Chapter 2

Empirical Log-Optimal Portfolio Selections: a Survey

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This chapter provides a survey of discrete time, multi-period, sequential investment strategies for financial markets. Under memoryless assumption on the underlying process generating the asset prices the best rebalancing is the log-optimal portfolio, which achieves the maximal asymptotic average growth rate. We show some examples (Kelly game, horse racing, St. Petersburg game) illustrating the surprising possibilities for rebalancing. Semi-log-optimal portfolio selection as a small computational complexity alternative of the log-optimal portfolio selection is studied both theoretically and empirically. For generalized dynamic portfolio selection, when asset prices are generated by a stationary and ergodic process, universally consistent empirical methods are shown. The empirical performance of the methods are illustrated for NYSE data.

2.1. Introduction

This chapter gives an overview on the investment strategies in financial stock markets inspired by the results of information theory, non-parametric statistics and machine learning. Investment strategies are allowed to use information collected from the past of the market and determine, at the beginning of a trading period, a portfolio, that is, a way to distribute their current capital among the available assets. The goal of the investor is to maximize his wealth in the long run without knowing the underlying distribution generating the stock prices. Under this assumption the asymptotic rate of growth has a well-defined maximum which can be achieved in full knowledge of the underlying distribution generated by the stock prices.

Both static (buy and hold) and dynamic (daily rebalancing) portfolio selections are considered under various assumptions on the behavior of the market process. In case of static portfolio selection, it was shown that every static portfolio asymptotically approximates the growth rate of the best asset in the study. One can achieve larger growth rate with daily rebalancing. Under memoryless assumption on the underlying process generating the asset prices, the log-optimal portfolio achieves the maximal asymptotic average growth rate, that is the expected value of the logarithm of the return for the best constant portfolio vector. Semi-log optimal portfolio selection as a small computational complexity alternative of the log-optimal portfolio selection is investigated both theoretically and empirically. Applying recent developments in nonparametric estimation and machine learning algorithms, for generalized dynamic portfolio selection, when asset prices are generated by a stationary and ergodic process, universal consistent (empirical) methods that achieve the maximal possible growth rate are shown. The spectacular empirical performance of the methods are illustrated for NYSE data.

Consider a market consisting of d assets. The evolution of the market in time is represented by a sequence of price vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots \in \mathbb{R}_+^d$, where

$$\mathbf{s}_n = (s_n^{(1)}, \ldots, s_n^{(d)})$$

such that the j-th component $s_n^{(j)}$ of $\mathbf{s}_n$ denotes the price of the j-th asset on the n-th trading period. In order to normalize, put $s_0^{(j)} = 1$. $\{\mathbf{s}_n\}$ has exponential trend:

$$s_n^{(j)} = e^{nW_n^{(j)}} \approx e^{nW^{(j)}},$$

with average growth rate (average yield)

$$W_n^{(j)} := \frac{1}{n} \ln s_n^{(j)}$$

and with asymptotic average growth rate

$$W^{(j)} := \lim_{n\to\infty} \frac{1}{n} \ln s_n^{(j)}.$$

The static portfolio selection is a single period investment strategy. A portfolio vector is denoted by $\mathbf{b} = (b^{(1)}, \ldots b^{(d)})$. (In Chapter 1 of this volume the components $b^{(j)}$ of this portfolio vector are called fractions and they are denoted by π_j.) The j-th component $b^{(j)}$ of $\mathbf{b}$ denotes the proportion of the investor's capital invested in asset j. We assume that the portfolio vector $\mathbf{b}$ has nonnegative components sum up to 1, that means that short selling is not permitted. The set of portfolio vectors is denoted by

$$\Delta_d = \left\{ \mathbf{b} = (b^{(1)}, \ldots, b^{(d)});\, b^{(j)} \geq 0,\, \sum_{j=1}^{d} b^{(j)} = 1 \right\}.$$

The aim of static portfolio selection is to achieve $\max_{1 \leq j \leq d} W^{(j)}$. The static portfolio is an index, for example, the S&P 500 such that at time $n = 0$ we distribute the initial capital S_0 according to a fix portfolio vector $\mathbf{b}$, i.e., if S_n denotes the wealth at the trading period n, then

$$S_n = S_0 \sum_{j=1}^{d} b^{(j)} s_n^{(j)}.$$

Apply the following simple bounds

$$S_0 \max_j b^{(j)} s_n^{(j)} \leq S_n \leq d S_0 \max_j b^{(j)} s_n^{(j)}.$$

If $b^{(j)} > 0$ for all $j = 1, \ldots, d$ then these bounds imply that

$$W := \lim_{n \to \infty} \frac{1}{n} \ln S_n = \lim_{n \to \infty} \max_j \frac{1}{n} \ln s_n^{(j)} = \max_j W^{(j)}.$$

Thus, any static portfolio selection achieves the growth rate of the best asset in the study, $\max_j W^{(j)}$, and so the limit does not depend on the portfolio $\mathbf{b}$. In case of uniform portfolio (uniform index) $b^{(j)} = 1/d$ and the convergence above is from below:

$$S_0 \max_j s_n^{(j)}/d \leq S_n \leq S_0 \max_j s_n^{(j)}.$$

The rest of the chapter is organized as follows. In Section 2.2 the constantly rebalanced portfolio is introduced, and the properties of log-optimal portfolio selection is analyzed in case of memoryless market. Next, a small computational complexity alternative of the log-optimal portfolio selection, the semi-log optimal portfolio is introduced. In Section 2.3 the general model of the dynamic portfolio selection is introduced and the basic features of the conditionally log-optimal portfolio selection in case of stationary and ergodic market are summarized. Using the principles of nonparametric statistics and machine learning, universal consistent, empirical investment strategies that are able to achieve the maximal asymptotic growth rate are introduced. Experiments on the NYSE data are given in Section 2.3.7.

2.2. Constantly rebalanced portfolio selection

In order to apply the usual prediction techniques for time series analysis one has to transform the sequence price vectors $\{\mathbf{s}_n\}$ into a more or less stationary sequence of return vectors (price relatives) $\{\mathbf{x}_n\}$ as follows:

$$\mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(d)})$$

such that

$$x_n^{(j)} = \frac{s_{n+1}^{(j)}}{s_n^{(j)}} \, .$$

Thus, the j-th component $x_n^{(j)}$ of the return vector $\mathbf{x}_n$ denotes the amount obtained after investing a unit capital in the j-th asset on the n-th trading period.

With respect to the static portfolio, one can achieve even higher growth rate for long run investments, if we make rebalancing, i.e., if the tuning of the portfolio is allowed *dynamically* after each trading period. The dynamic portfolio selection is a multi-period investment strategy, where at the beginning of each trading period we can rearrange the wealth among the assets. A representative example of the dynamic portfolio selection is the constantly rebalanced portfolio (CRP), which was introduced and studied by [Kelly (1956)], [Latané (1959)], [Breiman (1961)], [Markowitz (1976)], [Finkelstein and Whitley (1981)], [Móri (1982b)], [Móri and Székely (1982)] and [Barron and Cover (1988)]. For a comprehensive survey see also Chapter 1 of this volume, and Chapters 6 and 15 in [Cover and Thomas (1991)], and Chapter 15 in [Luenberger (1998)].

[Luenberger (1998)] summarizes the main conclusions as follows:

> "Conclusions about multi-period investment situations are not mere variations of single-period conclusions – rather they offer *reverse* those earlier conclusions. This makes the subject exciting, both intellectually and in practice. Once the subtleties of multi-period investment are understood, the reward in terms of enhanced investment performance can be substantial."

> "Fortunately the concepts and the methods of analysis for multi-period situation build on those of earlier chapters. Internal rate of return, present value, the comparison principle, portfolio design, and lattice and tree valuation all have natural extensions to general situations. But conclusions such as volatility is "bad" or diversification is "good" are no longer universal truths. The story is much more interesting."

In case of CRP we fix a portfolio vector $\mathbf{b} \in \Delta_d$, i.e., we are concerned with a hypothetical investor who neither consumes nor deposits new cash into his portfolio, but reinvests his portfolio each trading period. In fact, neither short selling, nor leverage is allowed. (Concerning short selling and leverage see Chapter 4 of this volume.) Note that in this case the investor has to rebalance his portfolio after each trading day to "corrigate" the daily price shifts of the invested stocks.

Let S_0 denote the investor's initial capital. Then at the beginning of the first trading period $S_0 b^{(j)}$ is invested into asset j, and it results in return $S_0 b^{(j)} x_1^{(j)}$, therefore at the end of the first trading period the investor's wealth becomes

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b}\, , \, \mathbf{x}_1 \rangle \, ,$$

where $\langle \cdot\, , \, \cdot \rangle$ denotes inner product. For the second trading period, S_1 is the new initial capital

$$S_2 = S_1 \cdot \langle \mathbf{b}\, , \, \mathbf{x}_2 \rangle = S_0 \cdot \langle \mathbf{b}\, , \, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}\, , \, \mathbf{x}_2 \rangle \, .$$

By induction, for the trading period n the initial capital is S_{n-1}, therefore

$$S_n = S_{n-1} \langle \mathbf{b}\, , \, \mathbf{x}_n \rangle = S_0 \prod_{i=1}^{n} \langle \mathbf{b}\, , \, \mathbf{x}_i \rangle \, .$$

The asymptotic average growth rate of this portfolio selection is

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \ln S_n &= \lim_{n\to\infty} \left(\frac{1}{n} \ln S_0 + \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b}\, , \, \mathbf{x}_i \rangle \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b}\, , \, \mathbf{x}_i \rangle \, , \end{aligned}$$

therefore without loss of generality one can assume in the sequel that the initial capital $S_0 = 1$.

2.2.1. *Log-optimal portfolio for memoryless market process*

If the market process $\{\mathbf{X}_i\}$ is memoryless, i.e., it is a sequence of independent and identically distributed (i.i.d.) random return vectors then we show that the best constantly rebalanced portfolio (BCRP) is the log-optimal portfolio:

$$\mathbf{b}^* := \arg\max_{\mathbf{b} \in \Delta_d} \mathbb{E}\{\ln \langle \mathbf{b}\, , \, \mathbf{X}_1 \rangle\}.$$

This optimality means that if $S_n^* = S_n(\mathbf{b}^*)$ denotes the capital after day n achieved by a log-optimal portfolio strategy $\mathbf{b}^*$, then for any portfolio strategy $\mathbf{b}$ with finite $\mathbb{E}\{(\ln \langle \mathbf{b}\,,\, \mathbf{X}_1\rangle)^2\}$ and with capital $S_n = S_n(\mathbf{b})$ and for any memoryless market process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n \le \lim_{n\to\infty} \frac{1}{n} \ln S_n^* \quad \text{almost surely}$$

and maximal asymptotic average growth rate is

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = W^* := \mathbb{E}\{\ln \langle \mathbf{b}^*\,,\, \mathbf{X}_1\rangle\} \quad \text{almost surely.}$$

The proof of the optimality is a simple consequence of the strong law of large numbers. Introduce the notation

$$W(\mathbf{b}) = \mathbb{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1\rangle\}.$$

Then

$$\begin{aligned}
\frac{1}{n} \ln S_n &= \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b}\,,\, \mathbf{X}_i\rangle \\
&= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_i\rangle\} + \frac{1}{n} \sum_{i=1}^{n} \left(\ln \langle \mathbf{b}\,,\, \mathbf{X}_i\rangle - \mathbb{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_i\rangle\}\right) \\
&= W(\mathbf{b}) + \frac{1}{n} \sum_{i=1}^{n} \left(\ln \langle \mathbf{b}\,,\, \mathbf{X}_i\rangle - \mathbb{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_i\rangle\}\right).
\end{aligned}$$

The strong law of large numbers implies that

$$\frac{1}{n} \sum_{i=1}^{n} \left(\ln \langle \mathbf{b}\,,\, \mathbf{X}_i\rangle - \mathbb{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_i\rangle\}\right) \to 0 \quad \text{almost surely,}$$

therefore

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n = W(\mathbf{b}) = \mathbb{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_1\rangle\} \quad \text{almost surely.}$$

Similarly,

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = W(\mathbf{b}^*) = \max_{\mathbf{b}} W(\mathbf{b}) \quad \text{almost surely.}$$

We have to emphasize the basic conditions of the model: assume that

(i) the assets are arbitrarily divisible, and they are available for buying and for selling in unbounded quantities at the current price at any given trading period,
(ii) there are no transaction costs,

(iii) the behavior of the market is not affected by the actions of the investor using the strategy under investigation.

Avoiding (ii), see Chapter 3 of this volume. For memoryless or Markovian market process, optimal strategies have been introduced if the distributions of the market process are known. For the time being, there is no asymptotically optimal, empirical algorithm taking into account the proportional transaction cost. Condition (iii) means that the market is inefficient.

The principle of log-optimality has the important consequence that

$$S_n(\mathbf{b}) \quad \text{is not close to} \quad \mathbb{E}\{S_n(\mathbf{b})\}.$$

We prove a bit more. The optimality property proved above means that, for any $\delta > 0$, the event

$$\left\{-\delta < \frac{1}{n}\ln S_n(\mathbf{b}) - \mathbb{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\} < \delta\right\}$$

has probability close to 1 if n is large enough. On the one hand, we have that

$$\begin{aligned}
&\left\{-\delta < \frac{1}{n}\ln S_n(\mathbf{b}) - \mathbb{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\} < \delta\right\}\\
&= \left\{-\delta + \mathbb{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\} < \frac{1}{n}\ln S_n(\mathbf{b}) < \delta + \mathbb{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\}\right\}\\
&= \left\{e^{n(-\delta+\mathbb{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\})} < S_n(\mathbf{b}) < e^{n(\delta+\mathbb{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\})}\right\},
\end{aligned}$$

therefore

$$S_n(\mathbf{b}) \quad \text{is close to} \quad e^{n\mathbb{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\}}.$$

On the other hand,

$$\mathbb{E}\{S_n(\mathbf{b})\} = \mathbb{E}\left\{\prod_{i=1}^{n}\langle \mathbf{b}\,,\,\mathbf{X}_i\rangle\right\} = \prod_{i=1}^{n}\langle \mathbf{b}\,,\,\mathbb{E}\{\mathbf{X}_i\}\rangle = e^{n\ln\langle \mathbf{b}\,,\,\mathbb{E}\{\mathbf{X}_1\}\rangle}.$$

By Jensen inequality,

$$\ln\langle \mathbf{b}\,,\,\mathbb{E}\{\mathbf{X}_1\}\rangle > \mathbb{E}\{\ln\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle\},$$

therefore

$$S_n(\mathbf{b}) \quad \text{is much less than} \quad \mathbb{E}\{S_n(\mathbf{b})\}.$$

Not knowing this fact, one can apply a naive approach

$$\arg\max_{\mathbf{b}} \mathbb{E}\{S_n(\mathbf{b})\}.$$

Because of

$$\mathbb{E}\{S_n(\mathbf{b})\} = \langle \mathbf{b}\,,\, \mathbb{E}\{\mathbf{X}_1\}\rangle^n\,,$$

this naive approach has the equivalent form

$$\arg\max_{\mathbf{b}} \mathbb{E}\{S_n(\mathbf{b})\} = \arg\max_{\mathbf{b}} \langle \mathbf{b}\,,\, \mathbb{E}\{\mathbf{X}_1\}\rangle\,,$$

which is called the mean approach. It is easy to see that $\arg\max_{\mathbf{b}} \langle \mathbf{b}\,,\, \mathbb{E}\{\mathbf{X}_1\}\rangle$ is a portfolio vector having 1 at the position, where the vector $\mathbb{E}\{\mathbf{X}_1\}$ has the largest component.

In his seminal paper [Markowitz (1952)] realized that the mean approach is inadequate, i.e., it is a dangerous portfolio. In order to avoid this difficulty he suggested a diversification, which is called mean-variance portfolio such that

$$\widetilde{\mathbf{b}} = \mathop{\arg\max}_{\mathbf{b}:\mathbf{Var}(\langle \mathbf{b}\,,\,\mathbf{X}_1\rangle)\le\lambda} \langle \mathbf{b}\,,\, \mathbb{E}\{\mathbf{X}_1\}\rangle\,,$$

where $\lambda > 0$ is the investor's risk aversion parameter.

For appropriate choice of λ, the performance (average growth rate) of $\widetilde{\mathbf{b}}$ can be close to the performance of the optimal $\mathbf{b}^*$, however, the good choice of λ depends on the (unknown) distribution of the return vector $\mathbf{X}$.

The calculation of $\widetilde{\mathbf{b}}$ is a quadratic programming (QP) problem, where a linear function is maximized under quadratic constraints.

In order to calculate the log-optimal portfolio $\mathbf{b}^*$, one has to know the distribution of $\mathbf{X}_1$. If this distribution is unknown then the empirical log-optimal portfolio can be defined by

$$\mathbf{b}_n^* = \arg\max_{\mathbf{b}} \frac{1}{n}\sum_{i=1}^{n} \ln \langle \mathbf{b}\,,\, \mathbf{X}_i\rangle$$

with linear constraints

$$\sum_{j=1}^{d} b^{(j)} = 1 \qquad \text{and} \qquad 0 \le b^{(j)} \le 1 \quad j = 1, \ldots, d\,.$$

The behavior of the empirical portfolio $\mathbf{b}_n^*$ and its modifications was studied by [Móri (1984, 1986)] and by [Morvai (1991, 1992)].

The calculation of $\mathbf{b}_n^*$ is a nonlinear programming (NLP) problem. [Cover (1984)] introduced an algorithm for calculating $\mathbf{b}_n^*$. An alternative possibility is the software routine DONLP2 of [Spellucci (1999)]. The routine is based on sequential quadratic programming method, which computes sequentially a local solution of NLP by solving a quadratic programming problem and it estimates the global maximum according to these local maximums.

2.2.2. *Examples for constantly rebalanced portfolio*

Next we show some examples of portfolio games.

Example 2.1. (Kelly game [Kelly (1956)])
Consider the example of $d = 2$ and $\mathbf{X} = (X^{(1)}, X^{(2)})$ such that the first component $X^{(1)}$ of the return vector $\mathbf{X}$ is the payoff of the Kelly game:

$$X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2, \end{cases} \tag{2.1}$$

and the second component $X^{(2)}$ of the return vector $\mathbf{X}$ is the cash:

$$X^{(2)} = 1.$$

Obviously, the cash has zero growth rate. Using the expectation of the first component

$$\mathbb{E}\{X^{(1)}\} = 1/2 \cdot (2 + 1/2) = 5/4 > 1,$$

Assume that we are given an i.i.d. sequence of Kelly payoffs $\{X_i^{(1)}\}_{i=1}^{\infty}$. One can introduce the sequential Kelly game $S_n^{(1)}$ such that there is a reinvestment:

$$S_n^{(1)} = \prod_{i=1}^{n} X_i^{(1)}.$$

The i.i.d. property of the payoffs $\{X_i^{(1)}\}_{i=1}^{\infty}$ implies that

$$\mathbb{E}\{S_n^{(1)}\} = \mathbb{E}\left\{\prod_{i=1}^{n} X_i^{(1)}\right\} = (5/4)^n, \tag{2.2}$$

therefore $\mathbb{E}\{S_n^{(1)}\}$ grows exponentially. However, it does not imply that the random variable $S_n^{(1)}$ grows exponentially, too. Let's calculate the growth rate $W^{(1)}$:

$$\begin{aligned} W^{(1)} := \lim_{n\to\infty} \frac{1}{n} \ln S_n^{(1)} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln X_i^{(1)} = \mathbb{E}\{\ln X^{(1)}\} \\ &= 1/2 \ln 2 + 1/2 \ln(1/2) = 0, \end{aligned}$$

a.s., which means that the first component $X^{(1)}$ of the return vector $\mathbf{X}$ has zero growth rate, too.

The following viewpoint may help explain this at first sight surprising property. First, we write the evolution of the wealth of the sequential Kelly game as follows: let $S_n^{(1)} = 2^{2B(n,1/2)-n}$, where $B(n,1/2)$ is a binomial random variable with parameters $(n,1/2)$ (it is easy to check if we choose $n=1$ then we return back to the one-step performance of the game). Now we write according to the Moivre-Laplace theorem (a special case of the central limit theorem for binomial distribution):

$$\mathbb{P}\left(\frac{2B(n,1/2)-n}{\sqrt{\mathbf{Var}(2B(n,1/2))}} \leq x\right) \simeq \phi(x),$$

where $\phi(x)$ is cumulative distribution function of the standard normal distribution. Rearranging the left-hand side we have

$$\begin{aligned}\mathbb{P}\left(\frac{2B(n,1/2)-n}{\sqrt{\mathbf{Var}(2B(n,1/2))}} \leq x\right) &= \mathbb{P}\left(2B(n,1/2)-n \leq x\sqrt{n}\right)\\ &= \mathbb{P}\left(2^{2B(n,1/2)-n} \leq 2^{x\sqrt{n}}\right)\\ &= \mathbb{P}\left(S_n^{(1)} \leq 2^{x\sqrt{n}}\right)\end{aligned}$$

that is

$$\mathbb{P}\left(S_n^{(1)} \leq 2^{x\sqrt{n}}\right) \simeq \phi(x)\ .$$

Now let x_ε choose so that $\phi(x_\varepsilon) = 1-\varepsilon$ then

$$\mathbb{P}\left(S_n^{(1)} \leq 2^{x_\varepsilon\sqrt{n}}\right) \simeq 1-\varepsilon$$

and for a fixed $\varepsilon > 0$ let n_0 be so that

$$2^{x_\varepsilon\sqrt{n}} < \mathbb{E}S_n^{(1)} = \left(\frac{5}{4}\right)^n$$

for all $n > n_0$ then we have

$$\mathbb{P}\left(S_n^{(1)} \geq \mathbb{E}S_n^{(1)}\right) \leq \mathbb{P}\left(S_n^{(1)} \geq 2^{x_\varepsilon\sqrt{n}}\right) \simeq \varepsilon.$$

It means that most of the values of $S_n^{(1)}$ are far smaller than its expected value $\mathbb{E}S_n^{(1)}$ (see in Figure 2.1).

Now let's turn back to the original problem and calculate the log-optimal portfolio for this return vector, where both components have zero growth rate. The portfolio vector has the form

$$\mathbf{b} = (b, 1-b).$$

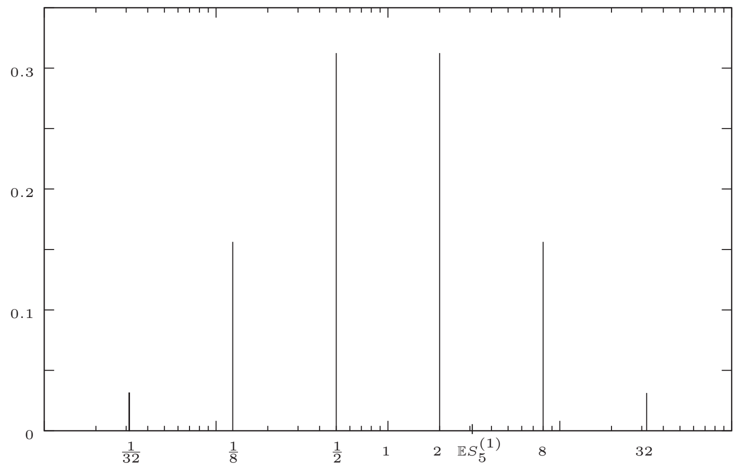

Fig. 2.1. The distribution of $S_n^{(1)}$ in case of $n = 5$

Then

$$
\begin{aligned}
W(\mathbf{b}) &= \mathbb{E}\{\ln \langle \mathbf{b}\, , \, \mathbf{X}\rangle\} \\
&= 1/2\,(\ln(2b + (1-b)) + \ln(b/2 + (1-b))) \\
&= 1/2\ln[(1+b)(1-b/2)].
\end{aligned}
$$

One can check that $W(\mathbf{b})$ has the maximum for $b = 1/2$, so the log-optimal portfolio is

$$\mathbf{b}^* = (1/2, 1/2),$$

and the asymptotic average growth rate is

$$W^* = \mathbb{E}\{\ln \langle \mathbf{b}^*\, , \, \mathbf{X}\rangle\} = 1/2\ln(9/8) = 0.059,$$

which is a positive growth rate.

Example 2.2. Consider the example of $d = 3$ and $\mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)})$ such that the first and the second components of the return vector $\mathbf{X}$ are Kelly payoffs of form (2.1), while the third component is the cash. One can show that the log-optimal portfolio is

$$\mathbf{b}^* = (0.46, 0.46, 0.08),$$

and the maximal asymptotic average growth rate is

$$W^* = \mathbb{E}\{\ln \langle \mathbf{b}^*\, , \, \mathbf{X}\rangle\} = 0.112.$$

Example 2.3. Consider the example of $d > 3$ and $\mathbf{X} = (X^{(1)}, X^{(2)}, \ldots, X^{(d)})$ such that the first $d-1$ components of the return vector $\mathbf{X}$ are Kelly payoffs of form (2.1), while the last component is the cash. One can show that the log-optimal portfolio is

$$\mathbf{b}^* = (1/(d-1), \ldots, 1/(d-1), 0),$$

which means that, for $d > 3$, according to the log-optimal portfolio the cash has zero weight. Let N denote the number of components of $\mathbf{X}$ equal to 2, then N is binomially distributed with parameters $(d-1, 1/2)$, and

$$\ln \langle \mathbf{b}^* \, , \, \mathbf{X} \rangle = \ln \left(\frac{2N + (d-1-N)/2}{d-1} \right) = \ln \left(\frac{3N}{2(d-1)} + \frac{1}{2} \right),$$

therefore

$$W^* = \mathbb{E}\{\ln \langle \mathbf{b}^* \, , \, \mathbf{X} \rangle\} = \mathbb{E} \left\{ \ln \left(\frac{3N}{2(d-1)} + \frac{1}{2} \right) \right\}.$$

For $d = 4$, the formula implies that the maximal asymptotic average growth rate is

$$W^* = \mathbb{E}\{\ln \langle \mathbf{b}^* \, , \, \mathbf{X} \rangle\} = 0.152,$$

while for $d \to \infty$,

$$W^* = \mathbb{E}\{\ln \langle \mathbf{b}^* \, , \, \mathbf{X} \rangle\} \to \ln(5/4) = 0.223,$$

which means that

$$S_n \approx e^{nW^*} = (5/4)^n,$$

so with many such Kelly components

$$S_n \approx \mathbb{E}\{S_n\}$$

(cf. (2.2)).

Example 2.4. (Horse racing [Cover and Thomas (1991)]) Consider the example of horse racing with d horses in a race. Assume that horse j wins with probability p_j. The payoff is denoted by o_j, which means that investing \$1 on horse j results in o_j if it wins, otherwise \$0. Then the return vector is of form

$$\mathbf{X} = (0, \ldots, 0, o_j, 0, \ldots, 0)$$

if horse j wins. For repeated races, it is a constantly rebalanced portfolio problem. Let's calculate the expected log-return:

$$W(\mathbf{b}) = \mathbb{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}\rangle\} = \sum_{j=1}^{d} p_j \ln(b^{(j)} o_j) = \sum_{j=1}^{d} p_j \ln b^{(j)} + \sum_{j=1}^{d} p_j \ln o_j,$$

therefore

$$\arg\max_{\mathbf{b}} \mathbb{E}\{\ln \langle \mathbf{b}\,,\, \mathbf{X}\rangle\} = \arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)}.$$

In order to solve the optimization problem

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)},$$

we introduce the Kullback-Leibler divergence of the distributions $\mathbf{p}$ and $\mathbf{b}$:

$$\mathrm{KL}(\mathbf{p}, \mathbf{b}) = \sum_{j=1}^{d} p_j \ln \frac{p_j}{b^{(j)}}.$$

The basic property of the Kullback-Leibler divergence is that

$$\mathrm{KL}(\mathbf{p}, \mathbf{b}) \geq 0,$$

and is equal to zero if and only if the two distributions are equal. The proof of this property is simple:

$$\mathrm{KL}(\mathbf{p}, \mathbf{b}) = -\sum_{j=1}^{d} p_j \ln \frac{b^{(j)}}{p_j} \geq -\sum_{j=1}^{d} p_j \left(\frac{b^{(j)}}{p_j} - 1\right) = -\sum_{j=1}^{d} b^{(j)} + \sum_{j=1}^{d} p_j = 0.$$

This inequality implies that

$$\arg\max_{\mathbf{b}} \sum_{j=1}^{d} p_j \ln b^{(j)} = \mathbf{p}.$$

Surprisingly, the log-optimal portfolio is independent of the payoffs, and

$$W^* = \sum_{j=1}^{d} p_j \ln(p_j o_j).$$

Knowing the distribution $\mathbf{p}$, the usual choice of payoffs is

$$o_j = \frac{1}{p_j},$$

and then

$$W^* = 0.$$

It means that, for this choice of payoffs, any gambling strategy has negative growth rate.

Example 2.5. (Sequential St. Petersburg games.)
Consider the simple St. Petersburg game, where the player invests 1 dollar and a fair coin is tossed until a tail first appears, ending the game. If the first tail appears in step k then the the payoff X is 2^k and the probability of this event is 2^{-k}:

$$\mathbb{P}\{X = 2^k\} = 2^{-k}.$$

Since $\mathbb{E}\{X\} = \infty$, this game has delicate properties (cf. [Aumann (1977)], [Bernoulli (1954)], [Durand (1957)], [Haigh (1999)], [Martin (2004)], [Menger (1934)], [Rieger and Wang (2006)] and [Samuelson (1960)].) In the literature, usually the repeated St. Petersburg game (called iterated St. Petersburg game, too) means multi-period game such that it is a sequence of simple St. Petersburg games, where in each round the player invest 1 dollar. Let X_n denote the payoff for the n-th simple game. Assume that the sequence $\{X_n\}_{n=1}^{\infty}$ is independent and identically distributed. After n rounds the player's wealth in the repeated game is

$$\tilde{S}_n = \sum_{i=1}^{n} X_i,$$

then

$$\lim_{n\to\infty} \frac{\tilde{S}_n}{n \log_2 n} = 1$$

in probability, where $\log_2$ denotes the logarithm with base 2 (cf. [Feller (1945)]). Moreover,

$$\liminf_{n\to\infty} \frac{\tilde{S}_n}{n \log_2 n} = 1 \qquad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{\tilde{S}_n}{n \log_2 n} = \infty \qquad \text{a.s.}$$

(cf. [Chow and Robbins (1961)]). Introducing the notation for the largest payoff

$$X_n^* = \max_{1\le i\le n} X_i$$

and for the sum with the largest payoff withheld

$$S_n^* = \tilde{S}_n - X_n^*,$$

one has that

$$\lim_{n\to\infty} \frac{S_n^*}{n\log_2 n} = 1$$

a.s. (cf. [Csörgő and Simons (1996)]). According to the previous results $\tilde{S}_n \approx n\log_2 n$. Next we introduce a multi-period game, called sequential St. Petersburg game, having exponential growth. The sequential St. Petersburg game means that the player starts with initial capital $S_0 = 1$ dollar, and there is an independent sequence of simple St. Petersburg games, and for each simple game the player reinvest his capital. If $S_{n-1}^{(c)}$ is the capital after the $(n-1)$-th simple game then the invested capital is $S_{n-1}^{(c)}(1-c)$, while $S_{n-1}^{(c)}c$ is the proportional cost of the simple game with commission factor $0 < c < 1$. It means that after the n-th round the capital is

$$S_n^{(c)} = S_{n-1}^{(c)}(1-c)X_n = S_0(1-c)^n \prod_{i=1}^n X_i = (1-c)^n \prod_{i=1}^n X_i.$$

Because of its multiplicative definition, $S_n^{(c)}$ has exponential trend:

$$S_n^{(c)} = e^{nW_n^{(c)}} \approx e^{nW^{(c)}},$$

with average growth rate

$$W_n^{(c)} := \frac{1}{n}\ln S_n^{(c)}$$

and with asymptotic average growth rate

$$W^{(c)} := \lim_{n\to\infty}\frac{1}{n}\ln S_n^{(c)}.$$

Let's calculate the the asymptotic average growth rate. Because of

$$W_n^{(c)} = \frac{1}{n}\ln S_n^{(c)} = \frac{1}{n}\left(n\ln(1-c) + \sum_{i=1}^n \ln X_i\right),$$

the strong law of large numbers implies that

$$W^{(c)} = \ln(1-c) + \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \ln X_i = \ln(1-c) + \mathbb{E}\{\ln X_1\}$$

a.s., so $W^{(c)}$ can be calculated via expected log-utility (cf. [Kenneth (1974)]). A commission factor c is called fair if

$$W^{(c)} = 0,$$

so the growth rate of the sequential game is 0. Let's calculate the fair c:

$$\ln(1-c) = -\mathbb{E}\{\ln X_1\} = -\sum_{k=1}^{\infty} k \ln 2 \cdot 2^{-k} = -2\ln 2,$$

i.e.,

$$c = 3/4.$$

[Györfi and Kevei (2009)] studied the portfolio game, where a fraction of the capital is invested in the simple fair St. Petersburg game and the rest is kept in cash. This is the model of the constantly rebalanced portfolio (CRP). Fix a portfolio vector $\mathbf{b} = (b, 1-b)$, with $0 \leq b \leq 1$. Let $S_0 = 1$ denote the player's initial capital. Then at the beginning of the portfolio game $S_0 b = b$ is invested into the fair game, and it results in return $bX_1/4$, while $S_0(1-b) = 1-b$ remains in cash, therefore after the first round of the portfolio game the player's wealth becomes

$$S_1 = S_0(bX_1/4 + (1-b)) = b(X_1/4 - 1) + 1.$$

For the second portfolio game, S_1 is the new initial capital

$$S_2 = S_1(b(X_2/4 - 1) + 1) = (b(X_1/4 - 1) + 1)(b(X_2/4 - 1) + 1).$$

By induction, for n-th portfolio game the initial capital is S_{n-1}, therefore

$$S_n = S_{n-1}(b(X_n/4 - 1) + 1) = \prod_{i=1}^{n}(b(X_i/4 - 1) + 1).$$

The asymptotic average growth rate of this portfolio game is

$$\begin{aligned} W(b) &:= \lim_{n\to\infty} \frac{1}{n} \log_2 S_n \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log_2(b(X_i/4 - 1) + 1) \\ &\to \mathbb{E}\{\log_2(b(X_1/4 - 1) + 1)\} \qquad \text{a.s.} \end{aligned}$$

The function ln is concave, therefore $W(b)$ is concave, too, so $W(0) = 0$ (keep everything in cash) and $W(1) = 0$ (the simple St. Petersburg game is fair) imply that for all $0 < b < 1$, $W(b) > 0$. Let's calculate

$$\max_b W(b).$$

We have that

$$\begin{aligned} W(b) &= \sum_{k=1}^{\infty} \log_2(b(2^k/4-1)+1) \cdot 2^{-k} \\ &= \log_2(1-b/2) \cdot 2^{-1} + \sum_{k=3}^{\infty} \log_2(b(2^{k-2}-1)+1) \cdot 2^{-k}. \end{aligned}$$

One can show that $\mathbf{b}^* = (0.385, 0.615)$ and $W^* = 0.149$.

Example 2.6. We can extend Example 2.5 such that in each round there are d St. Petersburg components, i.e., the return vector has the form

$$\mathbf{X} = (X^{(1)}, \ldots, X^{(d)}, X^{(d+1)}) = (X_1/4, \ldots, X_d/4, 1)$$

($d \geq 1$), where the first d i.i.d. components of $\mathbf{X}$ are fair St. Petersburg payoffs, while the last component is the cash. For $d = 2$, $\mathbf{b}^* = (0.364, 0.364, 0.272)$. For $d \geq 3$, the best portfolio is the uniform portfolio such that the cash has zero weight:

$$\mathbf{b}^* = (1/d, \ldots, 1/d, 0)$$

and the asymptotic average growth rate is

$$W_d^* = \mathbb{E}\left\{\log_2\left(\frac{1}{4d}\sum_{i=1}^{d} X_i\right)\right\}.$$

Here are the first few values:

Table 2.1. Numerical results

d	1	2	3	4	5	6	7	8
W_d^*	0.149	0.289	0.421	0.526	0.606	0.669	0.721	0.765

[Györfi and Kevei (2011)] proved that

$$W_d^* \approx \log_2 \log_2 d - 2 + \frac{\log_2 \log_2 d}{\ln 2 \log_2 d},$$

which results in some figures for large d:

Table 2.2. Simulation results

d	8	16	32	64
W_d^*	0.76	0.97	1.17	1.35

2.2.3. *Semi-log-optimal portfolio*

[Roll (1973)], [Pulley (1994)] and [Vajda (2006)] suggested an approximation of $\mathbf{b}^*$ and $\mathbf{b}_n^*$ using

$$h(z) := z - 1 - \frac{1}{2}(z-1)^2,$$

which is the second order Taylor expansion of the function $\ln z$ at $z = 1$. Then, the semi-log-optimal portfolio selection is

$$\bar{\mathbf{b}} = \arg\max_{\mathbf{b}} \mathbb{E}\{h(\langle \mathbf{b}\,,\, \mathbf{X}_1\rangle)\},$$

and the empirical semi-log-optimal portfolio is

$$\bar{\mathbf{b}}_n = \arg\max_{\mathbf{b}} \frac{1}{n}\sum_{i=1}^{n} h(\langle \mathbf{b}\,,\, \mathbf{x}_i\rangle).$$

In order to compute $\mathbf{b}_n^*$, one has to make an optimization over $\mathbf{b}$. In each optimization step the computational complexity is proportional to n. For $\bar{\mathbf{b}}_n$, this complexity can be reduced. We have that

$$\frac{1}{n}\sum_{i=1}^{n} h(\langle \mathbf{b}\,,\, \mathbf{x}_i\rangle) = \frac{1}{n}\sum_{i=1}^{n}(\langle \mathbf{b}\,,\, \mathbf{x}_i\rangle - 1) - \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}(\langle \mathbf{b}\,,\, \mathbf{x}_i\rangle - 1)^2.$$

If $\mathbf{1}$ denotes the all 1 vector, then

$$\frac{1}{n}\sum_{i=1}^{n} h(\langle \mathbf{b}\,,\, \mathbf{x}_i\rangle) = \langle \mathbf{b}\,,\, \mathbf{m}\rangle - \langle \mathbf{b}\,,\, \mathbf{C}\mathbf{b}\rangle\,,$$

where

$$\mathbf{m} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i - \mathbf{1})$$

and

$$\mathbf{C} = \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i - \mathbf{1})(\mathbf{x}_i - \mathbf{1})^T.$$

If we calculate the vector $\mathbf{m}$ and the matrix $\mathbf{C}$ beforehand then in each optimization step the complexity does not depend on n, so the running time for calculating $\bar{\mathbf{b}}_n$ is much smaller than for $\mathbf{b}_n^*$. The other advantage of the semi-log-optimal portfolio is that it can be calculated via quadratic programming, which is doable, e.g., using the routine QuadProg++ of [Gaspero (2006)]. This program uses Goldfarb-Idnani dual method for solving quadratic programming problems [Goldfarb and Idnani (1983)].

2.3. Time varying portfolio selection

For a general dynamic portfolio selection, the portfolio vector may depend on the past data. As before, $\mathbf{x}_i = (x_i^{(1)}, \dots x_i^{(d)})$ denotes the return vector on trading period i. Let $\mathbf{b} = \mathbf{b}_1$ be the portfolio vector for the first trading period. For initial capital S_0, we get that

$$S_1 = S_0 \cdot \langle \mathbf{b}_1 \, , \, \mathbf{x}_1 \rangle \, .$$

For the second trading period, S_1 is new initial capital, the portfolio vector is $\mathbf{b}_2 = \mathbf{b}(\mathbf{x}_1)$, and

$$S_2 = S_0 \cdot \langle \mathbf{b}_1 \, , \, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}(\mathbf{x}_1) \, , \, \mathbf{x}_2 \rangle \, .$$

For the nth trading period, a portfolio vector is $\mathbf{b}_n = \mathbf{b}(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) = \mathbf{b}(\mathbf{x}_1^{n-1})$ and

$$S_n = S_0 \prod_{i=1}^{n} \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle = S_0 e^{n W_n(\mathbf{B})}$$

with the average growth rate

$$W_n(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle .$$

2.3.1. *Log-optimal portfolio for stationary market process*

The fundamental limits, determined in [Móri (1982a)], in [Algoet and Cover (1988)], and in [Algoet (1992, 1994)], reveal that the so-called *(conditionally) log-optimal portfolio* $\mathbf{B}^* = \{\mathbf{b}^*(\cdot)\}$ is the best possible choice. More precisely, on trading period n let $\mathbf{b}^*(\cdot)$ be such that

$$\mathbb{E}\left\{ \ln \left\langle \mathbf{b}^*(\mathbf{X}_1^{n-1}) \, , \, \mathbf{X}_n \right\rangle \middle| \, \mathbf{X}_1^{n-1} \right\} = \max_{\mathbf{b}(\cdot)} \mathbb{E}\left\{ \ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}) \, , \, \mathbf{X}_n \right\rangle \middle| \, \mathbf{X}_1^{n-1} \right\} .$$

If $S_n^* = S_n(\mathbf{B}^*)$ denotes the capital achieved by a log-optimal portfolio strategy $\mathbf{B}^*$, after n trading periods, then for any other investment strategy $\mathbf{B}$ with capital $S_n = S_n(\mathbf{B})$ and with

$$\sup_n \mathbb{E}\left\{ (\ln \left\langle \mathbf{b}_n(\mathbf{X}_1^{n-1}) \, , \, \mathbf{X}_n \right\rangle)^2 \right\} < \infty,$$

and for any stationary and ergodic process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\limsup_{n \to \infty} \left(\frac{1}{n} \ln S_n - \frac{1}{n} \ln S_n^* \right) \leq 0 \quad \text{almost surely} \tag{2.3}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely,}$$

where

$$W^* := \mathbb{E}\left\{ \max_{\mathbf{b}(\cdot)} \mathbb{E}\left\{ \ln \left\langle \mathbf{b}(\mathbf{X}_{-\infty}^{-1})\,,\, \mathbf{X}_0 \right\rangle \middle| \, \mathbf{X}_{-\infty}^{-1} \right\} \right\}$$

is the maximal possible growth rate of any investment strategy. (Note that for memoryless markets $W^* = \max_{\mathbf{b}} \mathbb{E}\left\{\ln \langle \mathbf{b}\,,\, \mathbf{X}_0 \rangle\right\}$ which shows that in this case the log-optimal portfolio is the best constantly rebalanced portfolio.)

For the proof of this optimality we use the concept of martingale differences:

Definition 2.1. There are two sequences of random variables $\{Z_n\}$ and $\{X_n\}$ such that

- Z_n is a function of $X_1, \ldots, X_n$,
- $\mathbb{E}\{Z_n \mid X_1, \ldots, X_{n-1}\} = 0$ almost surely.

Then $\{Z_n\}$ is called martingale difference sequence with respect to $\{X_n\}$.

For martingale difference sequences, there is a strong law of large numbers: If $\{Z_n\}$ is a martingale difference sequence with respect to $\{X_n\}$ and

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\{Z_n^2\}}{n^2} < \infty$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Z_i = 0 \ \text{ a.s.}$$

(cf. [Chow (1965)], see also Theorem 3.3.1 in [Stout (1974)]).

In order to be self-contained, for martingale differences, we prove a weak law of large numbers. We show that if $\{Z_n\}$ is a martingale difference sequence with respect to $\{X_n\}$ then $\{Z_n\}$ are uncorrelated. Put $i < j$, then

$$\begin{aligned} \mathbb{E}\{Z_i Z_j\} &= \mathbb{E}\{\mathbb{E}\{Z_i Z_j \mid X_1, \ldots, X_{j-1}\}\} \\ &= \mathbb{E}\{Z_i \mathbb{E}\{Z_j \mid X_1, \ldots, X_{j-1}\}\} = \mathbb{E}\{Z_i \cdot 0\} = 0. \end{aligned}$$

It implies that

$$\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)^2\right\} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\{Z_i Z_j\} = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}\{Z_i^2\} \to 0$$

if, for example, $\mathbb{E}\{Z_i^2\}$ is a bounded sequence.

One can construct martingale difference sequence as follows: let $\{Y_n\}$ be an arbitrary sequence such that Y_n is a function of $X_1, \ldots, X_n$. Put

$$Z_n = Y_n - \mathbb{E}\{Y_n \mid X_1, \ldots, X_{n-1}\}.$$

Then $\{Z_n\}$ is a martingale difference sequence:

- Z_n is a function of $X_1, \ldots, X_n$,
- $\mathbb{E}\{Z_n | X_1, \ldots, X_{n-1}\} = \mathbb{E}\{Y_n - \mathbb{E}\{Y_n | X_1, \ldots, X_{n-1}\} | X_1, \ldots, X_{n-1}\} = 0$ almost surely.

Now we can prove of optimality of the log-optimal portfolio: introduce the decomposition

$$\begin{aligned}\frac{1}{n}\ln S_n &= \frac{1}{n}\sum_{i=1}^{n}\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\} \\ &\quad + \frac{1}{n}\sum_{i=1}^{n}\left(\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle - \mathbb{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}\right).\end{aligned}$$

The last average is an average of martingale differences, so it tends to zero a.s. Similarly,

$$\begin{aligned}\frac{1}{n}\ln S_n^* = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\} \\ + \frac{1}{n}\sum_{i=1}^{n}\left(\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle - \mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}\right).\end{aligned}$$

Because of the definition of the log-optimal portfolio we have that

$$\mathbb{E}\{\ln\left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\} \le \mathbb{E}\{\ln\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\},$$

and the proof is finished.

2.3.2. *Empirical portfolio selection*

The optimality relations proved above give rise to the following definition:

Definition 2.2. An empirical (data driven) portfolio strategy $\mathbf{B}$ is called **universally consistent with respect to a class $\mathcal{C}$ of stationary and ergodic processes $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,** if for each process in the class,

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n(\mathbf{B}) = W^* \quad \text{almost surely.}$$

It is not at all obvious that such universally consistent portfolio strategy exists. The surprising fact that there exists a strategy, universal with respect to a class of stationary and ergodic processes was proved by [Algoet (1992)].

Most of the papers dealing with portfolio selections assume that the distributions of the market process are known. If the distributions are unknown then one can apply a two stage splitting scheme.

1: In the first time period the investor collects data, and estimates the corresponding distributions. In this period there is no any investment.
2: In the second time period the investor derives strategies from the distribution estimates and performs the investments.

In the sequel we show that there is no need to make any splitting, one can construct sequential algorithms such that the investor can make trading during the whole time period, i.e., the estimation and the portfolio selection is made on the whole time period.

Let's recapitulate the definition of log-optimal portfolio:

$$\mathbb{E}\{\ln \left\langle \mathbf{b}^*(\mathbf{X}_1^{n-1}) \, , \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} = \max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}) \, , \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} \; .$$

For a fixed integer $k > 0$ large enough, we expect that

$$\mathbb{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}) \, , \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} \approx \mathbb{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}) \, , \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_{n-k}^{n-1}\}$$

and

$$\mathbf{b}^*(\mathbf{X}_1^{n-1}) \approx \mathbf{b}_k(\mathbf{X}_{n-k}^{n-1}) = \arg\max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}) \, , \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_{n-k}^{n-1}\}.$$

Because of stationarity

$$\begin{aligned}\mathbf{b}_k(\mathbf{x}_1^k) &= \arg\max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}) \, , \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_{n-k}^{n-1} = \mathbf{x}_1^k\} \\ &= \arg\max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \left\langle \mathbf{b}(\mathbf{x}_1^k) \, , \, \mathbf{X}_{k+1} \right\rangle \mid \mathbf{X}_1^k = \mathbf{x}_1^k\} \\ &= \arg\max_{\mathbf{b}} \mathbb{E}\{\ln \langle \mathbf{b} \, , \, \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k = \mathbf{x}_1^k\} \ ,\end{aligned}$$

which is the maximization of the regression function

$$m_{\mathbf{b}}(\mathbf{x}_1^k) = \mathbb{E}\{\ln \langle \mathbf{b} \, , \, \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k = \mathbf{x}_1^k\}.$$

Thus, a possible way for asymptotically optimal empirical portfolio selection is that, based on the past data, sequentially estimate the regression function $m_{\mathbf{b}}(\mathbf{x}_1^k)$, and choose the portfolio vector, which maximizes the regression function estimate.

2.3.3. *Regression function estimation*

Briefly summarize the basics of nonparametric regression function estimation. Concerning the details we refer to the book of [Györfi *et al.* (2002)] and to Chapter 5 of this volume. Let Y be a real valued random variable, and let X denote an observation vector taking values in $\mathbb{R}^d$. The regression function is the conditional expectation of Y given X:

$$m(x) = \mathbb{E}\{Y \mid X = x\}.$$

If the distribution of (X, Y) is unknown then one has to estimate the regression function from data. The data is a sequence of i.i.d. copies of (X, Y):

$$D_n = \{(X_1, Y_1), \dots, (X_n, Y_n)\}.$$

The regression function estimate is of form

$$m_n(x) = m_n(x, D_n).$$

An important class of estimates is the local averaging estimates

$$m_n(x) = \sum_{i=1}^{n} W_{n,i}(x; X_1, \dots, X_n) Y_i,$$

where usually the weights $W_{n,i}(x; X_1, \dots, X_n)$ are non-negative and sum up to 1. Moreover, $W_{n,i}(x; X_1, \dots, X_n)$ is relatively large if x is close to X_i, otherwise it is zero.

An example of such an estimate is the *partitioning estimate*. Here one chooses a finite or countably infinite partition $\mathcal{P}_n = \{A_{n,1}, A_{n,2}, \dots\}$ of $\mathbb{R}^d$ consisting of cells $A_{n,j} \subseteq \mathbb{R}^d$ and defines, for $x \in A_{n,j}$, the estimate by averaging Y_i's with the corresponding X_i's in $A_{n,j}$, i.e.,

$$m_n(x) = \frac{\sum_{i=1}^n I_{\{X_i \in A_{n,j}\}} Y_i}{\sum_{i=1}^n I_{\{X_i \in A_{n,j}\}}} \quad \text{for } x \in A_{n,j}, \tag{2.4}$$

where I_A denotes the indicator function of set A. Here and in the following we use the convention $\frac{0}{0} = 0$. In order to have consistency, on the one hand we need that the cells $A_{n,j}$ should be "small", and on the other hand the number of non-zero terms in the denominator of (2.4) should be "large". These requirements can be satisfied if the sequences of partition $\mathcal{P}_n$ is asymptotically fine, i.e., if

$$\operatorname{diam}(A) = \sup_{x,y \in A} \|x - y\|$$

denotes the diameter of a set such that $|| \cdot ||$ is the Euclidean norm, then for each sphere S centered at the origin

$$\lim_{n \to \infty} \max_{j : A_{n,j} \cap S \neq \emptyset} \operatorname{diam}(A_{n,j}) = 0$$

and

$$\lim_{n \to \infty} \frac{|\{j \,:\, A_{n,j} \cap S \neq \emptyset\}|}{n} = 0.$$

For the partition $\mathcal{P}_n$, the most important example is when the cells $A_{n,j}$ are cubes of volume h_n^d. For cubic partition, the consistency conditions above mean that

$$\lim_{n \to \infty} h_n = 0 \quad \text{and} \quad \lim_{n \to \infty} n h_n^d = \infty. \tag{2.5}$$

The second example of a local averaging estimate is the *Nadaraya–Watson kernel estimate*. Let $K : \mathbb{R}^d \to \mathbb{R}_+$ be a function called the kernel function, and let $h > 0$ be a bandwidth. The kernel estimate is defined by

$$m_n(x) = \frac{\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right) Y_i}{\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right)}$$

The kernel estimate is a weighted average of the Y_i, where the weight of Y_i (i.e., the influence of Y_i on the value of the estimate at x) depends on the distance between X_i and x. For the bandwidth $h = h_n$, the consistency conditions are (2.5). If one uses the so-called naïve kernel (or window kernel) $K(x) = I_{\{\|x\| \le 1\}}$, where $I_{\{\cdot\}}$ denotes the indicator function of the events in the brackets, that is, it equals 1 if the event is true and 0 otherwise. Then

$$m_n(x) = \frac{\sum_{i=1}^n I_{\{\|x-X_i\| \leq h\}} Y_i}{\sum_{i=1}^n I_{\{\|x-X_i\| \leq h\}}},$$

i.e., one estimates $m(x)$ by averaging Y_i's such that the distance between X_i and x is not greater than h.

Our final example of local averaging estimates is the *k-nearest neighbor* (k-NN) *estimate*l. Here one determines the k nearest X_i's to x in terms of distance $\|x - X_i\|$ and estimates $m(x)$ by the average of the corresponding Y_i's. More precisely, for $x \in \mathbb{R}^d$, let

$$(X_{(1)}(x), Y_{(1)}(x)), \ldots, (X_{(n)}(x), Y_{(n)}(x))$$

be a permutation of

$$(X_1, Y_1), \ldots, (X_n, Y_n)$$

such that

$$\|x - X_{(1)}(x)\| \leq \cdots \leq \|x - X_{(n)}(x)\|.$$

The k-NN estimate is defined by

$$m_n(x) = \frac{1}{k} \sum_{i=1}^{k} Y_{(i)}(x).$$

If $k = k_n \to \infty$ such that $k_n/n \to 0$ then the k-nearest-neighbor regression estimate is consistent.

We use the following correspondence between the general regression estimation and portfolio selection:

$$\begin{aligned} X &\sim \mathbf{X}_1^k, \\ Y &\sim \ln \langle \mathbf{b}\, , \, \mathbf{X}_{k+1} \rangle \, , \\ m(x) = \mathbb{E}\{Y \mid X = x\} &\sim m_{\mathbf{b}}(\mathbf{x}_1^k) = \mathbb{E}\{\ln \langle \mathbf{b}\, , \, \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k = \mathbf{x}_1^k\}. \end{aligned}$$

2.3.4. *Histogram based strategy*

Next we describe *histogram based strategy* due to [Györfi and Schäfer (2003)] and denote it by $\mathbf{B}^H$. We first define an infinite array of elementary strategies (the so-called *experts*) $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}^{(k,\ell)}(\cdot)\}$, indexed by the positive integers $k, \ell = 1, 2, \ldots$. Each expert $\mathbf{B}^{(k,\ell)}$ is determined by a period length k and by a partition $\mathcal{P}_\ell = \{A_{\ell,j}\}$, $j = 1, 2, \ldots, m_\ell$ of $\mathbb{R}_+^d$ into

m_ℓ disjoint cells. To determine its portfolio on the nth trading period, expert $\mathbf{B}^{(k,\ell)}$ looks at the return vectors $\mathbf{x}_{n-k},\dots,\mathbf{x}_{n-1}$ of the last k periods, discretizes this kd-dimensional vector by means of the partition $\mathcal{P}_\ell$, and determines the portfolio vector which is optimal for those past trading periods whose preceding k trading periods have identical discretized return vectors to the present one. Formally, let G_ℓ be the discretization function corresponding to the partition $\mathcal{P}_\ell$, that is,

$$G_\ell(\mathbf{x}) = j, \text{ if } \mathbf{x} \in A_{\ell,j} \ .$$

With some abuse of notation, for any n and $\mathbf{x}_1^n \in \mathbb{R}^{dn}$, we write $G_\ell(\mathbf{x}_1^n)$ for the sequence $G_\ell(\mathbf{x}_1),\dots,G_\ell(\mathbf{x}_n)$. Then define the expert $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}^{(k,\ell)}(\cdot)\}$ by writing, for each $n > k+1$,

$$\mathbf{b}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg\max_{\mathbf{b}\in\Delta_d} \prod_{i\in J_{k,l,n}} \langle \mathbf{b}\,,\,\mathbf{x}_i\rangle \ , \tag{2.6}$$

$$\text{where } J_{k,l,n} = \left\{k < i < n : G_\ell(\mathbf{x}_{i-k}^{i-1}) = G_\ell(\mathbf{x}_{n-k}^{n-1})\right\} \ ,$$

if $J_{k,l,n} \neq \emptyset$, and uniform $\mathbf{b}_0 = (1/d,\dots,1/d)$ otherwise. That is, $\mathbf{b}_n^{(k,\ell)}$ discretizes the sequence $\mathbf{x}_1^{n-1}$ according to the partition $\mathcal{P}_\ell$, and browses through all past appearances of the last seen discretized string $G_\ell(\mathbf{x}_{n-k}^{n-1})$ of length k. Then it designs a fixed portfolio vector optimizing the return for the trading periods following each occurrence of this string.

The problem left is how to choose k,ℓ. There are two extreme cases:

- small k or small ℓ implies that the corresponding regression estimate has large bias,
- large k and large ℓ implies that usually there are few matching, which results in large variance.

The good, data-driven choice of k and ℓ is doable borrowing recent techniques from machine learning. In online sequential machine learning setup k and ℓ are considered as parameters of the estimates, called experts. The basic idea of online sequential machine learning is the combination of the experts. The combination is an aggregated estimate, where an expert has large weight if its past performance is good (cf. [Cesa-Bianchi and Lugosi (2006)]).

The most appealing combination-type of the experts is exponential weighting due to its nice theoretical and practical properties. Combine the elementary portfolio strategies $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}_n^{(k,\ell)}\}$ as follows: let $\{q_{k,\ell}\}$ be a probability distribution on the set of all pairs (k,ℓ) such that for all k,ℓ, $q_{k,\ell} > 0$.

For a learning parameter $\eta > 0$, introduce the exponential weights

$$w_{n,k,\ell} = q_{k,\ell} e^{\eta \ln S_{n-1}(\mathbf{B}^{(k,\ell)})}.$$

For $\eta = 1$, it means that

$$w_{n,k,\ell} = q_{k,\ell} e^{\ln S_{n-1}(\mathbf{B}^{(k,\ell)})} = q_{k,\ell} S_{n-1}(\mathbf{B}^{(k,\ell)})$$

and

$$v_{n,k,\ell} = \frac{w_{n,k,\ell}}{\sum_{i,j} w_{n,i,j}}.$$

The combined portfolio $\mathbf{b}$ is defined by

$$\mathbf{b}_n(\mathbf{x}_1^{n-1}) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} v_{n,k,\ell} \mathbf{b}_n^{(k,\ell)}(\mathbf{x}_1^{n-1}).$$

This combination has a simple interpretation:

$$\begin{aligned}
S_n(\mathbf{B}^H) &= \prod_{i=1}^{n} \left\langle \mathbf{b}_i(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle \\
&= \prod_{i=1}^{n} \frac{\sum_{k,\ell} w_{i,k,\ell} \left\langle \mathbf{b}_i^{(k,\ell)}(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle}{\sum_{k,\ell} w_{i,k,\ell}} \\
&= \prod_{i=1}^{n} \frac{\sum_{k,\ell} q_{k,\ell} S_{i-1}(\mathbf{B}^{(k,\ell)}) \left\langle \mathbf{b}_i^{(k,\ell)}(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle}{\sum_{k,\ell} q_{k,\ell} S_{i-1}(\mathbf{B}^{(k,\ell)})} \\
&= \prod_{i=1}^{n} \frac{\sum_{k,\ell} q_{k,\ell} S_i(\mathbf{B}^{(k,\ell)})}{\sum_{k,\ell} q_{k,\ell} S_{i-1}(\mathbf{B}^{(k,\ell)})} \\
&= \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}).
\end{aligned}$$

The strategy $\mathbf{B}^H$ then arises from weighting the elementary portfolio strategies $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}_n^{(k,\ell)}\}$ such that the investor's capital becomes

$$S_n(\mathbf{B}^H) = \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}). \tag{2.7}$$

It is shown in [Györfi and Schäfer (2003)] that the strategy $\mathbf{B}^H$ is universally consistent with respect to the class of all ergodic processes such that $\mathbb{E}\{|\log X^{(j)}|\} < \infty$, for all $j = 1, 2, \ldots, d$ under the following two conditions on the partitions used in the discretization:

(a) the sequence of partitions is nested, that is, any cell of $\mathcal{P}_{\ell+1}$ is a subset of a cell of $\mathcal{P}_\ell$, $\ell = 1, 2, \ldots$;

(b) if $\mathrm{diam}(A) = \sup_{\mathbf{x},\mathbf{y}\in A} \|\mathbf{x}-\mathbf{y}\|$ denotes the diameter of a set, then for any sphere $S \subset \mathbb{R}^d$ centered at the origin,

$$\lim_{\ell\to\infty} \max_{j:A_{\ell,j}\cap S\neq\emptyset} \mathrm{diam}(A_{\ell,j}) = 0 \ .$$

2.3.5. *Kernel based strategy*

[Györfi *et al.* (2006)] introduced *kernel-based portfolio selection* strategies. Define an infinite array of experts $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}^{(k,\ell)}(\cdot)\}$, where k, ℓ are positive integers. For fixed positive integers k, ℓ, choose the radius $r_{k,\ell} > 0$ such that for any fixed k,

$$\lim_{\ell\to\infty} r_{k,\ell} = 0.$$

Then, for $n > k+1$, define the expert $\mathbf{b}^{(k,\ell)}$ by

$$\mathbf{b}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg\max_{\mathbf{b}\in\Delta_d} \prod_{\left\{k<i<n:\|\mathbf{x}_{i-k}^{i-1}-\mathbf{x}_{n-k}^{n-1}\|\le r_{k,\ell}\right\}} \langle \mathbf{b}\,,\, \mathbf{x}_i\rangle \ ,$$

if the sum is non-void, and $\mathbf{b}_0 = (1/d,\ldots,1/d)$ otherwise. These experts are mixed as in (2.7).

[Györfi *et al.* (2006)] proved that the portfolio scheme $\mathbf{B}^K = \mathbf{B}$ is universally consistent with respect to the class of all ergodic processes such that $\mathbb{E}\{|\ln X^{(j)}|\} < \infty$, for $j = 1, 2, \ldots d$.
Sketch of the proof: Because of the fundamental limit (2.3), we have to prove that

$$\liminf_{n\to\infty} W_n(\mathbf{B}) = \liminf_{n\to\infty} \frac{1}{n} \ln S_n(\mathbf{B}) \geq W^* \quad \text{a.s.}$$

We have that

$$\begin{aligned}
W_n(\mathbf{B}) &= \frac{1}{n} \ln S_n(\mathbf{B}) \\
&= \frac{1}{n} \ln \left(\sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}) \right) \\
&\geq \frac{1}{n} \ln \left(\sup_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}) \right) \\
&= \frac{1}{n} \sup_{k,\ell} \left(\ln q_{k,\ell} + \ln S_n(\mathbf{B}^{(k,\ell)}) \right) \\
&= \sup_{k,\ell} \left(W_n(\mathbf{B}^{(k,\ell)}) + \frac{\ln q_{k,\ell}}{n} \right).
\end{aligned}$$

Thus

$$\begin{aligned}\liminf_{n\to\infty} W_n(\mathbf{B}) &\geq \liminf_{n\to\infty} \sup_{k,\ell} \left(W_n(\mathbf{B}^{(k,\ell)}) + \frac{\ln q_{k,\ell}}{n} \right) \\ &\geq \sup_{k,\ell} \liminf_{n\to\infty} \left(W_n(\mathbf{B}^{(k,\ell)}) + \frac{\ln q_{k,\ell}}{n} \right) \\ &= \sup_{k,\ell} \liminf_{n\to\infty} W_n(\mathbf{B}^{(k,\ell)}) \\ &= \sup_{k,\ell} \epsilon_{k,\ell}.\end{aligned}$$

Because of $\lim_{\ell\to\infty} r_{k,\ell} = 0$, we can show that

$$\sup_{k,\ell} \epsilon_{k,\ell} = \lim_{k\to\infty} \lim_{l\to\infty} \epsilon_{k,\ell} = W^*.$$

2.3.6. *Nearest neighbor based strategy*

Define an infinite array of experts $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}^{(k,\ell)}(\cdot)\}$, where $0 < k$, ℓ are integers. Just like before, k is the window length of the near past, and for each ℓ choose $p_\ell \in (0,1)$ such that

$$\lim_{\ell\to\infty} p_\ell = 0. \tag{2.8}$$

Put

$$\hat{\ell} = \lfloor p_\ell n \rfloor.$$

At a given time instant n, the expert searches for the $\hat{\ell}$ nearest neighbor (NN) matches in the past. For fixed positive integers k, ℓ ($n > k + \hat{\ell} + 1$), introduce the set of the $\hat{\ell}$ nearest neighbor matches:

$$\begin{aligned}\hat{J}_n^{(k,\ell)} = \{i;\, & k+1 \leq i \leq n \text{ such that } \mathbf{x}_{i-k}^{i-1} \text{ is among the } \hat{\ell} \text{ NNs of } \mathbf{x}_{n-k}^{n-1} \\ & \text{in } \mathbf{x}_1^k, \ldots, \mathbf{x}_{n-k-1}^{n-2}\}.\end{aligned}$$

Define the expert by

$$\mathbf{b}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg\max_{\mathbf{b}\in\Delta_d} \prod_{i\in\hat{J}_n^{(k,\ell)}} \langle \mathbf{b}\,,\, \mathbf{x}_i \rangle\,.$$

That is, $\mathbf{b}_n^{(k,\ell)}$ is a fixed portfolio vector according to the returns following these nearest neighbors. These experts are mixed in the same way as in (2.7).

We say that a tie occurs with probability zero if for any vector $\mathbf{s} = \mathbf{s}_1^k$ the random variable

$$\|\mathbf{X}_1^k - \mathbf{s}\|$$

has continuous distribution function.

[Györfi *et al.* (2008)] proved the following theorem: assume (2.8) and that a tie occurs with probability zero, then the portfolio scheme $\mathbf{B}^{NN}$ is universally consistent with respect to the class of all stationary and ergodic processes such that $\mathbb{E}\{|\log X^{(j)}|\} < \infty$, for $j = 1, 2, \ldots d$.

2.3.7. *Numerical results on empirical portfolio selection*

The theoretical results above hold under the condition of stationarity. Obviously, the real data of returns (relative prices) are not stationary, therefore we performed some experiments for New-York Stock Exchange (NYSE) data . This section gives numerical results on empirical portfolio selection. At the web page [Gelencsér and Ottucsák (2006)] there are two benchmark data set from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years (5651 trading days ending in 1985). More precisely, the data set contains the daily price relatives, that was calculated from the nominal values of the *closing prices* corrected by the dividends and the splits for all trading day. This data set has been used for testing portfolio selection in [Cover (1991)], in [Singer (1997)], in [Györfi *et al.* (2006)], in [Györfi *et al.* (2008)] and in [Györfi *et al.* (2007)].
- The second data set contains 19 stocks and has length 44 years (11178 trading days ending in 2006) and it was generated same way as the previous data set (it was augmented by the last 22 years).

Our experiment is on the *second data set.* To make the analysis feasible, some simplifying assumptions are used that need to be taken into account. Remind the reader that we assume that

- the assets are arbitrarily divisible,
- the assets are available for buying or for selling in unbounded quantities at the current price at any given trading period,
- there are no transaction costs (in Chapter 3 of this volume we offer solutions to overcome this problem),
- the behavior of the market is not affected by the actions of the investor using the strategy under investigation.

For the 19 assets in the second data set, the average annual yield (AAY) of the static uniform portfolio (uniform index) is 14%, while the best asset was MORRIS (Philip Morris International Inc.) with AAY 20%. These yields match with theoretical consideration derived in Chapter 1 of this volume (Table 1.1), that is, that 44 years period is too "short" in the sense in order to show that the limit of the growth rate of the static portfolio coincides with that of the best asset.

Table 2.3. Comparison of the two algorithms for CRPs.

Stock's name	AAY	BCRP log	BCRP Semi-log
AHP	13%	0	0
ALCOA	9%	0	0
AMERB	14%	0	0
COKE	14%	0	0
DOW	12%	0	0
DUPONT	9%	0	0
FORD	9%	0	0
GE	13%	0	0
GM	7%	0	0
HP	15%	0.177	0.178
IBM	10%	0	0
INGER	11%	0	0
JNJ	16%	0	0
KIMBC	13%	0	0
MERCK	15%	0	0
MMM	11%	0	0
MORRIS	20%	0.747	0.746
PANDG	13%	0	0
SCHLUM	15%	0.076	0.076
AAY		20%	20%

Table 2.3 summarizes the numerical results for these 19 assets and for BCRP. The first column of Table 2.3 lists the stock's name, the second column shows the AAY. The third and the fourth columns present the weights of the stocks (the components of the best constant portfolio vector) using the log-optimal and semi-log-optimal algorithms. Surprisingly, the two portfolio vectors are almost the same, according to next-to-the-last row the growth rates are the same: 20%. Again, the 44 years period is "too small" in order to get that the growth rate of BCRP is much larger than the growth rate of the best asset. Here one can make the same observation as at the end of Chapter 1 of this volume, i.e., the BCRP is very undiversified,

only three assets have positive weight.

For the calculation of the optimal portfolio we use a recursive greedy GRADIENT ALGORITHM. Introduce the projection P of a vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$ to Δ_d:

$$P(\mathbf{b}) = \frac{\mathbf{b}}{\sum_{j=1}^{d} b^{(j)}}.$$

Put

$$W_n(\mathbf{b}) = \frac{1}{n} \sum_{i=1}^{n} \log \langle \mathbf{b} \, , \mathbf{x}_i \rangle \, ,$$

and let $\mathbf{e}_j$ be the j-th unit vector, i.e., its j-th component is 1, the other components are 0.

GRADIENT ALGORITHM

Parameters: number d of assets, initial portfolio $\mathbf{b}_0 = (1/d, \dots, 1/d)$, $V_0 = W_n(\mathbf{b}_0)$ and step size $\delta > 0$.

At iteration steps $k = 1, 2, 3, \dots$,

(1) Calculate

$$W_n(P(\mathbf{b}_{k-1} + \delta \cdot \mathbf{e}_j)) \quad j = 1, \dots, d \; ;$$

(2) If

$$V_{k-1} \geq \max_j W_n(P(\mathbf{b}_{k-1} + \delta \cdot \mathbf{e}_j))$$

then stop, and the result of the algorithm is $\mathbf{b}_{k-1}$.
Otherwise, put

$$V_k = \max_j W_n(P(\mathbf{b}_{k-1} + \delta \cdot \mathbf{e}_j))$$

and

$$\mathbf{b}_k = P(\mathbf{b}_{k-1} + \delta \cdot \mathbf{e}_{j^*}),$$

where

$$j^* = \arg\max_j W_n(P(\mathbf{b}_{k-1} + \delta \cdot \mathbf{e}_j)).$$

Go to (1);

Table 2.4. The average annual yields of the individual experts for the kernel strategy.

k / ℓ	1	2	3	4	5
1	31%	30%	24%	21%	26%
2	34%	31%	27%	25%	22%
3	35%	29%	26%	24%	23%
4	35%	30%	30%	32%	27%
5	34%	29%	33%	24%	24%
6	35%	29%	28%	24%	27%
7	33%	29%	32%	23%	23%
8	34%	33%	30%	21%	24%
9	37%	33%	28%	19%	21%
10	34%	29%	26%	20%	24%

Next we show experiments for time-varying portfolio selection. One can combine the kernel based portfolio selection and the principle of semi-log-optimal algorithm in Section 2.2.3, called kernel based semi-log-optimal portfolio (cf. [Györfi *et al.* (2007)]). We present some numerical results obtained by applying the kernel based semi-log-optimal algorithm to the second NYSE data set.

The proposed universally consistent empirical portfolio selection algorithms use an infinite array of experts. In practice we take a finite array of size $K \times L$. In our experiment we selected $K = 5$ and $L = 10$. Choose the uniform distribution $\{q_{k,\ell}\} = 1/(KL)$ over the experts in use, and the radius

$$r_{k,l}^2 = 0.0002 \cdot d \cdot k(1 + \ell/10),$$

($k = 1, \dots, K$ and $\ell = 1, \dots, L$).

Table 2.4 summarizes the average annual yield achieved by each expert at the last period when investing one unit for the kernel-based semi-log-optimal portfolio. Experts are indexed by $k = 1 \dots 5$ in columns and $\ell = 1 \dots 10$ in rows. The average annual yield of kernel based semi-log-optimal portfolio is 31%. According to Table 2.3, MORRIS had the best average annual yield, 20%, while the BCRP had average annual yield 20%, so with kernel based semi-log-optimal portfolio we have a spectacular improvement.

Another interesting feature of Table 2.4 is that for any fixed ℓ, the best k is equal to 1, so as far as empirical portfolio is concerned the Markovian modelling is appropriate. If the time horizon in the experiment were infinity, then the numbers in each fixed row would be monotonically increasing. Here we observe just the contrary, the reasoning of which is that in the k-th position of a row the dimension of the optimization problem is $19 \cdot k$, so for larger k the dimension is too large with respect to the length of the data, i.e., for larger k there are not enough data to "learn" the best portfolio. Again, the time varying portfolio is very undiversified such that the subset of assets with non-zero weight is changing from time to time, which makes the problem of transaction cost challenging.

Table 2.5. The average annual yields of the individual experts for the nearest neighbor strategy.

k / ℓ	1	2	3	4	5
50	31%	33%	28%	24%	35%
100	33%	32%	25%	29%	28%
150	38%	33%	26%	32%	27%
200	38%	28%	32%	32%	24%
250	37%	31%	37%	28%	26%
300	41%	35%	35%	30%	29%
350	39%	36%	31%	34%	32%
400	39%	35%	33%	32%	35%
450	39%	34%	34%	35%	37%
500	42%	36%	33%	38%	35%

We performed some experiments using nearest neighbor strategy. Again, we take a finite array of size $K \times L$ such that $K = 5$ and $L = 10$. Choose the uniform distribution $\{q_{k,\ell}\} = 1/(KL)$ over the experts in use. Table 2.5 summarizes the average annual yield achieved by each expert at the last period when investing one unit for the nearest neighbor portfolio strategy. Experts are indexed by $k = 1 \dots 5$ in columns and $\ell = 50, 100, \dots, 500$ in rows, where ℓ is the number of nearest neighbors. The average annual yield of nearest neighbor portfolio is 35% . Comparing Tables 2.4 and 2.5, one can conclude that the nearest neighbor strategy is more robust.

References

Algoet, P. (1992). Universal schemes for prediction, gambling, and portfolio selection, *Annals of Probability* **20**, pp. 901–941.

Algoet, P. (1994). The strong law of large numbers for sequential decisions under uncertainity, *IEEE Transactions on Information Theory* **40**, pp. 609–634.

Algoet, P. and Cover, T. (1988). Asymptotic optimality asymptotic equipartition properties of log-optimum investments, *Annals of Probability* **16**, pp. 876–898.

Aumann, R. J. (1977). The St. Petersburg paradox: A discussion of some recent comments, *Journal of Economic Theory* **14**, pp. 443–445.

Barron, A. R. and Cover, T. M. (1988). A bound on the financial value of information, *IEEE Transactions on Information Theory* **34**, pp. 1097–1100.

Bernoulli, D. (1954). Originally published in 1738; translated by L. Sommer. Exposition of a new theory on the measurement of risk, *Econometrica* **22**, pp. 22–36.

Breiman, L. (1961). Optimal gambling systems for favorable games, in *Proc. of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press, Berkeley), pp. 65–78.

Cesa-Bianchi, N. and Lugosi, G. (2006). *Prediction, Learning, and Games* (Cambridge University Press, Cambridge).

Chow, Y. S. (1965). Local convergence of martingales and the law of large numbers, *Annals of Mathematical Statistics* **36**, pp. 552–558.

Chow, Y. S. and Robbins, H. (1961). On sums of independent random variables with infinite moments and "fair" games, in *Proc. Nat. Acad. Sci*, Vol. 47 (USA), pp. 330–335.

Cover, T. (1984). An algorithm for maximizing expected log investment return, *IEEE Transactions on Information Theory* **30**, pp. 369–373.

Cover, T. and Thomas, J. A. (1991). *Elements of Information Theory* (Wiley, New York).

Cover, T. M. (1991). Universal portfolios, *Mathematical Finance* **1**, pp. 1–29.

Csörgő, S. and Simons, G. (1996). A strong law of large numbers for trimmed sums, with applications to generalized St. Petersburg games, *Statistics and Probability Letters* **26**, pp. 65–73.

Durand, D. (1957). Growth stocks and the Petersburg paradox, *The Journal of Finance* **12**, pp. 348–363.

Feller, W. (1945). Note on the law of large numbers and "fair" games, *Ann. Math. Statist.* **16**, pp. 301–304.

Finkelstein, M. and Whitley, R. (1981). Optimal strategies for repeated games, *Advances in Applied Probability* **13**, pp. 415–428.

Gaspero, L. (2006). QuadProg++, C++ porting of the Goldfarb-Idnani dual method, URL `http://www.diegm.uniud.it/digaspero/`.

Gelencsér, G. and Ottucsák, G. (2006). NYSE data sets at the log-optimal portfolio homepage, URL `www.szit.bme.hu/~oti/portfolio`.

Goldfarb, D. and Idnani (1983). A numerically stable dual method for solving strictly quadratic programs, *Mathematical Programming* **27**, pp. 1–33.

Györfi, L. and Kevei, P. (2009). St. Petersburg portfolio games, in R. Gavaldà, G. Lugosi, T. Zeugmann and S. Zilles (eds.), *Proceedings of Algorithmic Learning Theory 2009* (Lecture Notes in Artificial Intelligence 5809), pp. 83–96.

Györfi, L. and Kevei, P. (2011). On the rate of convergence of St. Petersburg games, *Periodica Mathematica Hungarica* **62**, pp. 13–37.

Györfi, L., Kohler, M., Krzyżak, A. and Walk, H. (2002). *A Distribution-Free Theory of Nonparametric Regression* (Springer, New York).

Györfi, L., Lugosi, G. and Udina, F. (2006). Nonparametric kernel based sequential investment strategies, *Mathematical Finance* **16**, pp. 337–357.

Györfi, L. and Schäfer, D. (2003). Nonparametric prediction, in J. A. K. Suykens, G. Horváth, S. Basu, C. Micchelli and J. Vandevalle (eds.), *Advances in Learning Theory: Methods, Models and Applications* (IOS Press, NATO Science Series), pp. 339–354.

Györfi, L., Udina, F. and Walk, H. (2008). Nonparametric nearest-neighbor-based empirical portfolio selection strategies, *Statistics and Decisions* **22**, pp. 145–157.

Györfi, L., Urbán, A. and Vajda, I. (2007). Kernel-based semi-log-optimal empirical portfolio selection strategies, *International Journal of Theoretical and Applied Finance* **10**, 5, pp. 505–516.

Haigh, J. (1999). *Taking Chances* (Oxford University Press, Oxford, UK).

Kelly, J. L. (1956). A new interpretation of information rate, *Bell System Technical Journal* **35**, pp. 917–926.

Kenneth, A. J. (1974). The use of unbounded utility functions in expected-utility maximization: Response, *Quarterly Journal of Economics* **88**, pp. 136–138.

Latané, H. A. (1959). Criteria for choice among risky ventures, *Journal of Political Economy* **38**, pp. 145–155.

Luenberger, D. G. (1998). *Investment Science* (Oxford University Press).

Markowitz, H. (1952). Portfolio selection, *Journal of Finance* **7**, 1, pp. 77–91.

Markowitz, H. (1976). Investment for the long run: New evidence for an old rule, *Journal of Finance* **31**, 5, pp. 1273–1286.

Martin, R. (2004). The St. Petersburg paradox, in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy (Fall 2004 Edition)* (Stanford University, Stanford, California).

Menger, K. (1934). Das Unsicherheitsmoment in der Wertlehre Betrachtungen im Anschlußan das sogenannte Petersburger Spiel, *Zeitschrift für Nationalökonomie* **5**, pp. 459–485.

Móri, T. F. (1982a). Asymptotic properties of empiricial strategy in favourable stochastic games, in *Proc. Colloquia Mathematica Societatis János Bolyai 36. Limit Theorems in Probability and Statistics*, pp. 777–790.

Móri, T. F. (1982b). On favourable stochastic games, *Annales Univ. Sci. Budapest. R. Eötvös Nom., Sect. Comput* **3**, pp. 99–103.

Móri, T. F. (1984). I-divergence geometry of distributions and stochastic games, in *Proc. of the 3rd Pannonian Symp. on Math. Stat* (Reidel, Dordrecht), pp. 231–238.

Móri, T. F. (1986). Is the empirical strategy optimal? *Statistics and Decisions* **4**, pp. 45–60.

Móri, T. F. and Székely, G. J. (1982). How to win if you can? in *Proc. Colloquia Mathematica Societatis János Bolyai 36. Limit Theorems in Probability and Statistics*, pp. 791–806.

Morvai, G. (1991). Empirical log-optimal portfolio selection, *Problems of Control and Information Theory* **20**, 6, pp. 453–463.

Morvai, G. (1992). Portfolio choice based on the empirical distribution, *Kybernetika* **28**, 6, pp. 484–493.

Pulley, L. B. (1994). Mean-variance approximation to expected logarithmic utility, *IEEE Transactions on Information Theory* **IT-40**, pp. 409–418.

Rieger, M. O. and Wang, M. (2006). Cumulative prospect theory and the St. Petersburg paradox, *Economic Theory* **28**, pp. 665–679.

Roll, R. (1973). Evidence on the "growth-optimum" model, *The Journal of Finance* **28**, pp. 551–566.

Samuelson, P. A. (1960). The St. Petersburg paradox as a divergent double limit, *International Economic Review* **1**, pp. 31–37.

Singer, Y. (1997). Switching portfolios, *International Journal of Neural Systems* **8**, pp. 445–455.

Spellucci, P. (1999). Donlp2, C Version 7/99. netlib, URL `http://www.netlib.org/`.

Stout, W. F. (1974). *Almost sure convergence* (Academic Press, New York).

Vajda, I. (2006). Analysis of semi-log-optimal investment strategies, in M. Huskova and M. Janzura (eds.), *Prague Stochastics 2006* (MATFYZPRESS, Prague).