

$$S_0 + S_0|b|(1 - x_1^{(j)}) = S_0 + S_0 b(x_1^{(j)} - 1).$$

In line with the previous example, assume that our investor has shorted 10 shares of IBM, at \$100. If the price drops \$10, he has to cover the short position at \$90, thus he gains 10 x \$10. If the price rises \$10, he has to cover at \$110, loosing 10 x \$10.

Let $\mathbf{b} = (b^{(0)}, b^{(1)}, \dots, b^{(d)})$ be the portfolio vector such that the 0-th component corresponds to cash. At the end of the first trading period the investor's wealth becomes

$$S_1 = S_0 \left(b^{(0)} + \sum_{j=1}^{d} \left[{b^{(j)}}^{+} x_1^{(j)} + {b^{(j)}}^{-} (x_1^{(j)} - 1) \right] \right)^{+}, \tag{4.12}$$

where $(.)^{-}$ denotes the negative part operation. In case of the investor's net wealth falling to zero or below he defaults. Negative wealth is not allowed in our framework, thus the outer positive part operation. Since only long positions cost money in this setup, we will constrain to portfolios such that $\sum_{j=0}^{d} {b^{(j)}}^{+} = 1$. Considering this it is also true that

$$S_1 = S_0 \left(\sum_{j=0}^{d} {b^{(j)}}^{+} + \sum_{j=1}^{d} \left[{b^{(j)}}^{+} (x_1^{(j)} - 1) + {b^{(j)}}^{-} (x_1^{(j)} - 1) \right] \right)^{+} \tag{4.13}$$

$$= S_0 \left(1 + \sum_{j=1}^{d} \left[b^{(j)} (x_1^{(j)} - 1) \right] \right)^{+}. \tag{4.14}$$

This shows that we gain as much as long positions raise and short positions fall.

We can see that short selling is a risky investment, because it is possible to default on total initial wealth without the default of any of the assets in the portfolio. The possibility of this would lead to a growth rate of minus infinity, thus we restrict our market according to

$$1 - B + \delta < x_n^{(j)} < 1 + B - \delta, j = 1, \ldots, d. \tag{4.15}$$

Besides aiming at no-ruin, the role of $\delta > 0$ is ensuring that rate of growth is finite for any portfolio vector (i.e. $> -\infty$).

For the usual stock market daily data, there exist $0 < a_1 < 1 < a_2 < \infty$ such that

$$a_1 \leq x_n^{(j)} \leq a_2$$

for all $j = 1, \ldots, d$, for example, $a_1 = 0.7$ and with $a_2 = 1.2$ (cf. [Fernholz (2000)]). Thus, we can choose $B = 0.3$.

Given (4.14) and (4.15) it is easy to see that maximal loss that we could suffer is $B \sum_{j=1}^{d} |b^{(j)}|$. This value has to be constrained to ensure no-ruin.

We denote the set of possible portfolio vectors by

$$\Delta_d^{(-B)} = \left\{ \mathbf{b} = (b^{(0)}, b^{(1)}, \ldots, b^{(d)});\, b^{(0)} \geq 0, \sum_{j=0}^{d} {b^{(j)}}^{+} = 1,\, B \sum_{j=1}^{d} |b^{(j)}| \leq 1 \right\}. \tag{4.16}$$

$\sum_{j=0}^{d} {b^{(j)}}^{+} = 1$ means that we invest all of our initial wealth into some assets – buying long – or cash. By $B \sum_{j=1}^{d} |b^{(j)}| \leq 1$, maximal exposure is limited such that ruin is not possible, and rate of growth it is finite. $b^{(0)}$ is not included in the latter inequality, since possessing cash does not pose risk. Notice that if $B \leq 1$ then $\Delta_{d+1} \subset \Delta_d^{(-B)}$, and so the achievable growth rate with short selling can not be smaller than in long only case.

According to (4.14) and (4.15) with $B \leq 1$ we show that ruin is impossible:

$$\begin{aligned}
& 1 + \sum_{j=1}^{d} \left[b^{(j)} (x_1^{(j)} - 1) \right] \\
& > 1 + \sum_{j=1}^{d} \left[{b^{(j)}}^{+} (1 - B + \delta - 1) + {b^{(j)}}^{-} (1 + B - \delta - 1) \right] \\
& = 1 - (B - \delta) \sum_{j=1}^{d} |b^{(j)}| \\
& \geq \delta \sum_{j=1}^{d} |b^{(j)}|.
\end{aligned}$$

If $\sum_{j=1}^{d} |b^{(j)}| = 0$ then $b^{(0)} = 1$, hence no-ruin. In any other case, $\delta \sum_{j=1}^{d} |b^{(j)}| > 0$, hence we have not only ensured no-ruin, but also

$$\mathbb{E} \ln \left(1 + \sum_{j=1}^{d} \left[b^{(j)} (X_1^{(j)} - 1) \right] \right)^{+} > -\infty.$$

4.3.2. *Optimality condition for short selling with cash account*

A problem with $\Delta_d^{(-B)}$ is its non-convexity. To see this consider

$$\mathbf{b}_1 = (0, 1) \in \Delta_1^{(-1)},$$

$$\mathbf{b}_2 = (1, -1/2) \in \Delta_1^{(-1)},$$

with

$$\frac{\mathbf{b}_1 + \mathbf{b}_2}{2} = (1/2, 1/4) \notin \Delta_1^{(-1)}.$$

This means we can not simply apply Kuhn-Tucker theorem on $\Delta_d^{(-B)}$.

Given cash balance, we can transform our non-convex $\Delta_d^{(-B)}$ to a convex region $\widetilde{\Delta}_d^{(-B)}$, where application of our tools established in long only investment becomes feasible. The new set of possible portfolio vectors is a convex region:

$$\widetilde{\Delta}_d^{(-B)} = \left\{ \widetilde{\mathbf{b}} = (\widetilde{b}^{(0+)}, \widetilde{b}^{(1+)}, \widetilde{b}^{(1-)}, \ldots, \widetilde{b}^{(d+)}, \widetilde{b}^{(d-)}) \in \mathbb{R}_0^{+\,2d+1}; \right.$$
$$\left. \sum_{j=0}^{d} \widetilde{b}^{(j+)} = 1, B \sum_{j=1}^{d} (\widetilde{b}^{(j+)} + \widetilde{b}^{(j-)}) \le 1 \right\}.$$

Mapping from $\Delta_d^{(-B)}$ to $\widetilde{\Delta}_d^{(-B)}$ happens by

$$\widetilde{\mathbf{b}} = (b^{(0)}, (b^{(1)})^{+}, |(b^{(1)})^{-}|, \ldots, (b^{(d)})^{+}, |(b^{(d)})^{-}|). \tag{4.17}$$

(4.12) implies that

$$S_1 = S_0 \left(\widetilde{b}^{(0+)} + \sum_{j=1}^{d} \left[\widetilde{b}^{(j+)} x_1^{(j)} + \widetilde{b}^{(j-)} (1 - x_1^{(j)}) \right] \right)^{+},$$

thus in line with the portfolio vector being transformed we transform the market vector too

$$\widetilde{\mathbf{X}} = (1, X^{(1)}, 1 - X^{(1)}, \ldots, X^{(d)}, 1 - X^{(d)}),$$

so that

$$S_1 = S_0 \left\langle \widetilde{\mathbf{b}}, \widetilde{\mathbf{X}} \right\rangle .$$

To use the Kuhn-Tucker theorem we enumerate linear, inequality type constraints over the search space

$$B \sum_{j=1}^{d} (\widetilde{b}^{(j_+)} + \widetilde{b}^{(j_-)}) \leq 1,$$

and

$$\widetilde{b}^{(0_+)} \geq 0, \widetilde{b}^{(i_+)} \geq 0, \widetilde{b}^{(i_-)} \geq 0,$$

for $i = 1, \ldots, d$. Our only equality type constraint is

$$\sum_{j=0}^{d} \widetilde{b}^{(j_+)} = 1.$$

The partial derivatives of the convex objective function $f_{\mathbf{X}}(\widetilde{\mathbf{b}}) = -\mathbb{E} \ln \left\langle \widetilde{\mathbf{b}}\, ,\, \widetilde{\mathbf{X}} \right\rangle$ are

$$\frac{\partial f_{\mathbf{X}}(\widetilde{\mathbf{b}})}{\partial \widetilde{b}^{(0_+)}} = -\mathbb{E} \frac{1}{\left\langle \widetilde{\mathbf{b}}, \widetilde{\mathbf{X}} \right\rangle},$$

$$\frac{\partial f_{\mathbf{X}}(\widetilde{\mathbf{b}})}{\partial \widetilde{b}^{(i_+)}} = -\mathbb{E} \frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}, \widetilde{\mathbf{X}} \right\rangle},$$

$$\frac{\partial f_{\mathbf{X}}(\widetilde{\mathbf{b}})}{\partial \widetilde{b}^{(i_-)}} = -\mathbb{E} \frac{1 - X^{(i)}}{\left\langle \widetilde{\mathbf{b}}, \widetilde{\mathbf{X}} \right\rangle},$$

for $i = 1, \ldots, d$.

According to Kuhn-Tucker theorem (KT), the portfolio vector $\widetilde{\mathbf{b}}^*$ is optimal if and only if, there are KT multipliers assigned to each of the former $2d + 3$ constraints $\mu_{0_+} \geq 0, \mu_{i_+} \geq 0, \mu_{i_-} \geq 0, \nu_B \geq 0$ $(i = 1, \ldots, d)$ and $\vartheta \in \mathbb{R}$, such that

$$-\mathbb{E} \frac{1}{\left\langle \widetilde{\mathbf{b}}^*, \mathbf{X} \right\rangle} + \vartheta - \mu_{0_+} = 0, \tag{4.18}$$

$$-\mathbb{E} \frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \mathbf{X} \right\rangle} + \vartheta - \mu_{i_+} + \nu_B B = 0,$$

$$-\mathbb{E} \frac{1 - X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \mathbf{X} \right\rangle} - \mu_{i_-} + \nu_B B = 0,$$

and

$$\mu_{0_+}\widetilde{b}^{*(0_+)} = 0,$$
$$\mu_{i_+}\widetilde{b}^{*(i_+)} = 0,$$
$$\mu_{i_-}\widetilde{b}^{*(i_-)} = 0,$$

for $i = 1, \ldots, d$, while

$$\nu_B[B\sum_{j=1}^{d}(\widetilde{b}^{(j_+)}+\widetilde{b}^{(j_-)}) - 1] = 0, \tag{4.19}$$
$$\nu_B B\sum_{j=1}^{d}(\widetilde{b}^{(j_+)}+\widetilde{b}^{(j_-)}) = \nu_B.$$

Summing up equations in (4.18) weighted by $\widetilde{b}^{*(0)}, \widetilde{b}^{*(i_+)}, \widetilde{b}^{*(i_-)}$, we obtain:

$$-\mathbb{E}\frac{\left\langle \widetilde{\mathbf{b}}^{*}, \widetilde{\mathbf{X}} \right\rangle}{\left\langle \widetilde{\mathbf{b}}^{*}, \widetilde{\mathbf{X}} \right\rangle} + \vartheta\sum_{j=0}^{d}\widetilde{b}^{*(j_+)} + \nu_B B\sum_{j=1}^{d}(\widetilde{b}^{(j_+)}+\widetilde{b}^{(j_-)}) = 0,$$
$$-1 + \vartheta + \nu_B = 0,$$
$$\vartheta = 1 - \nu_B, \tag{4.20}$$
$$\vartheta \le 1.$$

In case of $B\sum_{j=1}^{d}(\widetilde{b}^{(j_+)}+\widetilde{b}^{(j_-)}) < 1$, because of (4.19) and (4.20) we have that

$$\nu_B = 0, \text{ hence } \vartheta = 1.$$

This implies

$$-\mathbb{E}\frac{1}{\left\langle \widetilde{\mathbf{b}}^{*}, \widetilde{\mathbf{X}} \right\rangle} + 1 - \mu_{0_+} = 0,$$
$$-\mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^{*}, \widetilde{\mathbf{X}} \right\rangle} + 1 - \mu_{i_+} = 0,$$
$$-\mathbb{E}\frac{1 - X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^{*}, \widetilde{\mathbf{X}} \right\rangle} - \mu_{i_-} = 0.$$

These equations result in the following additional properties

$$\widetilde{b}^{*(0_+)} > 0 \Longrightarrow \mu_{0_+} = 0 \Longrightarrow \mathbb{E}\frac{1}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} = 1,$$

$$\widetilde{b}^{*(0_+)} = 0 \Longrightarrow \mu_{0_+} \geq 0 \Longrightarrow \mathbb{E}\frac{1}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq 1,$$

and

$$\widetilde{b}^{*(i_+)} > 0 \Longrightarrow \mu_{i_+} = 0 \Longrightarrow \mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} = 1,$$

$$\widetilde{b}^{*(i_+)} = 0 \Longrightarrow \mu_{i_+} \geq 0 \Longrightarrow \mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq 1,$$

and

$$\widetilde{b}^{*(i_-)} > 0 \Longrightarrow \mu_{i_-} = 0 \Longrightarrow \mathbb{E}\frac{1 - X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} = 0,$$

$$\widetilde{b}^{*(i_-)} = 0 \Longrightarrow \mu_{i_-} \geq 0 \Longrightarrow \mathbb{E}\frac{1 - X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq 0\ ,$$

for $i = 1, \ldots, d$.

We transform the vector $\widetilde{\mathbf{b}}^*$ to the vector $\mathbf{b}^*$ such that

$$b^{*(i)} = \widetilde{b}^{*(i_+)} - \widetilde{b}^{*(i_-)}$$

$(i = 1, \ldots, d)$, while

$$b^{*(0)} = \widetilde{b}^{*(0)} + \sum_{i=1}^{d} \min\{\widetilde{b}^{*(i_+)}, \widetilde{b}^{*(i_-)}\}.$$

This way

$$\sum_{j=1}^{d} |b^{*(j)}| = 1,$$

and we have the same market exposure with $\mathbf{b}^*$ as with $\widetilde{\mathbf{b}}$.

Due to the simple mapping (4.17), with regard to the original portfolio vector $\mathbf{b}^*$ this means

$$b^{*(0)} > 0 \Longrightarrow \mathbb{E}\frac{1}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} = 1, \tag{4.21}$$
$$b^{*(0)} = 0 \Longrightarrow \mathbb{E}\frac{1}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq 1.$$

Also

$$b^{*(i)} > 0 \Longrightarrow \mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} = 1 \text{ and } \mathbb{E}\frac{1 - X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq 0,$$

which is equivalent to

$$\mathbb{E}\frac{1}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq \mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} = 1,$$

and

$$b^{*(i)} = 0 \Longrightarrow \mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq 1 \text{ and } \mathbb{E}\frac{1 - X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq 0,$$

which is equivalent to

$$\mathbb{E}\frac{1}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq \mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq 1,$$

and

$$b^{*(i)} < 0 \Longrightarrow \mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq 1 \text{ and } \mathbb{E}\frac{1 - X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} = 0,$$

which is equivalent to

$$\mathbb{E}\frac{1}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq \mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^*, \widetilde{\mathbf{X}} \right\rangle} \leq 1,$$

for $i = 1, \ldots, d$.

4.4. Long only leveraged investment

4.4.1. *No-ruin condition*

In the leveraged frameworks we assume (4.15), thus market exposure can be increased over one without the possibility of ruin. Again, we denote the portfolio vector by

$$\mathbf{b} = (b^{(0)}, b^{(1)}, \dots, b^{(d)}),$$

where $b^{(0)} \geq 0$ stands for the cash balance, and since no short selling $b^{(i)} \geq 0, i = 1, \dots, d$.

Assume the investor can borrow money and invest it on the same rate r. Assume also that the maximal investable amount of cash $L_{B,r}$ (relative to initial wealth S_0), is always available for the investor. In the sequel we refer to $L_{B,r}$ as buying power. $L_{B,r}$ is chosen to be the maximal amount, investing of which ruin is not possible given 4.15. Because our investor decides over the distribution of his buying power

$$\sum_{j=0}^{d} b^{(j)} = L_{B,r}.$$

Unspent cash earns the same interest r, as the rate of lending. The market vector is defined as

$$\mathbf{X}_r = (X^{(0)}, X^{(1)}, \dots, X^{(d)}) = (1 + r, X^{(1)}, \dots, X^{(d)}),$$

so $X^{(0)} = 1 + r$. The feasible set of portfolio vectors is

$${}^{r}\Delta_d^{+B} = \left\{ \mathbf{b} = (b^{(0)}, b^{(1)}, \dots, b^{(d)}) \in {\mathbb{R}_0^+}^{d+1}, \sum_{j=0}^{d} b^{(j)} = L_{B,r} \right\},$$

where $b^{(0)}$ denotes unspent buying power. Market evolves according to

$$S_1 = S_0(\langle \mathbf{b}, \mathbf{X}_r \rangle - (L_{B,r} - 1)(1 + r))^+,$$

where $S_0 r(L_{B,r} - 1)$ is interest on borrowing $L_{B,r} - 1$ times initial wealth S_0.

To ensure no-ruin and finiteness of growth rate choose

$$L_{B,r} = \frac{1 + r}{B + r}. \tag{4.22}$$

This ensures that ruin is not possible:

$$\begin{aligned}
&\langle \mathbf{b}, \mathbf{X}_r \rangle - (L_{B,r} - 1)(1 + r) \\
&= \sum_{j=0}^{d} b^{(j)} X^{(j)} - (L_{B,r} - 1)(1 + r) \\
&= b^{(0)}(1 + r) + \sum_{j=1}^{d} b^{(j)} X^{(j)} - (L_{B,r} - 1)(1 + r) \\
&> b^{(0)}(1 + r) + \sum_{j=1}^{d} b^{(j)} (1 - B + \delta) - (L_{B,r} - 1)(1 + r) \\
&= b^{(0)}(1 + r) + (L_{B,r} - b^{(0)})(1 - B + \delta) - (L_{B,r} - 1)(1 + r) \\
&= b^{(0)}(r + B - \delta) - L_{B,r}(B - \delta + r) + 1 + r \\
&\geq -\frac{1 + r}{B + r}(B - \delta + r) + 1 + r \\
&= \delta \frac{1 + r}{B + r}.
\end{aligned}$$

4.4.2. *Kuhn-Tucker characterization*

Our convex objective function, the negative of asymptotic rate of growth is

$$f_{\mathbf{X}_r}^{+B}(\mathbf{b}) = -\mathbb{E} \ln(\langle \mathbf{b}, \mathbf{X}_r \rangle - (L_{B,r} - 1)(1 + r)).$$

The linear inequality type constraints are as follows:

$$-b^{(i)} \leq 0,$$

for $i = 0, \ldots, d$, while our only equality type constraint is

$$\sum_{j=0}^{d} b^{(j)} - L_{B,r} = 0.$$

The partial derivatives of the optimized function are

$$\frac{\partial f_{\mathbf{X}_r}^{+B}(\mathbf{b})}{\partial b^{(i)}} = -\mathbb{E} \frac{X^{(i)}}{\langle \mathbf{b}, \mathbf{X}_r \rangle - (L_{B,r} - 1)(1 + r)}.$$

According to the Kuhn-Tucker necessary and sufficient theorem, a portfolio vector $\mathbf{b}^*$, is optimal if and only if there are KT multipliers $\mu_j \geq 0$ $(j = 0, \ldots, d)$ and $\vartheta \in \mathbb{R}$, such that

$$-\mathbb{E} \frac{X^{(j)}}{\langle \mathbf{b}^*, \mathbf{X}_r \rangle - (L_{B,r} - 1)(1 + r)} - \mu_j + \vartheta = 0 \tag{4.23}$$

and

$$\mu_j b^{*(j)} = 0,$$

for $j = 0, \ldots, d$. Summing up (4.23) weighted by $b^{*(j)}$ we obtain:

$$-\mathbb{E}\frac{\langle \mathbf{b}^*, \mathbf{X}_r\rangle}{\langle \mathbf{b}^*, \mathbf{X}_r\rangle - (L_{B,r}-1)(1+r)} - \sum_{j=0}^{d}\mu_j b^{*(j)} + \sum_{j=0}^{d} b^{*(j)}\vartheta = 0,$$

$$1 + \mathbb{E}\frac{(L_{B,r}-1)(1+r)}{\langle \mathbf{b}^*, \mathbf{X}_r\rangle - (L_{B,r}-1)(1+r)} = L_{B,r}\vartheta,$$

$$\frac{1}{L_{B,r}} + \frac{(L_{B,r}-1)(1+r)}{L_{B,r}}\mathbb{E}\frac{1}{\langle \mathbf{b}^*, \mathbf{X}_r\rangle - (L_{B,r}-1)(1+r)} = \vartheta. \quad (4.24)$$

This means that

$$b^{*(j)} > 0 \Longrightarrow \mu_j = 0 \Longrightarrow \mathbb{E}\frac{X^{(j)}}{\langle \mathbf{b}^*, \mathbf{X}_r\rangle - (L_{B,r}-1)(1+r)} = \vartheta, \quad (4.25)$$

and

$$b^{*(j)} = 0 \Longrightarrow \mathbb{E}\frac{X^{(j)}}{\langle \mathbf{b}^*, \mathbf{X}_r\rangle - (L_{B,r}-1)(1+r)} \le \vartheta.$$

For the cash account this means

$$b^{*(0)} > 0 \Longrightarrow \mu_j = 0 \Longrightarrow \mathbb{E}\frac{1+r}{\langle \mathbf{b}^*, \mathbf{X}_r\rangle - (L_{B,r}-1)(1+r)} = \vartheta, \quad (4.26)$$

and

$$b^{*(0)} = 0 \Longrightarrow \mathbb{E}\frac{1+r}{\langle \mathbf{b}^*, \mathbf{X}_r\rangle - (L_{B,r}-1)(1+r)} \le \vartheta.$$

4.5. Short selling and leverage

For this case we need to use both tricks of the previous sections. The market evolves according to

$$S_1 = S_0\Big(b^{(0)}(1+r) + \sum_{j=1}^{d}\Big[b^{(j)+}x_1^{(j)} + b^{(j)-}(x_1^{(j)} - 1 - r)\Big] - (L_{B,r}-1)(1+r)\Big)^+,$$

over the non-convex region

$${}^{r}\Delta_d^{\pm B} = \left\{\mathbf{b} = (b^{(0)}, b^{(1)}, b^{(2)}, \ldots, b^{(d)});\ \sum_{j=0}^{d}|b^{(j)}| = L_{B,r}\right\},$$

where $L_{B,r}$ is the buying power defined in (4.22), and $b^{(0)}$ denotes unspent buying power. Again, one can check that the choice of $L_{B,r}$ ensures no-ruin and finiteness of growth rate.

With the help of our technique developed in the short selling framework, we convert to the following convex region:

$$
{}^{r}\widetilde{\Delta}_d^{\pm B} = \Big\{ \widetilde{\mathbf{b}} = (\widetilde{b}^{(0+)}, \widetilde{b}^{(1+)}, \widetilde{b}^{(1-)}, \dots, \widetilde{b}^{(d+)}, \widetilde{b}^{(d-)}) \in \mathbb{R}_0^{+\,2d+1}; \\
\widetilde{b}^{(0+)} + \sum_{j=1}^{d} (\widetilde{b}^{(j+)} + \widetilde{b}^{(j-)}) = L_{B,r} \Big\}
$$

such that

$$
\widetilde{\mathbf{b}} = (\widetilde{b}^{(0)}, \widetilde{b}^{(1+)}, \widetilde{b}^{(1-)} \dots, \widetilde{b}^{(d+)}, \widetilde{b}^{(d-)}) = (b^{(0)}, {b^{(1)}}^{+}, |{b^{(1)}}^{-}|, \dots, {b^{(d)}}^{+}, |{b^{(d)}}^{-}|).
$$

Similarly to the short selling case we introduce the transformed return vector. Given

$$
\mathbf{X} = (X^{(1)}, \dots, X^{(d)}),
$$

we introduce

$$
\mathbf{X}_{\pm r} = (1 + r, X^{(1)}, 2 - X^{(1)} + r, \dots, X^{(d)}, 2 - X^{(d)} + r).
$$

We introduce r in $2 - X^{(i)} + r$ terms, since short selling is free, hence buying power spent on short positions still earns interest. We use $2 - X^{(i)} + r$ instead of $1 - X^{(i)} + r$, since while short selling is actually free, it still limits our buying power, which is the basis of the convex formulation.

Because of ${}^{r}\widetilde{\Delta}_d^{\pm B} = {}^{r}\Delta_{2d}^{+B}$, we can easily apply (4.25) and (4.26), hence

$$
b^{*(0)} > 0 \Longrightarrow \mathbb{E} \frac{1 + r}{\left\langle \widetilde{\mathbf{b}}^{*}, \mathbf{X}_{\pm r} \right\rangle - (L_{B,r} - 1)(1 + r)} = \vartheta,
$$

$$
b^{*(0)} = 0 \Longrightarrow \mathbb{E} \frac{1 + r}{\left\langle \widetilde{\mathbf{b}}^{*}, \mathbf{X}_{\pm r} \right\rangle - (L_{B,r} - 1)(1 + r)} \le \vartheta,
$$

where ϑ is defined by (4.24) with $\mathbf{X}_r = \mathbf{X}_{\pm r}$ in place, and

$$
\begin{aligned}
& b^{*(i)} > 0 \Longrightarrow \\
\mathbb{E} \frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^{*}, \mathbf{X}_{\pm r} \right\rangle - (L_{B,r} - 1)(1 + r)} &= \vartheta, \\
\mathbb{E} \frac{2 - X^{(i)} + r}{\left\langle \widetilde{\mathbf{b}}^{*}, \mathbf{X}_{\pm r} \right\rangle - (L_{B,r} - 1)(1 + r)} &\le \vartheta
\end{aligned}
$$

and

$$
\begin{aligned}
b^{*(i)} &= 0 \Longrightarrow \\
\mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^{*}, \mathbf{X}_{\pm r} \right\rangle - (L_{B,r}-1)(1+r)} &\le \vartheta, \\
\mathbb{E}\frac{2 - X^{(i)} + r}{\left\langle \widetilde{\mathbf{b}}^{*}, \mathbf{X}_{\pm r} \right\rangle - (L_{B,r}-1)(1+r)} &\le \vartheta
\end{aligned}
$$

and

$$
\begin{aligned}
b^{*(i)} &< 0 \Longrightarrow \\
\mathbb{E}\frac{X^{(i)}}{\left\langle \widetilde{\mathbf{b}}^{*}, \mathbf{X}_{\pm r} \right\rangle - (L_{B,r}-1)(1+r)} &\le \vartheta, \\
\mathbb{E}\frac{2 - X^{(i)} + r}{\left\langle \widetilde{\mathbf{b}}^{*}, \mathbf{X}_{\pm r} \right\rangle - (L_{B,r}-1)(1+r)} &= \vartheta.
\end{aligned}
$$

Note, that in the special case of $L_{B,r} = 1$, we have $\vartheta = 1$ because of (4.24).

4.6. Experiments

Our empirical investigation consider three setups, each of which is considered in long only, short, leveraged and leveraged short cases. We examine the BCRP strategy, which chooses the best constant portfolio vector with hindsight, and its empirical causal counterpart, the causal i.i.d. strategy. The latter strategy uses the best portfolio based on past data, and it is asymptotically optimal for i.i.d. returns. The third algorithm is asymptotically optimal in case of Markovian time series. Using nearest-neighbor-based portfolio selection (cf. Chapter 2 of this volume) with 100 neighbors, we investigate whether shorting yields extra growth in case of dependent market returns.

The New York Stock Exchange (NYSE) data set [Gelencsér and Ottucsák (2006)] includes daily closing prices of 19 assets along a 44-year period ending in 2006. The same data is used in Chapter 2 of this volume, which facilitates comparison of algorithms.

Interest rate is constant in our experiments. We calculated effective daily yield over the 44 years based on Federal Reserve Fund Rate from the FRED database. The annual rate in this period is 6.3%, which is equivalent to $r = 0.000245$ daily interest.

Table 4.1. Average Annual Yields and optimal portfolios on NYSE data.

Asset	AAY	$\mathbf{b}^*$	$\mathbf{b}^*_{-B}$	$\mathbf{b}^*_{+B}$	$\mathbf{b}^*_{\pm B}$
Cash/Debt	–	0	0	-1.4991	-1.4991
AHP	13%	0	0	0	0
ALCOA	9%	0	0	0	0
AMERB	14%	0	0	0.01	0.01
COKE	14%	0	0	0	0
DOW	12%	0	0	0	0
DUPONT	9%	0	0	0	0
FORD	9%	0	0	0	0
GE	13%	0	0	0	0
GM	7%	0	0	0	0
HP	15%	0.17	0.17	0.32	0.32
IBM	10%	0	0	0	0
INGER	11%	0	0	0	0
JNJ	16%	0	0	0.48	0.48
KIMBC	13%	0	0	0.03	0.03
MERCK	15%	0	0	0.14	0.14
MMM	11%	0	0	0	0
MORRIS	20%	0.75	0.75	1.15	1.15
PANDG	13%	0	0	0	0
SCHLUM	15%	0.08	0.08	0.36	0.36
AAY		20%	20%	34%	34%

Table 4.2. Average Annual Yields.

Strategy		Annual Average Yield
BCRP		
	Long only	20%
	Short	20%
	Leverage	34%
	Short & Leverage	34%
IID		
	Long only	13%
	Short	11%
	Leverage	16%
	Short & Leverage	14%
Nearest Neighbor		
	Long only	32%
	Short	30%
	Leverage	66%
	Short & Leverage	83%

Regarding (4.15), we chose the conservative bound $B = 0.4$, as the largest one day change in asset value over the 44 years has been 0.3029. This bound implies that in the case of $r = 0$ the maximal leverage is $L_{B,r} = 2.5$ fold, while in case of $r = 0.000245$, $L_{B,r} = 2.4991$. Performance of BCRP algorithms improve further by decreasing B until $B = 0.2$, but this limit would not guarantee no-ruin. This property also implies that optimal leverage factor on our dataset is less than 5.

Given our convex formalism for the space of portfolio vector and convexity of log utility, we use Lagrange multipliers and active-set algorithms for the optimization.

Table 4.1 shows the results of the BCRP experiments. Shorting does not have any effect in this case, while leverage results in significant gain. Behavior of shorting strategies is in line with intuition, since taking permanently short position of an asset is not beneficial. The leveraged strategies use maximal leverage, and they do not only increase market exposure, but invest into more assets in order to reduce variation of the portfolio. This is in contrast with behavior of leveraged mean-variance optimal portfolios.

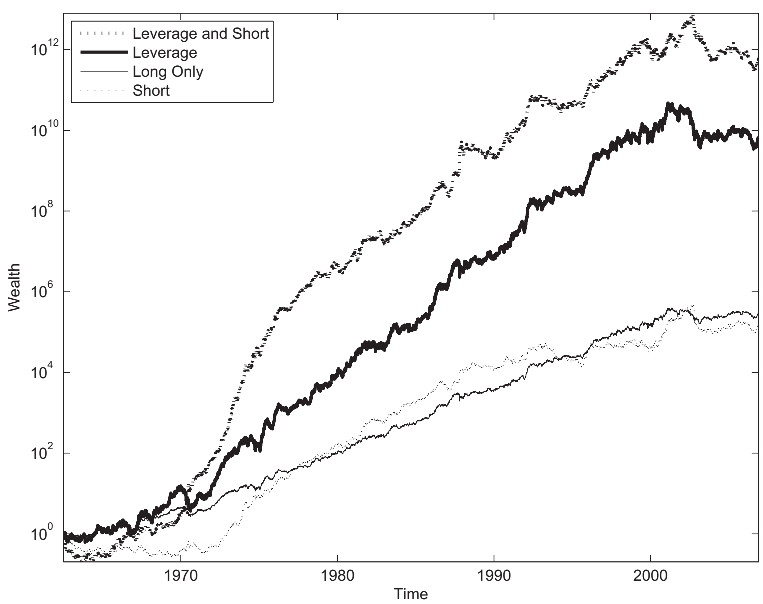

Fig. 4.1. Cumulative wealth of the nearest neighbor strategy starting from 1962.

Table 4.2 presents growth rates of the three setups we consider. BCRP being an optimistically anticipating estimate of possible growth, our i.i.d. strategies do significantly underperform, while the Average Annual Yields (AAYs) of the nearest neighbor strategies including leverage are spectacular. Figure 4.1 presents evolution of wealth in the latter case. While allowing short positions results in large drawdowns in the beginning, these algorithms catch up later. Figure 4.2 shows the result of the algorithms starting from 1980; it reveals that short selling does not offer any significant plus gain in this period benchmarking against the long-only approaches.

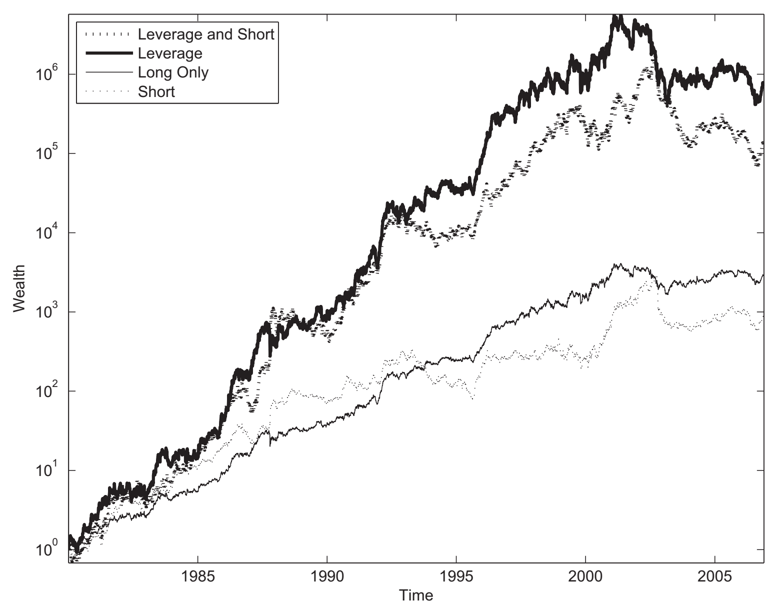

Fig. 4.2. Cumulative wealth of the nearest neighbor strategy starting from 1980.

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