Chapter 1

On the History of the Growth Optimal Portfolio

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The growth optimal portfolio (GOP) is a portfolio which has a maximal expected growth rate over any time horizon. As a consequence, this portfolio is sure to outperform any other significantly different strategy as the time horizon increases. This property in particular has fascinated many researchers in finance and mathematics created a huge and exciting literature on growth optimal investment. This chapter attempts to provide a comprehensive survey of the literature and applications of the GOP. In particular, the heated debate of whether the GOP has a special place among portfolios in the asset allocation decision is reviewed as this still seem to be an area where some misconceptions exists. The survey also provides an extensive review of the recent use of the GOP as a pricing tool, in for instance the so-called "benchmark approach". This approach builds on the numéraire property of the GOP, that is, the fact that any other asset denominated in units of the GOP become a supermartingale.

1.1. Introduction and a Historical Overview

Over the past 50 years a large number of papers have investigated the GOP. As the name implies this portfolio can be used by an investor to maximize the expected growth rate of his or her portfolio. However, this is only one among many uses of this object. In the literature it has been applied in as diverse connections as portfolio theory and gambling, utility theory, information theory, game theory, theoretical and applied asset pricing, insurance, capital structure theory, macro-economy and event studies. The ambition of the present chapter is to present a reasonably comprehensive

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review of the different connections in which the portfolio has been applied. An earlier survey in [Hakansson and Ziemba (1995)] focused mainly on the applications of the GOP for investment and gambling purposes. Although this will be discussed in Section 1.3, the present chapter has a somewhat wider scope.

The origins of the GOP have usually been tracked to the paper [Kelly (1956)], hence the name "Kelly criterion" , which is used synonymously. (The name Kelly criterion probably originates from [Thorp (1971)].) Kelly's motivation came from gambling and information theory, and his paper derived a striking but simple result: There is an optimal gambling strategy, such that with probability one, this optimal gambling strategy will accumulate more wealth than any other different strategy. Kelly's strategy was the growth optimal strategy and in this respect the GOP was discovered by him. However, whether this is the true origin of the GOP depends on a point of view. The GOP is a portfolio with several aspects, one of which is the maximization of the *geometric mean*. In this respect, the history might be said to have its origin in [Williams (1936)], who considered speculators in a multi-period setting and reached the conclusion that due to compounding, speculators should worry about the geometric mean and not the arithmetic ditto. Williams did not reach any result regarding the growth properties of this approach but was often cited as the earliest paper on the GOP in the seventies seemingly due to the remarks on geometric mean made in the appendix of his paper. Yet another way of approaching the history of the GOP is from the perspective of utility theory. As the GOP is the choice of a log-utility investor, one might investigate the origin of this utility function. In this sense the history dates even further back to the 18th century. The mathematician Pierre Rèmond Montemort challenged Nicolas Bernoulli with five problems, one of which was the famous St. Petersburg paradox. The St. Petersburg paradox refers to the coin tossing game, where returns are given as 2^{n-1}, where n is the number of games before "heads" come up the first time. The expected value of participating is infinite, but in Nicolas Bernoulli's words, no sensible man would pay 20 dollars for participating. Nicolas Bernoulli posed the problem to his cousin, Daniel Bernoulli, who suggested using a utility function to ensure that (rational) gamblers will use a more conservative strategy. Note that any unbounded utility function is subject to the generalized St. Petersburg paradox, obtained by scaling the outcomes of the original paradox sufficiently to provide infinite expected utility. For more information see e.g. [Bernoulli (1954)], [Menger (1967)], [Samuelson (1977)] or [Aase (2001)]. Nicolas Bernoulli conjectured that gamblers should be risk averse, but less so if they had high wealth. In particular, he suggested that marginal utility should be inverse proportional to wealth, which is tantamount to assuming log-utility. However, the choice of logarithm appears to have nothing to do with the growth properties of this strategy, as is sometimes suggested. The original article "Specimen Theoriae Nova de Mensura Sortis" from 1738 is reprinted in Econometrica [Bernoulli (1954)] and do not mention growth. (Concerning recent development on St. Petersburg portfolio games see Chapter 2 of this volume.) It should be noted that the St. Petersburg paradox was resolved even earlier by Cramer, who used the square-root function in a similar way. Hence log-utility has a history going back at least 250 years and in this sense, so has the GOP. It seems to have been Bernoulli who to some extent inspired the article [Latané (1959)]. Independent of Kelly's result, Latané suggested that investors should maximize the geometric mean of their portfolios, as this would maximize the probability that the portfolio would be more valuable than any other portfolio. The cited paper by Latané has a reference to Kelly's 1956 paper, but Latané mentions that he was unaware of Kelly's result before presenting the paper at an earlier conference in 1956. Independent of where the history of the GOP is said to start, the real interest in the GOP was not awoken until after the papers by Kelly and Latané. As will be described later on, the goal suggested by Latané caused a great deal of debate among economists which has not completely died out yet. The paper by Kelly caused a great deal of immediate interest in the mathematic and gambling community. [Breiman (1960, 1961)] expanded the analysis of [Kelly (1956)] and discussed applications for long term investment and gambling in a more general mathematical setting.

Calculating the growth optimal strategy is generally very difficult in discrete time and is treated in [Bellman and Kalaba (1957)], [Elton and Gruber (1974)] and [Maier *et al.* (1977b)] although the difficulties disappear whenever the market is complete. This is similar to the case when jumps in asset prices happen at random. In the continuous-time continuous-diffusion case, the problem is much easier and was solved in [Merton (1969)]. This problem along with a general study of the properties of the GOP have been studied for decades and is still being studied today. Mathematicians fascinated by the properties of the GOP has contributed to the literature with a significant number of theoretical articles spelling out the properties of the GOP in a variety of scenarios and increasingly generalized settings, including continuous time models based on semimartingale representation of asset prices. Today, solutions to the problem exist in a semi-explicit form and in the general case, the GOP can be characterized in terms of the semimartingale characteristic triplet. A non-linear integral equation must still be solved to get the portfolio weights. The properties of the GOP and the formulas required to calculate the strategy in a given set-up are discussed in Section 1.2. It has been split into two parts. Section 1.2.1 deals with the simple discrete time case, providing the main properties of the GOP without the need of demanding mathematical techniques. Section 1.2.2 deals with the fully general case, where asset price processes are modelled as semimartingales, and contains examples on important special cases.

The growth optimality and the properties highlighted in Section 1.2 inspired authors to recommend the GOP as a universally "best" strategy and this sparked a heated debate. In a number of papers Paul Samuelson and other academics argued that the GOP was only one among many other investment rules and any belief that the GOP was universally superior rested on a fallacy. The substance of this discussion is explained in details in Section 1.3.1. The debate from the late sixties and seventies contains some important lessons to be held in mind when discussing the application of the GOP as a long term investment strategy.

The use of the GOP became referred to as the *growth optimum theory* and it was introduced as an alternative to expected utility and the mean-variance approaches to asset pricing. It was argued that a theory for portfolio selection and asset pricing based on the GOP would have properties which are more appealing than those implied by the mean-variance approach developed by [Markowitz (1952)]. Consequently, a significant amount of the literature deals with comparing the two approaches. A discussion of the relation between the GOP and the mean-variance model is presented in Section 1.3.2. Since a main argument for applying the GOP is its ability to outperform other portfolios over time, authors have tried to estimate the time needed to be "reasonably" sure to obtain a better result using the GOP. Some answers to this question are provided in Section 1.3.3.

The fact that asset prices, when denominated in terms of the GOP, become supermartingales was realized quite early, appearing in a proof in [Breiman (1960)][Theorem 1]. It was not until 1990 in [Long (1990)] that this property was given a more thorough treatment. Although Long suggested this as a method for measuring abnormal returns in event studies and this approach has been followed recently in working papers by [Gerard *et al.* (2000)] and [Hentschel and Long (2004)], the consequences of the numéraire property stretches much further. It suggested a change of numéraire technique for asset pricing under which a change of probability measure would be unnecessary. The first time this is treated explicitly appears to be in [Bajeux-Besnaino and Portait (1997a)] in the late nineties. At first, the use of the GOP for derivative pricing purposes was essentially just the choice of a particular pricing operator in an incomplete market. Over the past five years, this idea became developed further in the benchmark framework of [Platen (2002)] and later papers, who emphasize the applicability of this idea in the absence of a risk-neutral probability measure. The use of the GOP as a tool for derivative pricing is reviewed in Section 1.4. This has motivated a substantial part of this paper, because it essentially challenges the approach of using some risk neutral measure for pricing derivatives. During this chapter I am going to conduct a (hopefully) thorough analysis of what arbitrage concepts are relevant in a mathematically consistent theory of derivative pricing and what role martingale measures play in this context. Section 1.4 gives a motivation and foreshadows some of the results I will derive later on. A complete survey of the benchmark approach is beyond the scope of this chapter, but may be found in [Platen (2006a)].

The suggestion that such GOP denominated prices could be martingales is important to the empirical work, since this provide a testable assumption which can be verified from market data. The Kuhn-Tucker conditions for optimum provides only the supermartingale property which may be a problem, see Section 1.2 and Section 1.5. Few empirical papers exist, and most appeared during the seventies. Some papers tried to obtain evidence for or against the assumption that the market was dominated by growth optimizers and to see how the growth optimum model compared to the mean-variance approach. Others try to document the performance of the GOP as an investment strategy, in comparison with other strategies. Section 1.5 deals with the existing empirical evidence related to the GOP.

Since an understanding of the properties of the GOP provides a useful background for analyzing the applications, the first task will be to present the relevant results which describe some of the remarkable properties of the GOP. The next section is separated into a survey of discrete time results which are reasonably accessible and a more mathematically demanding survey in continuous time. This is not just mathematically convenient but also fairly chronological. It also discusses the issues related to *solving* for the growth optimal portfolio strategy, which is a non-trivial task in the general case. Readers that are particularly interested in the GOP from an investment perspective may prefer to skip the general treatment in Section 1.2.2 with very little loss. However, most of this chapter relies extensively on the continuous time analysis and later sections builds on the results obtained in Section 1.2.2. Extensive references will be given in the notes at the end of each section and only the most important references are kept within the main text, in order to keep it fluent and short.

1.2. Theoretical Studies of the GOP

The early literature on the GOP was usually framed in discrete time and considered a restricted number of distributions. Despite the simplicity and loss of generality, most of the interesting properties of the GOP can be analyzed within such a framework. The more recent theory has almost exclusively considered the GOP in continuous time and considers very general set-ups, requiring the machinery of stochastic integration and sometimes applies a very general class of processes, semimartingales, which are well-suited for financial modelling. Although many of the fundamental properties of the GOP carry over to the general case, there are some quite technical, but very important differences to the discrete time case.

Section 1.2.1 reviews the major theoretical properties of the GOP in a discrete time framework, requiring only basic probability theory. Section 1.2.2, on the other hand, surveys the GOP problem in a very general semimartingale setting and places modern studies within this framework. It uses the theory of stochastic integration with

respect to semimartingales, but simpler examples have been provided for illustrative purposes. Both sections are structured around three basic issues. *Existence*, which is fundamental, particularly for theoretical applications. *Growth properties* are those that are exploited when using the GOP as an investment strategy. Finally, the *numéraire property* which is essential for the use of the GOP in derivative pricing.

1.2.1. *Discrete Time*

Consider a market consisting of a finite number of non-dividend paying assets. The market consists of $d+1$ assets, represented by a $d+1$ dimensional vector process, S, where

$$S = \left\{ S(t) = (S^{(0)}(t), \dots S^{(d)}(t)),\ t \in \{0, 1, \dots, T\} \right\}. \tag{1.1}$$

The first asset $S^{(0)}$ is sometimes assumed to be risk-free from one period to the next, i.e. the value $S^{(0)}(t)$ is known at time $t-1$. In other words, $S^{(0)}$ is a predictable process. Mathematically, let $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, \mathbb{P})$ denote a filtered probability space, where $\underline{\mathcal{F}} = (\mathcal{F}_t)_{t\in\{0,1,\ldots T\}}$ is an increasing sequence of information sets. Each price process $S^{(i)} = \{S^{(i)}(t),\ t \in \{0,1,\ldots T\}\}$ is assumed to be adapted to the filtration $\underline{\mathcal{F}}$. In words, the price of each asset is known at time t, given the information $\mathcal{F}_t$. Sometimes it will be convenient to work on an infinite time horizon in which case $T = \infty$. However, unless otherwise noted, T is assumed to be some finite number.

Define the *return* process

$$R = \left\{R(t) = (R^0(t), \ldots R^d(t)),\ t \in \{1,2,\ldots,T\}\right\}$$

by $R^i(t) \triangleq \frac{S^{(i)}(t)}{S^{(i)}(t-1)} - 1$. Often it is assumed that returns are independent over time, and for simplicity this assumption is made in this section.

Investors in such a market consider the choice of a *strategy*

$$\delta = \left\{\delta(t) = (\delta^{(0)}(t), \ldots \delta^{(d)}(t)),\ t \in \{0,\ldots,T\}\right\},$$

where $\delta^{(i)}(t)$ denotes the number of units of asset i that is being held during the period $(t, t+1]$. As usual some notion of "reasonable" strategy has to be used. Definition 1.1 makes this precise.

Definition 1.1. A trading strategy, δ, generates the portfolio value process $S^{(\delta)}(t) \triangleq \delta(t)\cdot S(t)$. The strategy is called *admissible* if it satisfies the three conditions

(1) Non-anticipative: The process δ is adapted to the filtration $\mathcal{F}$, meaning that $\delta(t)$ can only be chosen based on information available at time t.
(2) Limited liability: The strategy generates a portfolio process $S^{(\delta)}(t)$ which is non-negative.
(3) Self-financing: $\delta(t-1)\cdot S(t) = \delta(t)\cdot S(t)$, $t \in \{1,\ldots T\}$ or equivalently $\Delta S^{(\delta)}(t) = \delta(t-1)\cdot \Delta S(t)$.

The set of admissible portfolios in the market will be denoted $\Theta(S)$, and $\underline{\Theta}(S)$ will denote the strictly positive portfolios. It is assumed that $\underline{\Theta}(S) \neq \emptyset$.

Here, the notation $x \cdot y$ denotes the standard Euclidean inner product. These assumptions are fairly standard. The first part assumes that any investor is unable to look into the future, only the current and past information is available. The second part requires the investor to remain solvent, since his total wealth must always be non-negative. This requirement will prevent him from taking an unreasonably risky position. Technically, this constraint is not strictly necessary in the very simple set-up described in this subsection, unless the time horizon T is infinite. The third part requires that the investor re-invests all money in each time step. No wealth is withdrawn or added to the portfolio. This means that intermediate consumption is not possible. Although this is a restriction in generality, consumption can be allowed at the cost of slightly more complex statements. Since consumption is not important for the purpose of this survey, I have decided to leave it out altogether. The requirement that it should be possible to form a strictly positive portfolio is important, since the growth rate of any portfolio with a chance of defaulting will be minus infinity.

Consider an investor who invests a dollar of wealth in some portfolio. At the end of period T his wealth becomes

$$S^{(\delta)}(T) = S^{(\delta)}(0) \prod_{i=0}^{T-1} (1 + R^{(\delta)}(i))$$

where $R^{(\delta)}(t)$ is the return in period t. If the portfolio *fractions* are fixed during the period, the right-hand-side is the product of T independent and identically distributed (i.i.d.) random variables. The *geometric average* return over the period is then

$$\left(\prod_{i=0}^{T-1} (1 + R^{(\delta)}(i)) \right)^{\frac{1}{T}} .$$

Because the returns of each period are i.i.d., this average is a sample of the *geometric mean value* of the one-period return distribution. For discrete random variables, the geometric mean of a random variable X taking (not necessarily distinct) values $x_1, \dots x_S$ with equal probabilities is defined as

$$G(X) \triangleq \left(\Pi_{s=1}^{S} x_s \right)^{\frac{1}{S}} = \left(\Pi_{k=1}^{K} \tilde{x}_k^{f_k} \right) = \exp(\mathbb{E}[\log(X)]),$$

where $\tilde{x}_k$ is the distinct values of X and f_k is the frequency of which $X = x_k$, that is $f_k = \mathbb{P}(X = x_k)$. In other words, the geometric mean is the exponential function of the *growth rate* $g^{\delta}(t) \triangleq \mathbb{E}[\log(1 + R^{(\delta)})(t)]$ of some portfolio. Hence if Ω is discrete or more precisely if the σ-algebra $\mathcal{F}$ on Ω is countable, maximizing the geometric mean is equivalent to maximizing the expected growth rate. Generally, one defines the geometric mean of an arbitrary random variable by

$$G(X) \triangleq \exp(\mathbb{E}[\log(X)])$$

assuming the mean value $\mathbb{E}[\log(X)]$ is well defined. Over long stretches intuition dictates that each realized value of the return distribution should appear on average the number of times dictated by its frequency, and hence as the number of periods increase, it would hold that

$$\left(\prod_{i=0}^{T-1}(1+R^{(\delta)}(i))\right)^{\frac{1}{T}} = \exp\left(\frac{\sum_{i=1}^{T}\log(S^{(\delta)}(i))}{T}\right) \to G(1+R^{(\delta)}(1))$$

as $T \to \infty$. This states that the average growth rate converges to the expected growth rate. In fact this heuristic argument can be made precise by an application of the law of large numbers, but here I only need it for establishing intuition. In multi-period models, the geometric mean was suggested by [Williams (1936)] as a natural performance measure, because it took into account the effects from compounding. Instead of worrying about the average expected return, an investor who invests repeatedly should worry about the geometric mean return. As I will discuss later on, not everyone liked this idea, but it explains why one might consider the problem

$$\sup_{S^{(\delta)}(T)\in\Theta} \mathbb{E}\left[\log\left(\frac{S^{(\delta)}(T)}{S^{(\delta)}(0)}\right)\right]. \tag{1.2}$$

Definition 1.2. A solution, $S^{(\underline{\delta})}$, to (1.2) is called a GOP.

Hence the objective given by (1.2) is often referred to as the *geometric mean criteria*. Economists may view this as the maximization of expected terminal wealth for an individual with logarithmic utility. However, it is important to realize that the GOP was introduced into economic theory, not as a special case of a general utility maximization problem, but because it seems as an intuitive objective, when the investment horizon stretches over several periods. The next section will demonstrate the importance of this observation. For simplicity it is always assumed that $S^{(\delta)}(0) = 1$, i.e. the investors start with one unit of wealth.

If an investor can find an admissible portfolio having zero initial cost and which provides a strictly positive pay-off at some future date, a solution to (1.2) will not exist. Such a portfolio is called an *arbitrage* and is formally defined in the following way.

Definition 1.3. An admissible strategy δ is called an arbitrage strategy if

$$S^{(\delta)}(0) = 0 \quad \mathbb{P}(S^{(\delta)}(T) \geq 0) = 1 \quad \mathbb{P}(S^{(\delta)}(T) > 0) > 0.$$

It seems reasonable that this is closely related to the existence of a solution to problem (1.2), because the existence of a strategy that creates "something out of nothing" would provide an infinitely high growth rate.

In fact, in the present discrete time set-up, the two things are completely equivalent.

Theorem 1.1. *There exists a GOP, $S^{(\delta)}$, if and only if there is no arbitrage. If the GOP exists its value process is unique.*

The necessity of no arbitrage is straightforward as indicated above. The sufficiency will follow directly once the numéraire property of the GOP has been established, see Theorem 1.4 below. In a more general continuous time set-up, the equivalence between no arbitrage and the existence of a GOP, as predicted from Theorem 1.4, is not completely true and technically much more involved. The uniqueness of the GOP only concerns the value process, not the strategy. If there are redundant assets, the GOP strategy is not necessarily unique. Uniqueness of the value process will follow from the Jensen inequality, once the numéraire property has been established. The existence and uniqueness of a GOP plays only a minor role in the theory of investments, where it is more or less taken for granted. In the line of literature that deals with the application of the GOP for pricing purposes, establishing existence is essential.

It is possible to infer some simple properties of the GOP strategy, without further specifications of the model:

Theorem 1.2. *The GOP strategy has the following properties:*

(1) The fractions of wealth invested in each asset are independent of the level of total wealth.
(2) The invested fraction of wealth in asset i is proportional to the return on asset i.
(3) The strategy is myopic.

The first part is to be understood in the sense that the *fractions* invested are independent of current wealth. Moreover, the GOP strategy allocates funds in proportion to the excess return on an asset. Myopia means short-sighted and implies that the GOP strategy in a given period depends only on the distribution of returns in the next period. Hence the strategy is independent of the time horizon. Despite the negative flavor the word "myopic" can be given, it may for practical reasons be quite convenient to have a strategy which only requires the estimation of returns one period ahead. It seems reasonable to assume, that return distributions further out in the future are more uncertain. To see why the GOP strategy depends only on the distribution of asset returns one period ahead note that

$$\mathbb{E}\left[\log(S^{(\delta)}(T))\right] = \log(S^{(\delta)}(0)) + \sum_{i=1}^{T} \mathbb{E}\left[\log(1+R^{(\delta)}(i))\right].$$

In general, obtaining the strategy in an explicit closed form is not possible. This involves solving a non-linear optimization problem. To see this, I derive the first order conditions of (1.2). Since by Theorem 1.2 the GOP strategy is myopic and the invested fractions are independent of wealth, one needs to solve the problem

$$\sup_{\delta(t)} \mathbb{E}_t \left[\log\left(\frac{S^{(\delta)}(t+1)}{S^{(\delta)}(t)}\right)\right] \tag{1.3}$$

for each $t \in \{0, 1, \ldots, T-1\}$. Using the fractions $\pi^i_\delta(t) = \frac{\delta^{(i)}(t)S^{(i)}(t)}{S^{(\delta)}(t)}$ the problem can be written

$$\sup_{\pi_\delta(t)\in\mathbb{R}^d} \mathbb{E}\left[\log\left(1 + (1 - \sum_{i=1}^{n}\pi^i_\delta)R^0(t) + \sum_{i=1}^{n}\pi^i_\delta R^i(t)\right)\right]. \tag{1.4}$$

The properties of the logarithm ensures that the portfolio will automatically become admissible. By differentiation, the first order conditions become

$$\mathbb{E}_{t-1}\left[\frac{1+R^i(t)}{1+R^{\underline{\delta}}(t)}\right] = 1 \quad i \in \{0, 1, \ldots, n\}. \tag{1.5}$$

This constitutes a set of $d+1$ non-linear equation to be solved simultaneously such that one of which is a consequence of the others, due to the constraint that $\sum_{i=0}^{d} \pi^i_\delta = 1$. Although these equations do not generally posses an explicit closed-form solution, there are some special cases which can be handled:

Example 1.1 (Betting on events). *Consider a one-period model. At time $t = 1$ the outcome of the discrete random variable X is revealed. If the investor bets on this outcome, he receives a fixed number α times his original bet, which I normalize to one dollar. If the expected return from betting is negative, the investor would prefer to avoid betting, if possible. Let $A_i = \{\omega | X(\omega) = x_i\}$ be the sets of mutual exclusive possible outcomes, where $x_i > 0$. Some straightforward manipulations provide*

$$1 = \mathbb{E}\left[\frac{1+R^i}{1+R^{\underline{\delta}}}\right] = \mathbb{E}\left[\frac{1_{A_i}}{\pi^i_{\underline{\delta}}}\right] = \frac{\mathbb{P}(A_i)}{\pi^i_{\underline{\delta}}}$$

and hence $\pi^i_{\underline{\delta}} = \mathbb{P}(A_i)$. Consequently, the growth-maximizer bets proportionally on the probability of the different outcomes.

In the example above, the GOP strategy is easily obtained since there is a finite number of mutually exclusive outcomes and it was possible to bet on any of these outcomes. It can be seen by extending the example, that the odds for a given event has no impact on the *fraction* of wealth used to bet on the event. In other words, if all events have the same probability the pay-off if the event come true does not alter the optimal fractions.

Translated into a financial terminology, Example 1.1 illustrates the case when the market is *complete*. The market is complete whenever Arrow-Debreu securities paying one dollar in one particular state of the world can be replicated, and a bet on each event could be interpreted as buying an Arrow-Debreu security. Markets consisting of Arrow-Debreu securities are sometimes referred to as "horse race markets" because only one security, "the winner", will make a pay-off in a given state. (See also Example 4 in Chapter 2 of this volume.) In a financial setting, the securities are most often not modelled as Arrow-Debreu securities.

Example 1.2 (Complete Markets). *Again, a one-period model is considered. Assume that the probability space Ω is finite, and for $\omega_i \in \Omega$ there is a strategy δ_{ω_i} such that at time 1*

$$S^{(\delta_{\omega_i})}(\omega) = 1_{(\omega=\omega_i)}.$$

Then the growth optimal strategy, by the example above, is to hold a fraction of total wealth equal to $\mathbb{P}(\omega)$ in the portfolio $S^{(\delta_\omega)}$. In terms of the original securities, the investor needs to invest

$$\pi^i = \sum_{\omega \in \Omega} \mathbb{P}(\omega) \pi^i_{\delta_\omega}$$

where $\pi^i_{\delta_\omega}$ is the fraction of asset i held in the portfolio $S^{(\delta_\omega)}$.

The conclusion that a GOP can be obtained explicitly in a complete market is quite general. In an incomplete discrete time setting things are more complicated and no explicit solution will exist, requiring the use of numerical methods to solve the non-linear first order conditions. The non-existence of an explicit solution to the problem was mentioned by e.g. [Mossin (1973)] as a main reason for the lack of popularity of the Growth Optimum model in the seventies. Due to the increase in computational power over the past thirty years, time considerations have become unimportant. Leaving the calculations aside for a moment, I turn to the distinguishing properties of the GOP, which have made it quite popular among academics and investors searching for a utility independent criteria for portfolio selection. A discussion of the role of the GOP in asset allocation and investment decisions is postponed to Section 1.3.

Theorem 1.3. *The portfolio process $S^{(\underline{\delta})}(t)$ has the following properties*

(1) *If assets are infinitely divisible, the ruin probability, $\mathbb{P}(S^{(\underline{\delta})}(t) = 0$ for some $t \leq T)$, of the GOP is zero.*
(2) *If, additionally, there is at least one asset with non-negative expected growth rate, then the long-term ruin probability (defined below) of the GOP is zero.*
(3) *For any strategy δ it holds that $\limsup \frac{1}{t} \log \left(\frac{S^{(\delta)}(t)}{S^{(\underline{\delta})}(t)} \right) \leq 0$ almost surely.*
(4) *Asymptotically, the GOP maximizes median wealth.*

The no-ruin property critically depends on *infinite divisibility* of investments. This means that an arbitrary small amount of a given asset can be bought or sold. As wealth becomes low, the GOP will require a constant fraction to be invested and hence such a low absolute amount must be feasible. If not, ruin is a possibility. In general, any strategy which invests a fixed relative amount of capital will never cause the ruin of the investor in finite time as long as arbitrarily small amounts of capital can be invested. In the case, where the investor is guaranteed not to be ruined at some fixed time, the *long term ruin probability* of an investor following the strategy δ is defined as

$$\mathbb{P}(\liminf_{t \to \infty} S^{(\delta)}(t) = 0).$$

Only if the optimal growth rate is greater than zero can ruin in this sense be avoided. Note that seemingly rational strategies such as "bet such that $\mathbb{E}[X_t]$ is maximized" can be shown to ensure certain ruin, even in fair or favorable games. A simple example would be head or tail using a false coin, where chances of head are 90%. If a player bets all his money on head, then the chance that he will be ruined in n games will be $1 - 0.9^n \to 1$. Interestingly, certain portfolios selected by maximizing utility can have a long-term ruin probability of one, even if there exist portfolios with a strictly positive growth rate. This means that some utility maximizing investors are likely to end up with, on average, very little wealth. The third property is the distinguishing feature of the GOP. It implies that with probability one, the GOP will overtake the value of any other portfolio and stay ahead indefinitely. In other words, for *every* path taken, if the strategy δ is different from the GOP, there is an instant s such that $S^{(\underline{\delta})}(t) >$

$S^{(\delta)}(t)$ for every $t > s$. Hence, although the GOP is defined so as to maximize the *expected* growth rate, it also maximizes the long term growth rate in an *almost sure* sense. The proof in a simple case is due to [Kelly (1956)], more sources are cited in the notes. This property has led to some confusion: if the GOP outperforms any other portfolio at some point in time, it may be tempting to argue that long term investors should all invest in the GOP. This is, however, not literally true and I will discuss this in Section 1.3.1. The last part of the theorem has received less attention. Since the median of a distribution is unimportant to an investor maximizing expected utility, the fact that the GOP maximizes the median of wealth in the long run is of little theoretical importance, at least in the field of economics. Yet, for practical purposes it may be interesting, since for highly skewed distributions the median is quite useful as a measure of the most likely outcome. The property was recently shown by [Ethier (2004)].

Another performance criterion often discussed is the expected time to reach a certain level of wealth. In other words, if the investor wants to get rich fast, what strategy should he use? It is *not* generally true that the GOP is the strategy which minimizes this time, due to the problem of *overshooting*. If one uses the GOP, chances are that the target level is exceeded significantly. Hence a more conservative strategy might be better, if one wishes to attain a goal and there is no "bonus" for exceeding the target. To give a mathematical formulation define

$$\tau^{\delta}(x) \triangleq \inf\{t \mid S^{(\delta)}(t) \geq x\}$$

and let $g^{\delta}(t)$ denote the growth rate of the strategy δ, at time $t \in \{1, \ldots, \}$. Note that due to myopia, the GOP strategy does not depend on the final time, so it makes sense to define it even if $T = \infty$. Hence, $g^{\underline{\delta}}(t)$ denotes the expected growth rate using the GOP strategy. If returns are i.i.d., then $g^{\underline{\delta}}(t)$ is a constant, $g^{\underline{\delta}}$. Defining the stopping time $\tau^{(\delta)}(x)$ to be the first time the portfolio $S^{(\delta)}$ exceeds the level x, the following asymptotic result holds true.

Lemma 1.1 (Breiman, 1961). *Assume returns to be i.i.d. Then for any strategy* δ

$$\lim_{x \to \infty} \left(\mathbb{E}[\tau^{(\delta)}(x)] - \mathbb{E}[\tau^{(\underline{\delta})}(x)] \right) = \sum_{i \in \mathbb{N}} \left(1 - \frac{g^{\delta}(i)}{g^{\underline{\delta}}} \right).$$

In fact, a technical assumption needed is that the variables $\log(g^{(\delta)}(t))$ be *non-lattice*. A random variable X is lattice if there is some $a \in$

$\mathbb{R}$ and some $b > 0$ such that $\mathbb{P}(X \in a + b\mathbb{Z}) = 1$, where $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. As $g^{\underline{\delta}}$ is larger than g^{δ}, the right-hand side is non-negative, implying that the expected time to reach a goal is asymptotically minimized when using the GOP, as the desired level is increased indefinitely. In other words, for "high" wealth targets, the GOP will minimize the expected time to reach this target. Note that the assumption of i.i.d. returns implies that the expected growth rate is identical for all periods. For finite hitting levels, the problem of overshooting can be dealt with by introducing a "time rebate" when the target is exceeded. In this case, the GOP strategy remains optimal for finite levels. The problem of overshooting is eliminated in the continuous time diffusion case, because the diffusion can be controlled instantaneously and in this case the GOP will minimize the time to reach any goal, see [Pestien and Sudderth (1985)].

This ends the discussion of the properties that are important when considering the GOP as an investment strategy. Readers whose main interest is in this direction may skip the remainder of this section. Apart from the growth property, there is another property, of the GOP, the *numéraire property*, which I will explain below, and which is important in order to understand the role of the GOP in the fields of derivative/asset pricing. Consider equation (1.5) and assume there is a solution satisfying these first order conditions. It follows immediately that the resulting GOP will have the property that expected returns of any asset measured against the return of the GOP will be zero. In other words, if GOP denominated returns of any portfolio are zero, then GOP denominated prices become *martingales*, since

$$\mathbb{E}_t\left[\frac{1 + R^{\delta}(t+1)}{1 + R^{\underline{\delta}}(t+1)}\right] = \mathbb{E}_t\left[\frac{S^{(\delta)}(t+1)}{S^{(\underline{\delta})}(t+1)}\frac{S^{(\underline{\delta})}(t)}{S^{(\delta)}(t)}\right] = 1$$

which implies that

$$\mathbb{E}_t\left[\frac{S^{(\delta)}(t+1)}{S^{(\underline{\delta})}(t+1)}\right] = \frac{S^{(\delta)}(t)}{S^{(\underline{\delta})}(t)}.$$

If asset prices in GOP denominated units are martingales, then the empirical probability measure $\mathbb{P}$ is an equivalent martingale measure (EMM). This suggests a way of pricing a given pay-off. Measure it in units of the GOP and take the ordinary average. In fact this methodology was suggested recently and will be discussed in Section 1.4. Generally there is no guarantee that (1.5) has a solution. Even if Theorem 1.1 ensures the existence of a GOP, it may be that the resulting strategy does not satisfy (1.5). Mathematically, this is just the statement that an optimum need not be attained in an inner point, but can be attained at the boundary. Even in this case something may be said about GOP denominated returns - they become *strictly negative* - and the GOP denominated price processes become strict supermartingales.

Theorem 1.4. *The process* $\hat{S}^{(\delta)}(t) \triangleq \frac{S^{(\delta)}(t)}{S^{(\underline{\delta})}(t)}$ *is a supermartingale. If* $\pi^{\underline{\delta}}(t)$ *belongs to the interior of the set*

$$\{x \in \mathbb{R}^d | \textit{Investing the fractions } x \textit{ at time } t \textit{ is admissible}\},$$

then $\hat{S}^{(\delta)}(t)$ *is a true martingale.*

Note that $\hat{S}^{(\delta)}(t)$ can be a martingale even if the fractions are not in the interior of the set of admissible strategies. This happens in the (rare) cases where the first order conditions are satisfied on the boundary of this set. The fact that the GOP has the numéraire property follows by applying the bound $\log(x) \leq x - 1$ and the last part of the statement is obtained by considering the first order conditions for optimality, see Equation (1.5). The fact that the numéraire property of the portfolio $S^{(\underline{\delta})}$ implies that $S^{(\underline{\delta})}$ is the GOP is shown by considering the portfolio

$$S^{(\epsilon)}(t) \triangleq \epsilon S^{(\delta)}(t) + (1-\epsilon) S^{(\underline{\delta})}(t),$$

using the numéraire property and letting ϵ turn to zero.

The martingale condition has been used to establish a theory for pricing financial assets, see Section 1.4, and to test whether a given portfolio is the GOP, see Section 1.5. Note that the martingale condition is equivalent to the statement that returns denominated in units of the GOP become zero. A portfolio with this property was called a *numéraire portfolio* by [Long (1990)]. If one restricts the definition such that a numéraire portfolio only covers the case where such returns are exactly zero, then a numéraire portfolio need not exist. In the case where (1.5) has no solution, there is no numéraire portfolio, but under the assumption of no arbitrage there is a GOP and hence the existence of a numéraire portfolio is not a consequence of no arbitrage. This motivated the generalized definition of a numéraire portfolio, made by [Becherer (2001)], who defined a numéraire portfolio as a portfolio, $S^{(\underline{\delta})}$, such that for all other strategies, δ, the process $\frac{S^{(\delta)}(t)}{S^{(\underline{\delta})}(t)}$ would be a supermartingale. By Theorem 1.4 this portfolio is the GOP.

It is important to check that the numéraire property is valid, since otherwise the empirical tests of the martingale restriction implied by (1.5) become invalid. Moreover, using the GOP and the change of numéraire technique for pricing derivatives becomes unclear as will be discussed in Section 1.4.

A simple example illustrates the situation that GOP denominated asset prices may be supermartingales.

Example 1.3 (Becherer, 2001, Bühlmann and Platen, 2003). *Consider a simple one period model and let the market* $(S^{(0)}, S^{(1)})$ *be such that the first asset is risk free,* $S^{(0)}(t) = 1$, $t \in \{0, T\}$. *The second asset has a log-normal distribution* $\log(S^{(1)}(T)) \sim \mathcal{N}(\mu, \sigma^2)$ *and* $S^{(1)}(0) = 1$. *Consider an admissible strategy* $\delta = (\delta^{(0)}, \delta^{(1)})$ *and assume the investor has one unit of wealth. Since*

$$S^{(\delta)}(T) = \delta^{(0)} + \delta^{(1)} S^{(1)}(T) \geq 0$$

and $S^{(1)}(T)$ *is log-normal, it follows that* $\delta^{(i)} \in [0, 1]$ *in order for the wealth process to be non-negative. Now*

$$\mathbb{E}\left[\log\left(S^{(\delta)}(T)\right)\right] = \mathbb{E}\left[\log\left(1 + \delta^{(1)}(S^{(1)}(T) - S^{(0)}(T))\right)\right].$$

First order conditions imply that

$$\mathbb{E}\left[\frac{S^{(1)}(T)}{1 + \underline{\delta}^{(1)}(S^{(1)}(T) - S^{(0)}(T))}\right] = \mathbb{E}\left[\frac{S^{(0)}(T)}{1 + \underline{\delta}^{(1)}(S^{(1)}(T) - S^{(0)}(T))}\right] = 1.$$

It can be verified that there is a solution to this equation if and only if $|\mu| \leq \frac{\sigma^2}{2}$. *If* $\mu - \frac{\sigma^2}{2} \leq 0$ *then it is optimal to invest everything in* $S^{(0)}$. *The intuition is, that compared to the risk-less asset the risky asset has a negative growth rate. Since the two are independent it is optimal not to invest in the risky asset at all. In this case*

$$\hat{S}^{(0)}(T) = 1, \quad \hat{S}^{(1)}(T) = S^{(1)}(T).$$

it follows that $\hat{S}^{(0)}$ *is a martingale, whereas* $\hat{S}^{(1)}(T) = S^{(1)}(T)$ *is a strict supermartingale, since* $\mathbb{E}[S^{(1)}(T)|\mathcal{F}_0] \leq S^{(1)}(0) = 1$. *Conversely, if* $\mu \geq \frac{\sigma^2}{2}$ *then it is optimal to invest everything in asset 1, because the growth rate of the risk-free asset* relative *to the growth rate of the risky asset is negative. The word relative is important because the growth rate in absolute terms is zero. In this case*

$$\hat{S}^{(0)}(T) = \frac{1}{S^{(1)}(T)}, \quad \hat{S}^{(1)}(T) = 1$$

and hence, $\hat{S}^{(0)}$ *is a supermartingale, whereas* $\hat{S}^{(1)}$ *is a martingale.*

The simple example shows that there is economic intuition behind the case when GOP denominated asset prices become true martingales. It happens in two cases. Firstly, it may happen if the growth rate of the risky asset is low. In other words, the market price of risk is very low and investors cannot create short positions due to limited liability to short the risky asset. Secondly, it may happen if the risky asset has a high growth rate, corresponding to the situation where the market price of risk is high. In the example this corresponds to $\mu \geq \frac{\sigma^2}{2}$. Investors cannot have arbitrary long positions in the risky assets, because of the risk of bankruptcy. The fact that investors avoid bankruptcy is not a consequence of Definition 1.1, it will persist even without this restriction. Instead, it derives from the fact that the logarithmic utility function turns to minus infinity as wealth turns to zero. Consequently, any strategy that may result in zero wealth with positive probability will be avoided. One may expect to see the phenomenon in more general continuous-time models, in cases where investors are facing portfolio constraints or if there are jumps which may suddenly reduce the value of the portfolio. I will return to this issue in the next section.

Notes

The assumption of independent returns can be loosened, see [Hakansson and Liu (1970)] and [Algoet and Cover (1988)]. Although strategies in such set-ups should be based on previous information, not just the information of the current realizations of stock prices, it can be shown that the growth and numéraire property remains intact in this set-up.

That no arbitrage is necessary seems to have been noted quite early by [Hakansson (1971a)], who formulated this as a "no easy money" condition, where "easy money" is defined as the ability to form a portfolio whose return dominates the risk free interest rate almost surely. The one-to-one relation to arbitrage appears in [Maier *et al.* (1977b)][Theorem 1 and 1'] and although they do not mention arbitrage and state price densities (SPD) explicitly, their results could be phrased as the equivalence between the existence of a solution to problem 1.2 and the existence of an SPD [Theorem 1] and the absence of arbitrage [Theorem 1']. The first time the relation is mentioned explicitly is in [Long (1990)]. Long's Theorem 1, as stated, is *not* literally true, although it would be if numéraire portfolio was replaced by GOP. Uniqueness of the value process, $S^{(\underline{\delta})}(t)$, was remarked in [Breiman (1961)][Proposition 1].

The properties of the GOP strategy, in particular the myopia was analyzed in [Mossin (1968)]. Papers addressing the problem of obtaining a solution to the problem include [Bellman and Kalaba (1957)], [Ziemba (1972)], [Elton and Gruber (1974)], [Maier *et al.* (1977b)] and [Cover (1984)]. The methods are either approximations or based on non-linear optimization models.

The proof of the second property of Theorem 1.3 dates back to [Kelly (1956)] for a very special case of Bernoulli trials but was noted independently by [Latané (1959)]. The results where refined in [Breiman (1960, 1961)] and extended to general distributions in [Algoet and Cover (1988)].

The expected time to reach a certain goal was considered in [Breiman (1961)] and the inclusion of a rebate in [Aucamp (1977)] implies that the GOP will minimize this time for finite levels of wealth.

The numéraire property can be derived from the proof of [Breiman (1961)][Theorem 3]. The term numéraire portfolio is from [Long (1990)]. The issue of supermartingality was apparently overlooked until explicitly pointed out in [Kramkov and Schachermayer (1999)][Example 5.1]. A general treatment which takes this into account is found in [Becherer (2001)], see also [Korn and Schäl (1999, 2009)] and [Bühlmann and Platen (2003)] for more in a discrete time setting.

1.2.2. *Continuous Time*

In this section some of the results are extended to a general continuous time framework. The main conclusions of the previous section stand, although with some important modifications, and the mathematical exposition is more challenging. For this reason, the results are supported by examples. Most conclusions from the continuous case are important for the treatment in Section 1.4 and the remainder of this chapter which is held in continuous time.

The mathematical object used to model the financial market given by (1.1), is now a $d+1$-dimensional semimartingale, S, living on a filtered probability space $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, \mathbb{P})$, satisfying the usual conditions, see [Protter (2004)]. Being a semimartingale, S can be decomposed as

$$S(t) = A(t) + M(t)$$

where A is a finite variation process and M is a local martingale. The reader is encouraged to think of these as *drift* and *volatility* respectively, but should beware that the decomposition above is not always unique. If A can be chosen to be predictable, then the decomposition is unique. This is exactly the case when S is a special semimartingale, see [Protter (2004)].

Following standard conventions, the first security is assumed to be the numéraire, and hence it is assumed that $S^{(0)}(t) = 1$ almost surely for all $t \in [0, T]$. The investor needs to choose a strategy, represented by the $d+1$ dimensional process

$$\delta = \{\delta(t) = (\delta^{(0)}(t), \ldots, \delta^{(d)}(t)), t \in [0, T]\}.$$

The following definition of admissibility is the natural counterpart to Definition 1.1

Definition 1.4. An admissible trading strategy, δ, satisfies the three conditions:

(1) δ is an S-integrable, predictable process.
(2) The resulting portfolio value $S^{(\delta)}(t) \triangleq \sum_{i=0}^{d} \delta^{(i)}(t) S^{(i)}(t)$ is non-negative.
(3) The portfolio is self-financing, that is $S^{(\delta)}(t) = \int_0^t \delta(s) dS(s)$.

Here, predictability can be loosely interpreted as left-continuity, but more precisely, it means that the strategy is adapted to the filtration generated by all left-continuous $\underline{\mathcal{F}}$-adapted processes. In economic terms, it means that the investor cannot change his portfolio to guard against jumps that occur randomly. For more on this and a definition of integrability with respect to a semimartingale, see [Protter (2004)]. The second requirement is important in order to rule out simple, but unrealistic, strategies leading to arbitrage, as for instance doubling strategies. The last requirement states that the investor does not withdraw or add any funds. Recall that $\underline{\Theta}(S)$ denotes the set of non-negative portfolios, which can be formed using the elements of S. It is often convenient to consider portfolio fractions, i.e.

$$\pi_\delta = \{\pi_\delta(t) = (\pi_\delta^0(t), \ldots, \pi_\delta^d(t))^\top, \quad t \in [0, \infty)\}$$

with coordinates defined by:

$$\pi_\delta^i(t) \triangleq \frac{\delta^{(i)}(t) S^{(i)}(t)}{S^{(\delta)}(t)}. \tag{1.6}$$

One may define the GOP, $S^{(\underline{\delta})}$, as in Definition 1.2, namely as the solution to the problem

$$S^{(\underline{\delta})} \triangleq \arg \sup_{S^{(\delta)} \in \underline{\Theta}(S)} \mathbb{E}[\log(S^{(\delta)}(T))]. \tag{1.7}$$

This of course only makes sense if the expectation is uniformly bounded on $\underline{\Theta}(S)$ although alternative and economically meaningful definitions exist which circumvent the problem of having

$$\sup_{S^{(\delta)} \in \underline{\Theta}(S)} \mathbb{E}[\log(S^{(\delta)}(T))] = \infty.$$

For simplicity, I use the following definition.

Definition 1.5. A portfolio is called a GOP if it satisfies (1.7).

In discrete time, there was a one-to-one correspondence between no arbitrage and the existence of a GOP. Unfortunately, this breaks down in continuous time. Here several definitions of arbitrage are possible. A key existence result is based on the article [Kramkov and Schachermayer (1999)], who used the notion of *No Free Lunch with Vanishing Risk* (NFLVR). The essential feature of NFLVR is the fact that it implies the existence of an equivalent martingale measure, see [Delbaen and Schachermayer (1994, 1998)]. More precisely, if asset prices are locally bounded, the measure is an equivalent local martingale measure and if they are unbounded, the measure becomes an equivalent sigma martingale measure. Here, these measures will all be referred to collectively as equivalent martingale measures (EMM).

Theorem 1.5. *Assume that*

$$\sup_{S^{(\delta)} \in \underline{\Theta}(S)} \mathbb{E}[\log(S^{(\delta)}(T))] < \infty$$

and that NFLVR holds. Then there is a GOP.

Unfortunately, there is no clear one-to-one correspondence between the existence of a GOP and no arbitrage in the sense of NFLVR. In fact, the GOP may easily exist, even when NFLVR is not satisfied, and NFLVR does not guarantee that the expected growth rates are bounded. Moreover, the choice of numéraire influences whether or not NFLVR holds. A less stringent and numéraire invariant condition is the requirement that the market should have a *martingale density*. A martingale density is a strictly positive process Z, such that $\int S dZ$ is a local martingale. In other words, a Radon-Nikodym derivative of some EMM is a martingale density, but a martingale density is only the Radon-Nikodym derivative of an EMM if it is a true martingale. Modifying the definition of the GOP slightly, one may show that:

Corollary 1.1. *There is a GOP if and only if there is a martingale density.*

The reason why this addition to the previous existence result may be important is discussed in Section 1.4.

To find the growth optimal strategy in the current setting can be a non-trivial task. Before presenting the general result an important, yet simple, example is presented.

Example 1.4. Let the market consist of two assets, a stock and a bond. Specifically the SDEs describing these assets are given by

$$\begin{aligned} dS^{(0)}(t) &= S^{(0)}(t)rdt \\ dS^{(1)}(t) &= S^{(1)}(t)\left(adt + \sigma dW(t)\right) \end{aligned}$$

where W is a Wiener process and r, a, σ are constants. Using fractions, any admissible strategy can be written

$$dS^{(\delta)}(t) = S^{(\delta)}(t)\left((r + \pi(t)(a-r))dt + \pi(t)\sigma dW(t)\right).$$

Applying Itô's lemma to $Y(t) = \log(S^{(\delta)}(t))$ provides

$$dY(t) = \left((r + \pi(t)(a-r) - \frac{1}{2}\pi(t)^2\sigma^2)dt + \pi(t)\sigma dW(t)\right).$$

Hence, assuming the local martingale with differential $\pi(t)\sigma dW(t)$ to be a true martingale, it follows that

$$\mathbb{E}[\log(S^{(\delta)}(T)] = \mathbb{E}\left[\int_0^T (r + \pi(t)(a-r) - \frac{1}{2}\pi(t)^2\sigma^2)dt\right],$$

so by maximizing the expression for each (t, ω) the optimal fraction is obtained as

$$\pi_{\underline{\delta}}(t) = \frac{a-r}{\sigma^2}.$$

Hence, inserting the optimal fractions into the wealth process, the GOP is described by the SDE

$$\begin{aligned} dS^{(\underline{\delta})}(t) &= S^{(\underline{\delta})}(t)\left((r + (\frac{a-r}{\sigma})^2)dt + \frac{a-r}{\sigma}dW(t)\right) \\ &\triangleq S^{(\underline{\delta})}(t)\left((r+\theta^2)dt + \theta dW(t)\right) . \end{aligned}$$

The parameter $\theta = \frac{a-r}{\sigma}$ is the market price of risk process.

The example illustrates how the myopic properties of the GOP makes it relatively easy to derive the portfolio fractions. Although the method seems heuristic, it will work in very general cases and when asset prices are continuous, an explicit solution is always possible. This however, is not true in the general case. A very general result was provided in [Goll and Kallsen (2000, 2003)], who showed how to obtain the GOP in a setting with intermediate consumption and consumption takes place according to a (possibly random) consumption clock. Here the focus will be on the GOP strategy and its corresponding wealth process, whereas the implications for optimal consumption will not be discussed. In order to state the result, the reader is reminded of the semimartingale *characteristic triplet*, see [Jacod and Shiryaev (1987)]. Fix a truncation function, h, i.e. a bounded function with compact support, $h : \mathbb{R}^d \to \mathbb{R}^d$, such that $h(x) = x$ in a neighborhood around zero. For instance, a common choice would be $h(x) = x1_{(|x|\leq 1)}$. For such truncation function, there is a triplet (A, B, ν), describing the behavior of the semimartingale. One may choose a "good version" that is, there exists a locally integrable, increasing, predictable process, $\hat{A}$, such that (A, B, ν) can be written as

$$A = \int a d\hat{A}, \quad B = \int b d\hat{A}, \quad \text{and} \quad \nu(dt, dv) = d\hat{A}_t F(t, dv).$$

The process A is related to the finite variation part of the semimartingale, and it can be thought of as a generalized drift. The process B is similarly interpreted as the quadratic variation of the continuous part of S, or in other words it is the square volatility where volatility is measured in absolute terms. The process ν is the compensated jump measure, interpreted as the expected number of jumps with a given size over a small interval and F essentially characterizes the jump size. Note that A depends on the choice of truncation function.

Example 1.5. Let $S^{(1)}$ be as in Example 1.4, i.e. geometric Brownian Motion. Then $\hat{A} = t$ and

$$dA(t) = S^{(1)}(t)a dt \quad dB(t) = (S^{(1)}(t)\sigma)^2 dt.$$

Theorem 1.6 (Goll and Kallsen, 2000). *Let S have a characteristic triplet (A, B, ν) as described above. Suppose there is an admissible strategy $\underline{\delta}$ with corresponding fractions $\pi_{\underline{\delta}}$, such that*

$$a^j(t) - \sum_{i=1}^{d} \frac{\pi^i_{\underline{\delta}}(t)}{S^{(i)}(t)}(t) b^{i,j}(t) + \int_{\mathbb{R}^d} \left(\frac{x^j}{1 + \sum_{i=1}^{d} \frac{\pi^i_{\underline{\delta}}(t)}{S^{(i)}(t)} x^i} - h(x) \right) F(t, dx) = 0 \tag{1.8}$$

for $\mathbb{P} \otimes d\hat{A}$ almost all $(\omega, t) \in \Omega \times [0, T]$, where $j \in \{0, \ldots, d\}$ and $\otimes$ denotes the standard product measure. Then $\underline{\delta}$ is the GOP strategy.

Essentially, equation (1.8) represent the first order conditions for optimality and they would be obtained easily if one tried to solve the problem in a pathwise sense, as done in Example 1.4. From Example 1.3 in the previous section it is clear, that such a solution need not exist, because there may be a "corner solution".

The following examples show how to apply Theorem 1.6.

Example 1.6. Assume that discounted asset prices are driven by an m-dimensional Wiener process. The locally risk free asset is used as numéraire, whereas the remaining risky assets evolve according to

$$dS^{(i)}(t) = S^{(i)}(t)a^i(t)dt + \sum_{j=1}^{m} S^{(i)}(t)b^{i,j}(t)dW^j(t)$$

for $i \in \{1, \ldots, d\}$. Here $a^i(t)$ is the excess return above the risk free rate. From this equation, the decomposition of the semimartingale S follows directly. Choosing $\hat{A} = t$, a good version of the characteristic triplet becomes

$$(A, B, \nu) = \left(\int a(t)S(t)dt, \int S(t)b(t)(S(t)b(t))^\top dt, 0 \right).$$

Consequently, in vector form and after division by $S^{(i)}(t)$ equation (1.8) yields that

$$a(t) - (b(t)b(t)^\top)\pi_{\underline{\delta}}(t) = 0.$$

In the particular case where $m = d$ and the matrix b is invertible, I get the well-known result that

$$\pi(t) = b^{-1}(t)\theta(t),$$

where $\theta(t) = b^{-1}(t)a(t)$ is the market price of risk.

Generally, whenever the asset prices can be represented by a continuous semimartingale, a closed form solution to the GOP strategy may be found. The cases where jumps are included are less trivial as shown in the following example.

Example 1.7 (Poissonian Jumps). *Assume that discounted asset prices are driven by an m-dimensional Wiener process, W, and an $n-m$ dimensional Poisson jump process, N, with intensity $\lambda \in \mathbb{R}^{n-m}$. Define the compensated Poisson process $q(t) \triangleq N(t) - \int_0^t \lambda(s)ds$. Then asset prices evolve as*

$$dS^{(i)}(t) = S^{(i)}(t)a^i(t)dt + \sum_{j=1}^{m} S^{(i)}(t)b^{i,j}(t)dW^j(t) + \sum_{j=m+1}^{n} S^{(i)}(t)b^{i,j}(t)dq^j(t)$$

for $i \in \{1, \ldots, d\}$. *If it is assumed that* $n = d$, *then an explicit solution to the first order conditions may be found. Assume that* $b(t) = \{b^{i,j}(t)\}_{i,j \in \{1,\ldots,d\}}$ *is invertible. This follows if it is assumed that no arbitrage exists. Define*

$$\theta(t) \triangleq b^{-1}(t)(a^1(t), \ldots, a^d(t))^\top.$$

If $\theta^j(t) \geq \lambda^j(t)$ *for* $j \in \{m+1, \ldots, d\}$, *then there is an arbitrage, so it can be assumed that* $\theta^j(t) < \lambda^j(t)$. *In this case, the GOP fractions satisfy the equation*

$$\begin{aligned}&(\pi^1(t), \ldots, \pi^d(t))^\top \\ &= (b^\top)^{-1}(t)\left(\theta^1(t), \ldots, \theta^m(t), \frac{\theta^{m+1}(t)}{\lambda^{m+1}(t) - \theta^{m+1}(t)}, \ldots, \frac{\theta^d(t)}{\lambda^d(t) - \theta^d(t)}\right)^\top.\end{aligned}$$

It can be seen that the optimal fractions are no longer linear in the market price of risk. This is because when jumps are present, investments cannot be scaled arbitrarily, since a sudden jump may imply that the portfolio becomes non-negative. Note that the market price of jump risk needs to be less than the intensity for the expression to be well-defined. If the market is complete, then this restriction follows by the assumption of no arbitrage.

In general when jumps are present, there is no explicit solution in an incomplete market. In such cases, it is necessary to use numerical methods to solve equation (1.8). As in the discrete case, the assumption of complete markets will enable the derivation of a fully explicit solution of the problem. In the case of more general jump distributions, where the jump measure does not have a countable support set, the market cannot be completed by any finite number of assets. The jump uncertainty which appears in this case can then be interpreted as driven by a Poisson process of an infinite dimension. In this case, one may still find an explicit solution if the definition of a solution is generalized slightly as in [Christensen and Larsen (2007)].

As in discrete time the GOP can be characterized in terms of its growth properties.

Theorem 1.7. *The GOP has the following properties:*

(1) The GOP maximizes the instantaneous growth rate of investments.
(2) In the long term, the GOP will have a higher realized growth rate than any other strategy, i.e.

$$\limsup_{T\to\infty} \frac{1}{T} \log(S^{(\delta)}(T)) \leq \limsup_{T\to\infty} \frac{1}{T} \log(S^{(\underline{\delta})}(T))$$

for any other admissible strategy $S^{(\delta)}$.

The instantaneous growth rate is the drift of $\log(S^{(\delta)}(t))$.

Example 1.8. In the context of Example 1.4 the instantaneous growth rate, $g^{\delta}(t)$, of a portfolio $S^{(\delta)}$ was found by applying the Itô formula to get

$$dY(t) = \left((r + \pi(t)(a - r) - \frac{1}{2}\pi(t)^2\sigma^2)dt + \pi(t)\sigma dW(t) \right).$$

Hence, the instantaneous growth rate is

$$g^{\delta}(t) = r + \pi(t)(a - r) - \frac{1}{2}\pi(t)^2\sigma^2.$$

In example 1.4 I derived the GOP, exactly by maximizing this expression and so the GOP maximized the instantaneous growth rate by construction.

As mentioned, the procedure of maximizing the instantaneous growth rate may be applied in a straightforward fashion in more general settings. In the case of a Wiener driven diffusion with deterministic parameters, the second claim can be obtained directly by using the law of large numbers for Brownian motion. The second claim does not rest on the assumption of continuous asset prices although this was the setting in which it was proved. The important thing is that other portfolios measured in units of the GOP become supermartingales. Since this is shown below for the general case of semimartingales, the proof in [Karatzas (1989)] will also apply here as shown in [Platen (2004a)].

As in the discrete setting, the GOP enjoys the numéraire property. However, there are some subtle differences.

Theorem 1.8. *Let* $S^{(\delta)}$ *denote any admissible portfolio process and define* $\hat{S}^{(\delta)}(t) \triangleq \frac{S^{(\delta)}(t)}{S^{(\underline{\delta})}(t)}$. *Then*

(1) $\hat{S}^{(\delta)}(t)$ *is a supermartingale if and only if* $S^{(\underline{\delta})}(t)$ *is the GOP.*
(2) The process $\frac{1}{\hat{S}^{(\delta)}(t)}$ *is a submartingale.*
(3) If asset prices are continuous, then $\hat{S}^{(\delta)}(t)$ *is a local martingale.*

In the discrete case, it was shown that prices denominated in units of the GOP could become strict supermartingales. In the case of (unpredictable) jumps, this may also happen, practically for the same reasons as before. If there is a large expected return on some asset and a very slim chance of reaching values close to zero, the log-investor is implicitly restricted from taking large positions in this asset, because by doing so he would risk ruin at some point. This is related to the structure of the GOP in a complete market as explained in Example 1.7.

There is a small but important difference to the discrete time setting. It may be that GOP denominated prices become *strict local* martingales, which is a local martingale that is not a martingale. This is a special case of being a strict supermartingale since, due to the Fatou lemma, a non-negative local martingale may become a supermartingale. This case does not arise because of any implicit restraints on the choice of portfolios and the threat of being illiquid. Instead, it has to do with the fact that not all portfolios "gets the biggest bang for the buck" as will be explained in Section 1.4.

Example 1.9 (Example 1.4 continued). *Assume a market as in Example 1.4 and define the processes $\hat{S}^{(0)}$ and $\hat{S}^{(1)}$ as in Theorem 1.8 above. An application of the Itô formula implies that*

$$d\hat{S}^{(0)}(t) = -\hat{S}^{(0)}(t)\theta dW(t)$$

and

$$d\hat{S}^{(1)}(t) = \hat{S}^{(1)}(t)(b(t) - \theta)dW(t).$$

The processes above are local martingales since they are Itô integrals with respect to a Wiener process. A sufficient condition for a local martingale to be a true martingale is given by the so-called Novikov condition, see [Novikov (1973)] requiring

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta(t)^2 dt\right)\right] < \infty,$$

which is satisfied in this case since θ is a constant. However, in more general models θ can be a stochastic process. Several examples exist where the Novikov condition is not satisfied and hence the processes $\hat{S}^{(0)}$ and $\hat{S}^{(1)}$ become true supermartingales. A simple example is the situation where $S^{(1)}$ is a Bessel process of dimension three. The inverse of this process is the standard example of a local martingale which is not a martingale.

The fact that local martingales need not be martingales is important in the theory of arbitrage free pricing and will be discussed in Section 1.4. In these cases, the numéraire may be outperformed by trading the GOP.

The growth properties indicating that in the long run the GOP will outperform all other portfolios have made it very interesting in the literature on asset allocation and it has been argued that the GOP is a universally "best" investment strategy in the long run. This application and the debate it has raised in the academic community is reviewed in the next section. A second and more recent application is the numéraire property, particularly interesting in the literature on arbitrage pricing, is reviewed subsequently.

Notes

The literature on the properties of the GOP is huge and only a few have been discussed here. The properties of this section have been selected because they have attracted the most interest in the literature. Using the logarithm as utility function often provides very tractable results, so the GOP arises implicitly in a large number of papers which, for simplicity, use this function as part of the theory. To manage the literature on the subject, I have focused on papers which deal explicitly with the GOP. Theorem 1.5 appears in [Becherer (2001)][Theorem 4.5] and is a straightforward application of [Kramkov and Schachermayer (1999)][Theorem 2.2]. In some papers the GOP is defined in a pathwise sense, see [Platen (2002, 2004b)] and [Christensen and Platen (2005)], which circumvents the problem of infinite expected growth rates. An alternative solution is to define the GOP in terms of relative growth rates, see [Algoet and Cover (1988)]. An alternative existence proof, which is more direct, but does not relate explicitly to the notion of arbitrage, can be found in [Aase (1988)] and [Aase and Oksendal (1988)]. [Long (1990)][Appendix B, page 58] claims that the existence of the GOP follows from no arbitrage alone, but this is in general incorrect. The proof that the existence of a GOP is equivalent to the existence of a martingale density is found in [Christensen and Larsen (2007)].

Theorem 1.6 was proved by [Goll and Kallsen (2000)] and expanded to stochastic consumption clocks in [Goll and Kallsen (2003)]. The solution in a complete Wiener driven set-up with constant parameters dates back to [Merton (1969)], extended in [Merton (1971, 1973)]. [Aase (1988)] introduced the problem in a jump-diffusion setting and derived a similar formula in the context of a model with Wiener and Poisson noise using the Bellman principle. This has been extended in [Aase (1984, 1986, 1988)], [Browne (1999)], [Korn *et al.* (2003)]. [Yan *et al.* (2000)] and [Hurd (2004)] study exponential Levy processes and [Platen (2004b)] obtains a fully explicit solution in the case of a complete Poisson/Wiener market, similar to Example 1.7. It was noted by [Aase (1984)] that equation (1.8) would follow from a pathwise optimization problem. [Christensen and Platen (2005)] follows this procedure in a general marked point process setting and express the solution in terms of the market price of risk. I show that a generalized version of the GOP can be characterized explicitly and approximated by a sequence of portfolios in approximately complete markets. In an abstract framework, relying on the duality result of [Kramkov and Schachermayer (1999)] and the decomposition of [Schweizer (1995)], a general solution was obtained in [Christensen and Larsen (2007)].

The problem of determining the GOP can be extended to the case of portfolio constraints, see, for instance, [Cvitanić and Karatzas (1992)] and in particular [Goll and Kallsen (2003)]. The case of transaction costs is considered in [Serva (1999)], [Cover and Iyengar (2000)] and [Aurell and Muratore-Ginanneschi (2004)]. Cases where the growth optimizer has access to a larger filtration are treated by, for instance, [Ammendinger *et al.* (1998)], who show how expanding the set of available information increases the maximal growth rate. In the setting of continuous asset prices, [Larsen and Zitković (2008)] show that the existence of a GOP when the filtration is enlarged, guarantees the price process will remain a semimartingale, which is convenient since arbitrage may arise in models, where this property is not guaranteed. A model free approach to the maximization of portfolio growth rate is derived in [Cover (1991)], and the literature on "universal portfolios".

Theorem 1.7(1) has often been used as the definition of the GOP. (2) was proved in [Karatzas (1989)] in the setting of continuous diffusions. For further results on the theoretical long-term behavior of the GOP in continuous time, the reader is referred to [Pestien and Sudderth (1985)], [Heath *et al.* (1987)] and [Browne (1999)]. Analysis of the long term behavior and ruin probability is conducted in [Aase (1986)].

The numéraire property in continuous time was shown initially by [Long (1990)]. The issue of whether GOP denominated prices become supermartingales is discussed in [Becherer (2001)], [Korn *et al.* (2003)], [Hurd (2004)] and [Christensen and Larsen (2007)]. The fact that the GOP is a submartingale in any other denomination is shown by for instance [Aase (1988)]. For examples of models, where the inverse GOP is not a true local martingale, see [Delbaen and Schachermayer (1995a)] for a very simple example and [Heath and Platen (2002a)] for a more elaborate one. The standard (mathematical) textbook reference for such processes is [Revuz and Yor (1991)].

1.3. The GOP as an Investment Strategy

When the GOP was introduced to the finance community, it was not as the result of applying a logarithmic utility function, but in the context of maximizing growth. Investment theory based on growth is an alternative to utility theory and is directly applicable because the specification of the goal is quite simple. The popularity of the mean-variance approach is probably not to be found in its theoretical foundation, but rather the fact that it suggested a simple trade-off between return and uncertainty. Mean-variance based portfolio choice left one free parameter, the amount of variance acceptable to the individual investor. The theory of growth optimal investment suggests the GOP as an investment tool for long horizon investors because of the properties stated in the previous section, in particular because it will almost surely dominate other investment strategies in terms of wealth as the time horizon increases. Hence, in the literature of portfolio management, the GOP has often been, and is still, advocated as a useful investment strategy, because utility maximization is a somewhat abstract investment goal. For example, [Roy (1952)][Page 433] states that

> "In calling in a utility function to our aid, an appearance of generality is achieved at the cost of a loss of practical significance, and applicability in our results. A man who seeks advice about his actions will not be grateful for the suggestion that he maximize expected utility."

In these words lies the potential strength of the growth optimal approach to investment. However, utility theory being a very influential if not *the* dominating paradigm, is a challenge to alternative, normative, theories of portfolio selection. If investors are correctly modelled as individuals who maximize some (non-logarithmic) utility function, then the growth rate per se is of no importance and it makes no sense to recommend the GOP to such individuals.

In this section three issues will be discussed. Firstly, the lively debate on how widely the GOP can be applied as an investment strategy is reviewed in detail. This debate contains several points which may be useful to keep in mind, since new papers in this line of literature often express the point of view that the GOP deserves a special place in the universe of investment strategies. In Section 1.3.1, the discussion of whether the GOP can replace or proxy other investment strategies when the time horizon of the investor is long is presented. The section is aimed to be a chronological review of the pros and cons of the GOP as seen by different authors.

Secondly, because the strategy of maximizing growth appeared as a challenge to the well-established mean-variance dogma, and because a large part of the literature has compared the two, Section 1.3.2 will deal with the relation between growth optimal investments and mean-variance efficient investments. Finally, because the main argument for the GOP has been its growth properties, some theoretical insight into the ability of the GOP to dominate other strategies over time will be provided in Section 1.3.3.

Before commencing, let me mention that the GOP has found wide applications in gambling and to some extent horse racing. In these disciplines a main issue is how to "gain an edge", i.e. to create a favorable game with non-negative expected growth rate of wealth. Obviously, if the game cannot be made favorable, i.e. there is no strategy, such that the expected pay-off is larger than the bet, the growth optimal strategy is of course simply to walk away. If, on the other hand, it is possible to turn the game into a favorable game, then applying the growth optimal strategy is possible. This can be done in e.g. Black Jack since simple card counting strategies can be applied to shift the odds slightly. Similarly, this may be done in horse-racing by playing different bookmakers, see [Hausch and Ziemba (1990)]. There are literally hundreds of papers on this topic. Growth maximizing strategies are in this stream of literature predominantly denoted "Kelly strategies". It appears that Kelly strategies or fractional Kelly strategies are quite common in the theory of gambling and despite the striking similarity with investment decisions, the gambling literature appears to pay limited attention to the expected utility paradigm in general. Perhaps because gamblers by nature are much less risk averse than "common investors". A general finding which may be interesting in the context of asset allocation is that model uncertainty generally leads to over-betting. Hence, if one wishes to maximize the growth rate of investment one might wish to apply a fractional Kelly strategy, because the model indicated strategy could be "too risky".

Notes

Some further references for applying the GOP in gambling can be found in [Blazenko *et al.* (1992)], and in particular the survey [Hakansson and Ziemba (1995)] and the paper [Thorp (1998)]. See also the papers [Ziemba (2003, 2004)] for some easy-to-read accounts. A standard reference in gambling is the book [Thorp (1966)], while the book [Poundstone (2005)] and the edited volume [Maclean *et al.* (2010)] contain popular treatment of the application of Kelly-strategies in gambling and investment.

1.3.1. *Is the GOP Better? - The Samuelson Controversy*

The discussion in this section is concerned with whether the different attributes of the growth optimal investment constitute a reasonable criteria for selecting portfolios. More specifically, I discuss whether the GOP can be said to be "better" in any strict mathematical sense and whether the GOP is an (approximately) optimal decision rule for investors with a long time horizon. Due to the chronological form of this section and the extensive use of quotes, most references are given in the text, but further references may be found in the notes.

It is a fact that the GOP attracted interest primarily due to the properties stated in Theorem 1.3. A strategy, which in the long run will beat any other strategy in terms of wealth sounds intuitively attractive, in particular to the investor who is not concerned with short term fluctuations, but has a long horizon. Such an investor can lean back and watch his portfolio grow and eventually dominate all others. From this point of view it may sound as if any investor would prefer the GOP, if only his investment horizon is sufficiently long.

Unfortunately, things are not this easy as was initially pointed out by [Samuelson (1963)]. Samuelson argues in his 1963 paper, that if one is not willing to accept one bet, then one will never rationally accept a sequence of that bet, no matter the probability of winning. In other words, if one does not follow the growth optimal strategy over one period, then it will not be rational to follow the rule when there are many periods. His article is not addressed directly to anyone in particular, rather it is written to "dispel a fallacy of wide currency", see [Samuelson (1963)][p. 50]. However, whether it was intended or not, Samuelson's paper serves as a counterargument to the proposed strategy in [Latané (1959)]. Latané had suggested as the criteria for portfolio choice, see [Latané (1959)][p. 146], that one chooses

> "...the portfolio that has a greater probability (P') of being as valuable or more valuable than any other significantly different portfolio at the end of n years, n being large."

Latané had argued that this was logical long-term goal, but that it "would not apply to one-in-a-lifetime choices" [p. 145]. This view is repeated in [Latané and Tuttle (1967)]. It would be reasonable to assume that this is the target of Samuelsons critique. Indeed, Samuelson argues that to use this goal is counter logical, first of all because it does not provide a transitive ordering and secondly as indicated above it is not rational to change objective just because the investment decision is repeated in a number of periods. This criticism is valid to a certain extent, but it is based on the explicit assumption that "acting rationally" means maximizing an expected utility of a certain class. Samuelson's statement is meant as a normative statement. Experimental evidence shows that investors may act inconsistently, see for instance [Benartzi and Thaler (1999)]. Note that one may construct utility functions, such that two games are accepted, but one is not. An example is in fact given by Samuelson himself (sic) in the later paper [Samuelson (1984)]. Further references to this discussion are cited in the notes. However, Latané never claimed his decision rule to be consistent with utility theory. In fact, he seems to be aware of this, as he states

> "For certain utility functions and for certain repeated gambles, no amount of repetition justifies the rule that the gamble which is almost sure to bring the greatest wealth is the preferable one."

See [Latané (1959)][p. 145, footnote 3]. [Thorp (1971)] clarifies the argument made by Samuelson that making choices based on the probability that some portfolio will do better or worse than others is non-transitive. However, in the limit, the property characterizing the GOP is that it dominates all other portfolios almost surely. This property, being equal almost surely, clearly is transitive. Moreover, Thorp argues that even in the case where transitivity does not hold, a related form of "approximate transitivity" does, see [Thorp (1971)][p. 217]. Consequently he does not argue against Samuelson (at least not directly), but merely points out that the objections made by Samuelson do not pose a problem for his theory. One may wish to emphasize that to compare the outcomes as the number of repetitions turn to infinity, requires the limit $S^{(\delta)}(t)$ to be well-defined, something which is usually not the case whenever the expected growth rate is non-negative. However, from Theorem 1.7, the limit

$$\lim_{t\to\infty} \hat{S}^{(\delta)}(t)$$

is well-defined and less than one almost surely. Hence the question of transitivity depends on whether "n-large" means *in the limit*, in which case it holds or it means for certain *finite* but large n, in which case it does not hold. Second, as pointed out above "acting rationally" is in the language of Samuelson to have preferences that are consistent with a single Von-Neumann, Morgenstern utility function. Whether investors who act consistently according to the same utility function ever existed is a questionable and this is not assumed by the proponents of the GOP, who intended the GOP as a normative investment rule.

A second question is whether due to the growth properties there may be some way to say that "in the long run, everyone should use the GOP".

In this discussion Samuelson points directly to [Williams (1936)], [Kelly (1956)] and [Latané (1959)]. The main point is that just because the GOP in the long run will end up dominating the value of any other portfolio, it will not be true, over any horizon however long, that the GOP is better for all investors. In Samuelson's own words, see [Samuelson (1971)][p. 2494]:

> "...it is tempting to believe in the truth of the following false corollary: *False Corollary.* If maximizing the geometric mean almost certainly leads to a better outcome, then the expected utility of its outcome exceeds that of any other rule, provided that T is sufficiently large."

Such an interpretation of the arguments given by for instance Latané may be possible, see [Latané (1959)][footnote on page 151]. Later it becomes absolutely clear that Samuelson did indeed interpret Latané in this way, but otherwise it is difficult to find any statement in the literature which explicitly expresses the point of view which is inherent in the false corollary of [Samuelson (1971)]. Possibly the view point expressed in [Markowitz (1959)] could be interpreted along these lines. Markowitz finds it irrational that long-term investors would not choose the GOP - he does not argue that investors with other utility functions would not do it, but rather he argues that one should not have other utility functions in the very long run. This is criticized by [Thorp (1971)], who points out that the position taken by [Markowitz (1959)] cannot be supported mathematically. Nevertheless, this point of view is somewhat different to that expressed by the false corollary. Whether believers in the false corollary ever existed is questioned by [Thorp (1971)][p. 602]. The point is that one cannot exchange the limits in the following way: if

$$\lim_{t\to\infty} \frac{S^{(\delta)}(t)}{S^{(\underline{\delta})}(t)} \leq 1,$$

then it does not hold that

$$\lim_{t\to\infty} \mathbb{E}[U(S^{(\delta)}(t))] \leq \lim_{t\to\infty} \mathbb{E}[U(S^{(\underline{\delta})}(t))],$$

given some utility function U. This would require, for instance, the existence of the pointwise limit $S^{(\underline{\delta})}(\infty)$ and uniform integrability of the random variables $U(S^{(\underline{\delta})}(t))$. Even if the limit and the expectation operator can be exchanged, one might have $\mathbb{E}[U(S^{(\delta)}(t))] > \mathbb{E}[U(S^{(\underline{\delta})}(t))]$ for all finite t

and equality in the limit. The intuitive reason is that even if the GOP dominates another portfolio with a very high probability, i.e.

$$\mathbb{P}(S^{(\delta)}(t) < S^{(\underline{\delta})}(t)) = 1 - \epsilon,$$

then the probability of the outcomes where the GOP performs poorly may still be unacceptable to an investor who is more risk averse than a log-utility investor. In other words, the left tail distribution of the GOP may be too "thick" for an investor who is more risk averse than the log-utility investor. It seems that a large part of the dispute is caused by claims which argue that the aversion towards such losses is "irrational" because the probability becomes arbitrarily small, whereas the probability of doing better than everyone else becomes large. Whether or not such an attitude is "irrational" is certainly a debatable subject and is probably more a matter of opinion than a matter of mathematics.

When it became clear that the GOP would not dominate other strategies in any crystal clear sense, several approximation results where suggested. The philosophy was that as the time horizon increased, the GOP would *approximate* the maximum expected utility of other utility functions. However, even this project failed. [Merton and Samuelson (1974a)] pointed out a flaw in an argument in [Hakansson (1971b)] and [Samuelson (1971)], that a log-normal approximation to the distribution of returns over a long period can be made. [Hakansson (1974)] admits to this error, but points out that this has no consequences for the general statements of his paper. Moreover, [Merton and Samuelson (1974a)] remarks that a conjecture made by [Samuelson (1971)] and [Markowitz (1972)], that over a long horizon the GOP will equal or be a good approximation to the optimal policy when investors have bounded utility, is incorrect. Presumably, this unpublished working paper, referred to by [Merton and Samuelson (1974a)] is an older version of [Markowitz (1976)]. Firstly, they remark that [Markowitz (1972)] did not define precisely what a "good approximation" was. Secondly, [Goldman (1974)] gives a counter example showing that following the GOP strategy can lead to a large loss in terms of certainty equivalents, even when investors have a bounded utility function. If U is a bounded utility function, then certainly $U(S^{(\underline{\delta})}(t))$ is a family of uniformly integrable variables and consequently, any converging sequence also converges in mean. This means that it is true that

$$\lim_{T\to\infty} \mathbb{E}[U(S^{(\delta)}(T)] = \lim_{T\to\infty} \mathbb{E}[U(S^{(\underline{\delta})}(T)],$$

but the argument in, for instance, [Goldman (1974)] is that $\mathbb{E}[U(S^{(\underline{\delta})}(t))]$

converges much slower as $t \to \infty$ than for the optimal policy. In other words, if δ_U is the optimal policy for an investor with utility function U, then [Goldman (1974)] provides an example such that

$$\lim_{t\to\infty} \frac{\mathbb{E}[U(S^{(\underline{\delta})}(t))]}{\mathbb{E}[U(S^{(\delta_u)}(t))]} = 0.$$

So even though the absolute difference in utility levels when applying the GOP instead of the optimal strategy is shrinking, the GOP does infinitely worse than the optimal strategy in terms of relative utility. Similarly, one may investigate the certainty equivalent for an investor who is forced to invest in the GOP. The certainty equivalent measures the amount of extra wealth needed to obtain the same level of utility when using a suboptimal strategy. The certainty equivalent when using the GOP in place of the optimal strategy is usually not decreasing as time goes by. [Markowitz (1976)] argues that the criterion for asymptotic optimality adopted by [Merton and Samuelson (1974a)] and [Goldman (1974)] is unacceptable, because it violates the notion that only the normalized form of the game is necessary for comparing strategies. The "bribe", which is described as a concept similar to certainty equivalent, cannot be inferred by the normalized form of the game. Markowitz moves on to define utility on a sequence of games and concludes that if the investor is facing two sequences X and X' and prefers X to X' if $X_n \geq X'_n$ from a certain n with probability one, then such an investor should choose the GOP. A very similar support of the max-expected-growth-rate point of view is given by [Miller (1975)], who shows that if the utility function depends only on the tail of the wealth sequence of investments, then the GOP is optimal. In technical terms, if $(X_n)_{n\in\mathbb{N}}$ is a sequence such that X_n represents wealth after n periods, then $U : \mathbb{R}^\infty \to \mathbb{R}$ is such that, $U(x_1, \ldots, x_n, \ldots) \geq U(x'_1, \ldots, x'_n, \ldots)$ whenever $x_{n+j} \geq x'_{n+j}$ for some $n \in \mathbb{N}$ and all $j \geq n$. This abstract notion implies that the investor will only care about wealth effects, that are "far out in the future". It is unclear whether such a criterion can be given an axiomatic foundation, although it does have some resemblance to the Ramsey-Weizsäcker overtaking criterion used in growth theory, see [Brock (1970)] for the construction of an axiomatic basis.

It seems that the debate on this subject was somewhat obstructed because there was some disagreement about the correct way to measure whether something is "a good approximation". The concept of "the long run" is by nature not an absolute quantity and depends on the context. Hence, the issue of how long the long run is will be discussed later on.

In the late seventies the discussion became an almost polemic repetition of the earlier debate. [Ophir (1978)] repeats the arguments of Samuleson and provides examples where the GOP strategy as well as the objective suggested by Latané will provide unreasonable outcomes. In particular, he notes the lack of transitivity when choosing the investment with the highest probability of providing the best outcome. [Latané (1978)] counter-argues that nothing said so far invalidates the usefulness of the GOP and that he never committed to the fallacies mentioned in Samuelsons paper. As for his choice of objective Latané refers to the discussion in [Thorp (1971)] regarding the lack of transitivity. Moreover, Latané points out that a goal which he advocates for use when making a long sequence of investment decisions, is being challenged by an example involving only one *unique* decision. In [Latané (1959)], Latané puts particular emphasis on the point that goals can be different in the short and long run. As mentioned, this was exactly the reasoning which Samuelson attacks in [Samuelson (1963)]. [Ophir (1979)] refuses to acknowledge that goals should depend on circumstances and once again establishes that Latanés objective is inconsistent with the expected utility paradigm. [Samuelson (1979)] gets the last word in his, rather amusing, article which is held in words of only one syllabus (apart from the word syllabus itself!). In two pages he disputes that the GOP has any special merits, backed by his older papers. The polemic nature of these papers emphasizes that parts of the discussion for and against maximizing growth rates depend on a point of view and is not necessarily supported by mathematical necessities.

To sum up this discussion, there seems to be complete agreement that the GOP can neither proxy for nor dominate other strategies in terms of expected utility, and no matter how long (finite) horizon the investor has, utility based preferences can make other portfolios more attractive because they have a more appropriate risk profile. However, it should be understood that the GOP was recommended as an alternative to expected utility and as a *normative* rather than *descriptive* theory. In other words, authors that argued pro the GOP did so because they believed growth optimality to be a reasonable investment goal, with attractive properties that would be relevant to long horizon investors. They recommended the GOP because it seems to manifest the desire of getting as much wealth as fast as possible. On the other hand, authors who disagreed did so because they did not believe that every investor could be described as log-utility maximizing investors. Their point is that if an investor can be *described* as utility maximizing, it is pointless to *recommend* a portfolio which provides less utility than would be the case, should he choose optimally. Hence, the disagreement has its roots in two very fundamental issues, namely whether or not utility theory is a reasonable way of approaching investment decisions in practice and whether utility functions, different from the logarithm, is a realistic description of individual long-term investors. The concept of utility based portfolio selection, although widely used, may be criticized by the observation that investors may be unaware of their own utility functions. Even the three axioms required in the construction of utility functions, see [Kreps (1988)] have been criticized, because there is some evidence that choices are not made in the coherent fashion suggested by these axioms. Moreover, to say that one strategy provides higher utility than another strategy may be "business as usual" to the economist. Nevertheless it is a very abstract statement, whose content is based on deep assumptions about the workings of the minds of investors. Consequently, although utility theory is a convenient and consistent theoretical approach it is not a fundamental law of nature. Neither is it strongly supported by empirical data and experimental evidence. (See for instance the monograph [Bossaerts (2002)] for some of the problems that asset pricing theory that builds on CAPM and other equilibrium models are facing and how some may be explained by experimental evidence on selection.) After the choice of portfolio has been made it is important to note that only one path is ever realized. It is practically impossible to verify *ex post* whether some given portfolio was "the right choice". In contrast, the philosophy of maximizing growth and the long-run growth property are formulated in dollars, not in terms of utility, and so when one evaluates the portfolio performance ex post, there is a greater likelihood that the GOP will come out as a "good idea", because the GOP has a high probability of being more valuable than any other portfolio. It seems plausible that individuals, who observe their final wealth will not care that their wealth process is the result of an ex-ante correct portfolio choice, when it turns out that the performance is only mediocre compared to other portfolios.

Every once in a while articles continue the debate about the GOP as a very special strategy. These can be separated into two categories. The first, can be represented by [McEnally (1986)] who agrees that the criticism raised by Samuelson is valid. However, he argues that for practical purposes, in particular when investing for pension, the probability that one will realize a gain is important to investors. Consequently, Latané's subgoal is not without merit in McEnally's point of view. Hence this category consists of those who simply believe the GOP to be a tool of practical importance and this view reflects the conclusions I have drawn above.

The second category does not acknowledge the criticism to the same extent and is characterized by statements such as

> "... Kelly has shown that repetition of the investment many times gives an objective meaning to the statement that the Growth-optimal strategy is the best, regardless to the subjective attitude to risk or other psychological considerations."

see [Aurell *et al.* (2000b)][Page 4]. The contributions of this specific paper lie within the theory of derivative pricing and will be considered in Section 1.4. Here I simply note that they argue in contrary to the conclusions of my previous analysis. In particular, they seem to insist on an interpretation of Kelly, which has been disproved. Their interpretation is even more clear in [Aurell *et al.* (2000a)][Page 5], stating:

> "Suppose some agents want to maximize non-logarithmic utility... and we compare them using the growth optimal strategy, they would almost surely end up with less utility according to their own criterion.",

which appears to be a misconception and in general the statement will not hold literally as explained previously. Hence some authors still argue that every *rational* long term investor should choose the GOP. They seem to believe that either other preferences will yield the same result, which is incorrect, or that other preferences are irrational, which is a viewpoint that is difficult to defend on purely theoretical grounds. A related idea which is sometimes expressed is that it does not make sense to be more risk-seeking than the logarithmic investor. This viewpoint was expressed and criticized very early in the literature. Nevertheless, it seems to have stuck and is found in many papers discussing the GOP as an investment strategy. Whether it is true depends on the context. Although unsupported by utility theory, the viewpoint finds support within the context of growth-based investment. Investors who invest more in risky securities than the fraction warranted by the GOP will, by definition, obtain a lower growth rate over time and at the same time they will face more risk. Since the added risk does not imply a higher growth rate of wealth it constitutes a choice which is "irrational", but only in the same way as choosing a non-efficient portfolio within the mean-variance framework. It is similar to the discussion of whether, in the long run, stocks are better than bonds. In many models, stocks will outperform bonds almost surely as time goes to infinity. Whether long-horizon investors should invest more in stocks depends: from utility based portfolio selection the answer may be no. If the pathwise properties of the wealth distribution is emphasized, then the answer may be yes. As was the case in this section, arguments supporting the last view will often be incompatible with the utility based theories for portfolio selection. Similar is the argument that "relative risk goes to zero as time goes to infinity" because portfolio values will often converge to infinity as the time horizon increases. Hence, risk measures such as VaR will converge to zero as time turns to infinity, which is somewhat counterintuitive, see [Treussard (2005)].

In conclusion, many other unclarities in the finance relate to the fact that a pathwise property may not always be reflected when using expected utility to derive the true portfolio choice. It is a trivial exercise to construct a sequence of random variables that converge to zero, and yet the mean value converges to infinity. In other words, a portfolio may converge to zero almost surely and still be preferred to a risk-free asset by a utility maximizing agent. Intuition dictates that one should never apply such a portfolio over the long term, whereas the utility maximization paradigm says differently. Similarly, if one portfolio beats others almost surely over a long horizon, then intuition suggests that this may be a good investment. Still utility maximization refuses this intuition. It is those highly counterintuitive results which have caused the debate among economists and which continue to cast doubt on the issue of choosing a long term investment strategy.

As a way of investigating the importance of the growth property of the GOP, Section 1.3.3 sheds light on how long it will take before the GOP gets ahead of other portfolios. I will document that choosing the GOP because it outperforms other portfolios may not be a strong argument because it may take hundreds of years before the probability of outperformance becomes high.

Notes

The criticism by Samuelson and others can be found in the papers, [Samuelson (1963, 1969, 1971, 1979, 1991)], [Merton and Samuelson (1974a,b)] and [Ophir (1978, 1979)]. The sequence of papers provides a very interesting criticism. Although they do point out certain factual flaws, some of the viewpoints may be characterized as (qualified) opinions rather than truth in any objective sense.

Some particularly interesting references which explicitly take a different stand in this debate is [Latané (1959, 1978)], [Hakansson (1971a,b)] and [Thorp (1971, 1998)], which are all classics. Some recent support is found in [McEnally (1986)], [Aurell *et al.* (2000b)], [Michaud (2003)] and [Platen

(2005b)]. The view that investment of more than 100% in the GOP is irrational is common in the gambling literature - referred to as "overbetting" and is found for instance in [Ziemba (2003, 2004, 2005)]. In a finance context the argument is voiced in [Platen (2005c)]. Game theoretic arguments in favor of using the GOP is found in [Bell and Cover (1980, 1988)]. [Rubinstein (1976)] argues that using generalized logarithmic utility has practical advantages to other utility functions, but does not claim superiority of investment strategies based on such assumptions. The "fallacy of large numbers" problem is considered in numerous papers, for instance [Samuelson (1984)], [Ross (1999)], [Brouwer and den Spiegel (2001)] and [Vivian (2003)]. It is shown in [Ross (1999)] that if utility functions have a bounded first order derivative near zero, then they may indeed accept a long sequence of bets, while rejecting a single one.

A recent working paper, [Rotar (2004)], considers investors with distorted beliefs, that is, investors who maximize expected utility not with respect to the real world measure, but with respect to some transformation. Conditions such that selected portfolios will approximate the GOP as the time horizon increases to infinity are given.

1.3.2. *Capital Growth and the Mean-Variance Approach*

In the early seventies, the mean-variance approach developed in [Markowitz (1952)] was the dominating theory for portfolio selection. Selecting portfolios by maximizing growth was much less used, but attracted significant attention from academics and several comparisons of the two approaches can be found in the literature from that period. Of particular interest was the question of whether or not the two approaches could be united or if they were fundamentally different. I will review the conclusion from this investigation along with a comparison of the two approaches. In general, growth maximization and mean-variance based portfolio choice are two different things. This is unsurprising, since it is well-known that mean-variance based portfolio selection is not consistent with maximizing a given utility function except for special cases. Given the theoretically more solid foundation of the growth optimum theory compared to mean-variance based portfolios selection, I will try to explain why the growth optimum theory became much less widespread. Most parts of the discussion are presented in discrete time, but in the second part of this section, the continuous time parallel will be considered since the conclusions here are very different.

Discrete time

Consider the discrete time framework of Section 1.2.1. Recall that a mean-variance efficient portfolio, is a portfolio, such that any other portfolio having the same mean return will have equal or higher variance. These portfolios are obtained as the solution to a quadratic optimization program. It is well-known that the theoretical justification of this approach requires either a quadratic utility function or some fairly restrictive assumption on the class of return distribution, the most common being the assumption of normally distributed returns. The reader is assumed to be familiar with the method, but sources are cited in the notes. Comparing this method for portfolio selection to the GOP yields the following general conclusion.

- The GOP is in general not mean-variance efficient. [Hakansson (1971a)] construct examples such that the GOP lies very far from the efficient frontier. These examples are quite simple and involve only a few assets with two point distributions but illustrate the fact that the GOP may be far from the mean-variance efficient frontier. This is perhaps not surprising given the fact that mean-variance selection can be related to quadratic utility, whereas growth optimality is related to logarithmic utility. Only for specific distributions will the GOP be efficient. Note that if the distribution has support on the entire real axis, then the GOP is trivially efficient, since all money will be put in the risk-free asset. This is the case for normally distributed returns.
- Mean-variance efficient portfolios risk ruin. From Theorem 1.3 and the subsequent discussion, it is known that if the growth rate of some asset is positive and the investment opportunities are infinitely divisible, then the GOP will have no probability of ruin, neither in the short or the long run sense. This is not the case for mean-variance efficient portfolios, since there are efficient portfolios which can become negative and some which have a negative expected growth rate. Although portfolios with a negative expected growth rate need not become negative, such portfolios will converge to zero as the number of periods turn to infinity.
- Mean-variance efficient portfolio choice is inconsistent with first order stochastic dominance. Since the quadratic utility function is decreasing from a certain point onwards, a strategy which provides more wealth almost surely may not be preferred to one that brings less wealth. Since the logarithmic function is increasing, the GOP will not be dominated by any portfolio.

The general conclusions above leave the impression that the growth based investment strategies and the mean-variance efficient portfolios are very different. This view is challenged by authors who show that approximations of the geometric mean by the first and second moment can be quite accurate. Given the difficulties of calculating the GOP, such approximations were sometimes used to simplify the optimization problem of finding the portfolio with the highest geometric mean, see for instance [Latané and Tuttle (1967)]. Moreover, the empirical results of Section 1.5 indicate that it can be difficult to tell whether the GOP is in fact mean-variance efficient or not.

In the literature, it has been suggested to construct different trade-offs between growth and security in order for investors with varying degrees of risk aversion to invest more conservatively. These ideas have the same intuitive content as the mean-variance efficient portfolios. One chooses a portfolio which has a desired risk level and which maximizes the growth rate given this restriction. Versions of this trade-off include the compound return mean-variance model, which is in a sense a multi-period version of the original one-period mean-variance model. In this model, the GOP is the only efficient portfolio in the long run. More direct trade-offs between growth and security include models where security is measured as the probability of falling short of a certain level, the probability of falling below a certain path, the probability of losing before winning etc.

Interpreted in the context of general equilibrium, the mean-variance approach has been further developed into the well-known CAPM, postulating that the market portfolio is mean-variance efficient. A similar theory was developed for the capital growth criteria by [Budd and Litzenberger (1971)] and [Kraus and Litzenberger (1975)]. If all agents are assumed to maximize the expected logarithm of wealth, then the GOP becomes the market portfolio and from this an equilibrium asset pricing model appears. This is not different from what could be done with any other utility function, but the conclusions of the analysis provide empirically testable predictions and are therefore of some interest. At the heart of the equilibrium model appearing from assuming log-utility is the martingale or numéraire condition. Recall that $R^i(t)$ denotes the return on asset i between time $t-1$ and time t and $R^{\underline{\delta}}$ is the return process for the GOP. Then the equilibrium condition is

$$1 = \mathbb{E}_{t-1}\left[\frac{1+R^i(t)}{1+R^{\underline{\delta}}(t)}\right], \tag{1.9}$$

which is simply the first order conditions for a logarithmic investor. Assume a setting with a finite number of states, that is, $\Omega = \{\omega_1, \ldots, \omega_n\}$, and define $p_i = \mathbb{P}(\{\omega_i\})$. Then, if $S^{(i)}(t)$ is an Arrow-Debreu security, paying off one unit of wealth at time $t+1$, substituting into equation (1.9) provides

$$S^{(i)}(t) = \mathbb{E}_t\left[\frac{1_{(\omega=\omega_i)}}{1+R^{\underline{\delta}}(t+1)}\right] \tag{1.10}$$

and consequently summing over all states provides an equilibrium condition for the risk free interest rate

$$1 + r(t,t+1) = \mathbb{E}_t\left[\frac{1}{1+R^{\underline{\delta}}(t+1)}\right]. \tag{1.11}$$

Combining equations (1.9) and (1.11), defining $\bar{R}^i \triangleq R^i - r$ and performing some basic, but lengthy manipulations, yield

$$\mathbb{E}_t[\bar{R}^i(t+1)] = \beta^i_t \mathbb{E}_t[\bar{R}^{(\underline{\delta})}(t+1)], \tag{1.12}$$

where

$$\beta^i_t = \frac{\operatorname{cov}\left(\bar{R}^i(t+1), \frac{\bar{R}^{\underline{\delta}}(t+1)}{R^{\underline{\delta}}(t+1)}\right)}{\operatorname{cov}\left(\bar{R}^{\underline{\delta}}(t+1), \frac{\bar{R}^{\underline{\delta}}(t+1)}{R^{\underline{\delta}}(t+1)}\right)}.$$

This is similar to the CAPM, apart from the β which in the CAPM has the form

$$\beta_{\text{CAPM}} = \frac{\operatorname{cov}(R^i, R^*)}{\operatorname{var}(R^*)} .$$

In some cases, the CAPM and the CAPM based on the GOP will be very similar. For instance, when the characteristic lines are linear or trading intervals the two approaches are indistinguishable and should be perceived as equivalent theories. Later in this section, I will show the continuous time version and here the GOP is always instantaneously mean-variance efficient.

Since the growth based approach to portfolio choice has some theoretically nice features compared to the mean-variance theory and the "standard" CAPM, one may wonder why this approach did not find more widespread use. The main reason is presumably the strength of simplicity. Mean-variance based portfolio choice has an intuitive appeal as it provides a simple trade-off between expected return and variance. This trade-off can be parameterized in a closed form, requiring only the estimation of a variance-covariance matrix of returns and the ability to invert this matrix. Although choosing a portfolio which is either a fractional Kelly strategy or logarithmic mean-variance efficient provides the same trade-off, it is computationally much more involved. In Section 1.2.1 I pointed out the fact that determining the GOP in a discrete time setting is potentially difficult and no closed form solution is available. Although this may be viewed as a rather trivial matter today, it certainly was a challenge to the computational power available 35 years ago. Second, the theory was attacked immediately for the lack of economic justification. Finally, the empirical data presented in Section 1.5 show that it is very hard to separate the CAPM tangency portfolio and the GOP in practice.

Continuous time

The assumption that trading takes place continuously is the foundation of the *Intertemporal CAPM* of [Merton (1973)]. Here, the price process of the risky asset $S^{(i)}$, $i \in \{1,\ldots,d\}$, is modelled as continuous time diffusions of the form

$$dS^{(i)}(t) = S^{(i)}\left(a^i(t)dt + \sum_{j=1}^{m} b^{i,j}(t)dW^j(t)\right)$$

where W is an m-dimensional standard Wiener process. The process $a^i(t)$ can be interpreted as the instantaneous mean return and $\sum_{j=1}^{m}(b^{i,j}(t))^2$ is the instantaneous variance. One may define the instantaneous mean-variance efficient portfolios as solutions to the problem

$$\begin{array}{c} \sup_{\delta \in \underline{\Theta}(S)} a^{\delta}(t) \\ \text{s.t. } b^{\delta}(t) \leq k(t), \end{array}$$

where $k(t)$ is some non-negative adapted process. To characterize such portfolios, define the *minimal market price of risk* vector,

$$\theta^p = \{\theta^p(t) = ((\theta^p)^1(t),\ldots,(\theta^p)^m(t))^\top, t \in [0,T]\},$$

by

$$\theta^p(t) \triangleq b(t)(b(t)b(t)^\top)^{-1}(t)(a(t) - r(t)\mathbf{1}). \tag{1.13}$$

Denote the Euclidean norm by,

$$||\theta^p(t)|| = \left(\sum_{j=1}^{m}(\theta^p)^j(t)^2\right)^{\frac{1}{2}}.$$

Then, instantaneously mean-variance efficient portfolios have fractions which are solutions to the equation

$$(\pi^1(t),\ldots,\pi^N(t))^\top b(t) = \alpha(t)\theta(t) \tag{1.14}$$

for some non-negative process α, and the corresponding SDE for such portfolios is given by

$$dS^{(\delta(\alpha))}(t) = S^{(\delta(\alpha))}(t)\left((r(t)+\alpha(t)||\theta(t)||^2)dt + \alpha(t)\sum_{j=1}^{m}\theta^j(t)dW^j(t)\right). \tag{1.15}$$

From Example 1.6 it can be verified that in this case, the GOP is in fact instantaneously mean-variance efficient, corresponding to the choice of $\alpha = 1$. In other words, the GOP belongs to the class of instantaneous Sharpe ratio maximizing strategies, where the Sharpe ratio, $s^{(\delta)}$, of some strategy δ is defined as

$$s^{(\delta)}(t) = \frac{a^{\delta}(t) - r(t)}{\sum_{j=1}^{m}(b^{\delta,j}(t))^2}.$$

Here $a^{\delta}(t) = \delta^{(0)}(t)r(t) + \sum_{i=1}^{n}\delta^{(i)}(t)a^i(t)$ and similarly $b^{\delta,j}(t) = \sum_{i=1}^{n}\delta^{(i)}(t)b^{i,j}(t)$.

Note that the instantaneously mean-variance efficient portfolios consist of a position in the GOP and the rest in the risk-free asset, in other words a fractional Kelly strategy. Under certain conditions, for instance if the market price of risk and the interest rate are deterministic processes, it can be shown that any utility maximizing investor will choose a Sharpe ratio maximizing strategy and in such cases, fractional Kelly strategies will be optimal for any investor. This result can be generalized to the case where the short rate and the total market price of risk, $||\theta^p(t)||$, are adapted to the filtration generated by the noise source that drives the GOP. It is, however, well-known that if the short rate or the total market price of risk is driven by factors which can be hedged in the market, some investors will choose to do so and consequently not choose a fractional Kelly strategy. When jumps are added to asset prices, the GOP will again become instantaneously mean-variance *inefficient*, except for very special cases. The conclusion is shown to depend strongly on the pricing of event risk and completeness of markets.

If the representative investor has logarithmic utility, then in equilibrium the GOP will become the market portfolio. Otherwise this will not be the case. Since the conditions under which the GOP becomes exactly the market portfolio are thus fairly restrictive, some authors have suggested that the GOP may be very similar to the market portfolio under a set of more general assumptions. For instance, it has been shown in [Platen (2003, 2005a)] that sufficiently diversified portfolios will approximate the GOP under certain regularity conditions. (In Chapter 2 of this volume a special mean-variance approach, called semi-log-optimal portfolio, is a good approximation of the GOP.) It should be noted that the circumstances under which the GOP approximates the market portfolio do not rely on the preferences of individual investors. The regularity conditions consist of a limit to the amount of volatility not mirroring that of the GOP. This condition may be difficult to verify empirically.

In the end, whether the GOP is close to the market portfolio and whether the theory based on this assumption holds true remains an empirical question, which will be considered later on. Foreshadowing these conclusions, the general agreement from the empirical analysis is that if anything, the GOP is more risky than the market portfolio, but rejecting the hypothesis that the GOP is a proxy for the market portfolio is on the other hand very difficult.

Notes

The mean-variance portfolio technique is found in most finance textbooks. For proofs and a reasonably rigorous introduction, see [Huang and Litzenberger (1988)]. The main results of the comparison, between mean-variance and growth optimality is found in [Hakansson (1971b,a)], see also [Hakansson (1974)]. The compound return mean-variance trade-off was introduced in [Hakansson (1971b)]. A critique of this model is found in [Merton and Samuelson (1974a,b)], but some justification is given by [Luenberger (1993)]. Papers discussing the growth-security trade-off include [Blazenko *et al.* (1992)], [Li (1993)], [Li *et al.* (2005)], [Michaud (2003)] and [MacLean *et al.* (2004)]. In the gambling literature, the use of fractional Kelly strategies is widespread. For more references, the reader is referred to [Hakansson and Ziemba (1995)]. An earlier comparison between mean-variance and the GOP is found in [Bickel (1969)]. [Thorp (1969)] recommends that the Kelly-criterion replaces the mean-variance criterion for portfolio selection, due to the sometimes improper choices made by the latter. For approximations of geometric means see [Trent and Young (1969)] and [Elton and Gruber (1974)]. They conclude that the first two moments can provide reasonable approximations, in particular if the distribution does not have very fat tails. In continuous time a recent discussion of a growth security trade-off and the CAPM formula which appears, can be found in [Bajeux-Besnaino and Portait (1997a)] and [Platen (2002, 2006b, 2005c)]. An application of the GOP for asset pricing purposes can be found in [Ishijima (1999)] and [Ishima *et al.* (2004)].

Versions of the result that in continuous time a two-fund separation result will imply that investors choose fractional Kelly strategies have been shown at different levels of generality in for instance [Merton (1971, 1973)], [Khanna and Kulldorff (1999)], [Nielsen and Vassalou (2002, 2004)], [Platen (2002)], [Zhao and Ziemba (2003)] and [Christensen and Platen (2005)]. Some general arguments that the GOP will approximate or be identical to the market portfolio is provided in [Platen (2004c, 2005a)]. [Christensen (2005)] shows that when the risky asset can be dominated, investors must stay "reasonably close to the GOP" when the market conditions become favorable. However, this is a relatively weak approximation result as I will make clear. Further results in the case where asset prices are of a more general class are treated in [Platen (2006b)].

In an entirely different literature, the so-called *evolutionary finance* literature, [Blume and Easley (1992)] show that using the GOP will result in market dominance. The conclusion is, however, not stable to more general set-ups as shown in [Amir *et al.* (2004)], where market prices are determined endogenously and the market is incomplete.

1.3.3. *How Long Does it Take for the GOP to Outperform other Portfolios?*

As the GOP was advocated, not as a particular utility function, but as an alternative to utility theory relying on its ability to outperform other portfolios over time, it is important to document this ability over horizons relevant to actual investors. In this section, I will assume that investors are interested in the GOP because they hope it will outperform other competing strategies. This goal may not be a "rational" investment goal from the point of view of expected utility, but it is investigated because it is the predominant reason why the GOP was recommended as an investment strategy, as explained previously.

To get a feeling for the time it takes for the GOP to dominate other assets, consider the following illustrative example.

Example 1.10. Assume a set-up similar to Example 1.4. This is a two-asset Black-Scholes model with constant parameters. By solving the differential equation, the savings account with a risk-free interest rate of r is given by

$$S^{(0)}(t) = \exp(rt)$$

and solving the SDE, the stock price is given as

$$S^{(1)}(t) = \exp((a - \frac{1}{2}\sigma^2)t + \sigma W(t)).$$

By Example 1.4, the GOP is given by the process

$$S^{(\underline{\delta})}(t) = \exp((r + \frac{1}{2}\theta^2)t + \theta W(t))$$

where $\theta = \frac{a-r}{\sigma}$. Some simple calculations imply that the probability

$$P_0(t) \triangleq \mathbb{P}(S^{(\underline{\delta})}(t) \geq S^{(0)}(t))$$

of the GOP outperforming the savings account over a period of length t and the probability

$$P_1(t) \triangleq \mathbb{P}(S^{(\underline{\delta})}(t) \geq S^{(1)}(t))$$

of the GOP outperforming the stock over a period of length t are given by

$$P_0(t) = N\left(\frac{1}{2}\theta\sqrt{t}\right) ,$$

and

$$P_1(t) = N\left(\frac{1}{2}|\theta - \sigma|\sqrt{t}\right) .$$

Here $N(\cdot)$ denotes the cumulative distribution function of the standard Gaussian distribution. Clearly, these probabilities are independent of the short rate. This would remain true even if the short rate was stochastic, as long as the short rate does not influence the market price of risk and volatility of the stock. Moreover, they are increasing in the market price of risk and time horizon. The probabilities converge to one as the time horizon increases to infinity, which is a manifestation of the growth properties of the GOP. Table 1.1 shows the time horizon needed for outperforming the savings account at certain confidence levels. If θ is interpreted as $|\theta - \sigma|$, then the results can be interpreted as the time horizon needed to outperform the stock.

Table 1.1. Time horizon needed for outperforming the risk free asset at certain confidence levels.

Conf. level	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$
99%	8659	2165	346	87
95%	4329	1082	173	43
90%	2628	657	105	26

Table 1.1 shows that if the market price of risk is 0.25 then over a 105 year period the GOP will provide a better return than the risk free asset with a 90% level of confidence. This probability is equal to the probability of outperforming a moderately risky stock with a volatility of 50% per year. Figure 1.1 below show how the outperformance probability depends on the time horizon.

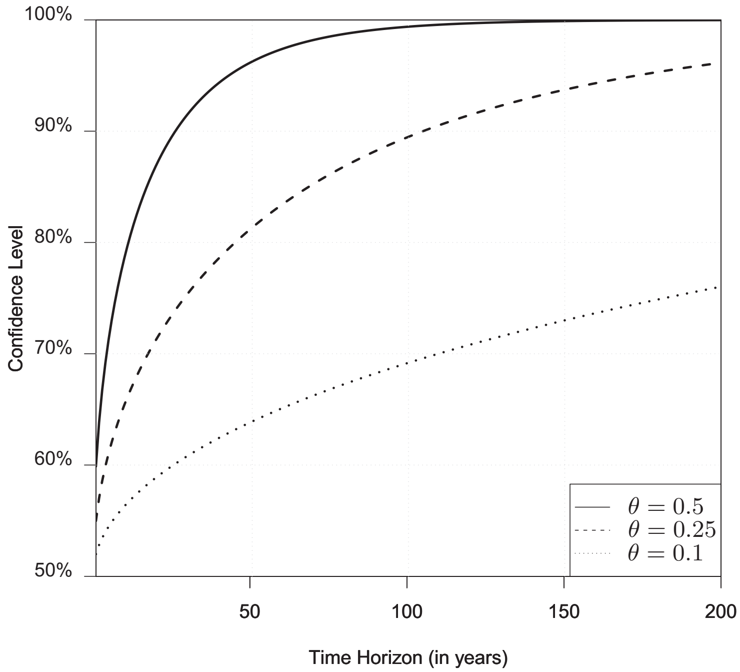

Fig. 1.1. Outperformance Likelihood.

The preliminary conclusion based on these simple results is that the long run may be very long indeed. A Sharpe ratio of 0.5 is a reasonably high one, for instance this would be the result of a strategy, with an expected excess rate of return above the risk free rate of 20% and a volatility of 40%. Even with such a strategy, it would take almost 30 years to beat the risk-free bond with a 90% probability.

Similar experiments have been conducted in the literature. For instance, [Aucamp (1993)] considers an application of the GOP strategy to the St.

Petersburg game and calculates the probability of outperforming a competing strategy. It is analyzed how many games are necessary for the GOP to outperform other strategies at a 95% confidence level. It is shown that this takes quite a number of games. For instance, if the alternative is "do nothing", then it takes the growth optimal betting strategy 87 games. Making the alternative strategy more competitive (i.e. comparing to a conservative betting strategy) makes the number of games required grow very fast. If it takes a long time before the GOP dominates alternative investment strategies, then the argument that one should choose the GOP to maximize the probability of doing better than other portfolios is somewhat weakened. Apart from the discussion of whether this property is interesting or not, it requires an unrealistically long time horizon to obtain any high level of confidence. In order to be really useful would require the GOP, when calibrated to market data, to outperform, say, an index over a (relatively) short horizon - 10 or 20 years. In this case, given the absence of a clearly specified utility function it might be useful to consider the GOP strategy. Hence, in order to see how long it will take the GOP to outperform a given alternative strategy one needs to conduct a further systematic analysis. The analysis needs to be conducted in a more realistic model calibrated to actual market data. There appears to be no available results in this direction in the literature.

Notes

Some papers that include studies of the wealth distribution when applying the GOP includes [Hakansson (1971a)], [Gressis *et al.* (1974)], [Michaud (1981)] and [Thorp (1998)]. Somewhat related to this is the study by [Jean (1980)], which relates the GOP to n-th order stochastic dominance. He shows that if a portfolio X exhibits n-th order stochastic dominance against a portfolio Y for any given n, then X needs to have a higher geometric mean than Y.

Example 1.10 is similar to [Rubinstein (1991)], who shows that to be 95% sure of beating an all-cash strategy will require 208 years; to be 95% sure of beating an all-stock strategy will require 4,700 years.

Note that the empirical evidence is mixed, see for instance the results in [Thorp (1971)], [Hunt (2005)] and the references in Section 1.5. The existing attempts to apply the GOP seem to have been very successful, but this has the character of "anecdotal evidence" and does not constitute a formal proof that the period required to outperform competing strategies is relatively short.

1.4. The GOP and the Pricing of Financial Assets and Derivatives

The numéraire property of the GOP and Theorem 1.4 has made several authors suggest that it could be used as a convenient pricing tool for derivatives in complete and incomplete markets. Although different motivations and different economic interpretations are possible for this methodology, the essence is very simple. This section is fairly important since it motivates a large part of the analysis in later sections. The set-up in this section is similar to the general set-up described in Section 1.2.2. A set of $d+1$ assets is given as semimartingales and it is assumed that the GOP, $S^{(\underline{\delta})}$, exists as a well-defined, non-explosive portfolio process on the interval $[0,T]$. I make the following assumption:

Assumption 1.1. For $i \in \{0,\ldots,d\}$ the process

$$\hat{S}^{(i)}(t) \triangleq \frac{S^{(i)}(t)}{S^{(\underline{\delta})}(t)}$$

is a local martingale.

Hence, I rule out the cases where the process is a supermartingale but not a local martingale, see Example 1.3. The reason why this is done will become clear shortly. Assumption 1.1 implies that the GOP gives rise to a *martingale density*, in the sense that for any $S^{(\delta)} \in \underline{\Theta}(S)$ it holds that

$$\hat{S}^{(\delta)}(t) \triangleq \frac{S^{(\delta)}(t)}{S^{(\underline{\delta})}(t)} = \int \frac{S^{(\delta)}(t)}{S^{(0)}(t)} dZ(t)$$

is a local martingale, where

$$Z(t) = \frac{S^{(0)}(t)}{S^{(\underline{\delta})}(t)} = \hat{S}^{(0)}(t)\ .$$

So the process $Z(t)$ can under regularity conditions be interpreted as the Radon-Nikodym derivative of the usual risk neutral measure. However, some of these processes may be strict local martingales, not true martingales. In particular, if the GOP denominated savings account is a true local martingale, then the classical risk-neutral martingale measure will not exist as will be discussed below.

Definition 1.6. Let H be any $\mathcal{F}_T$-measurable random variable. This random variable is interpreted as the pay-off of some financial asset at time T.

Assume that

$$\mathbb{E}\left[\frac{|H|}{S^{(\underline{\delta})}(T)}\right] < \infty.$$

The *fair price* process of the pay-off H is then defined as

$$H(t) = S^{(\underline{\delta})}(t)\mathbb{E}\left[\frac{H}{S^{(\underline{\delta})}(T)}|\mathcal{F}_t\right]. \tag{1.16}$$

The idea is to define the fair price in such a way that the numéraire property of the GOP is undisturbed. In other words, the GOP remains a GOP after the pay-off H is introduced in the market. There are two primary motivations for this methodology. Firstly, the market may not be complete, in which case there may not be a replicating portfolio for the pay-off H. Second, the market may be complete, but there need not exist an equivalent risk neutral measure, which is usually used for pricing. In the case of complete markets which have an equivalent risk neutral measure the fair pricing concept is equivalent to pricing using the standard method.

Lemma 1.2. *Suppose the market has an equivalent martingale measure, that is, a probability measure Q such that $P \sim Q$ and discounted asset prices are Q-local martingales. Then the risk-neutral price given by*

$$\tilde{H}(t) = S^{(0)}(t)\mathbb{E}^{Q}\left[\frac{H}{S^{(0)}(T)}|\mathcal{F}_t\right]$$

is identical to the fair price, i.e. $H(t) = \tilde{H}(t)$ almost surely, for all $t \in [0, T]$.

The following example illustrates why fair pricing is restricted as suggested by Assumption 1.1.

Example 1.11 (Example 1.3 continued). *Recall that the market is given such that the first asset is risk free, $S^{(0)}(t) = 1$, $t \in \{0, T\}$ and the second asset has a log-normal distribution $\log(S^{(1)}(T)) \sim \mathcal{N}(\mu, \sigma^2)$ and $S^{(1)}(0) = 1$.*

Suppose that $\hat{S}^{(0)}(t)$ is a strict supermartingale. What happens if the fair pricing concept is applied to a zero-coupon bond? The price of the zero coupon bond in the market is simply $S^{(0)}(0) = 1$. The fair price on the other hand is

$$S^{(1)}(0)\mathbb{E}\left[\frac{1}{S^{(1)}(T)}\right] < 1.$$

Hence, introducing a fairly priced zero coupon bond in this market produces an arbitrage opportunity. More generally this problem will occur in all cases, where some primary assets denoted in units of the GOP are strict supermartingales, and not local martingales.

Below I consider the remaining cases in turn. In the incomplete market case, I show how the fair price defined above is related to other pricing methodologies in an incomplete market. Then I consider markets without a risk-neutral measure and discuss how and why the GOP can be used in this case.

1.4.1. *Incomplete Markets*

Fair pricing as defined above was initially suggested as method for pricing derivatives in incomplete markets, see [Bajeux-Besnaino and Portait (1997a)] and the sources cited in the notes. In this subsection, markets are assumed to be incomplete, but to keep things separate, it is assumed that the set of martingale measures is non-empty. In particular, the process $\hat{S}^{(0)}$ is assumed to be a true martingale. When markets are incomplete and there is no portfolio which replicates the pay-off, H, arbitrage theory is silent on how to price this pay-off. From the seminal work of [Harrison and Pliska (1981)] it is well-known that this corresponds to the case of an infinite number of candidate martingale measures. Any of these measures will price financial assets in accordance with no-arbitrage and there is no a priori reason for choosing one over the other. In particular, no arbitrage considerations does not suggest that one might use the martingale measure Q defined by $\frac{dQ}{dP} = \hat{S}^{(0)}(T)$, which is the measure induced by applying the GOP. One might assume that investors maximized the growth rate of their investments. Then it could be argued that a "reasonable" price of the pay-off, H, should be such that the maximum growth rate obtainable from trading the derivative and the existing assets should not be higher than trading the existing assets alone. Otherwise, the derivative would be in positive net-demand, as investors applied it to obtain a higher growth rate. It can be shown that the only pricing rule which satisfies this property is the fair pricing rule. Of course, whether or not growth rates are interesting to investors has been a controversial issue. Indeed, as outlined in the previous sections, the growth rate is only in directly relevant to an investor with logarithmic utility and the argument that the maximal growth rate should not increase after the introduction of the derivative is generally not backed by an equilibrium argument, except for the case where the representative investor is assumed to have logarithmic utility. Although there may be no strong theoretical argument behind the selection of the GOP as the pricing operator in an incomplete market, its application is fully consistent with arbitrage free pricing. Consequently, it is useful to compare this method to a few of the pricing alternatives presented in the literature.

Utility Based Pricing: This approach to pricing assumes agents to be endowed with some utility function U. The *utility indifference price* at time t of k units of the pay-off H, is then defined as the price $p_H(k,t)$ such that

$$\sup_{S^{(\delta)}(T),S^{(\delta)}(t)=x-p_H(k,t)} \mathbb{E}\left[U(S^{(\delta)}(T)+kH)\right] = \sup_{S^{(\delta)}(T),S^{(\delta)}(t)=x} \mathbb{E}\left[U(S^{(\delta)}(T))\right].$$

This price generally depends on k, i.e. on the number of units of the pay-off, in a non-linear fashion, due to the concavity of U. Supposing that the function $p_H(k,t)$ is smooth, one may define the *marginal price* as the limit

$$p_H(t) = \lim_{k\to 0} \frac{p_H(k,t)}{k},$$

which is the utility indifference price for obtaining a marginal unit of the pay-off, when the investor has none to begin with. If one uses logarithmic utility, then the marginal indifference price is equal to the fair price, i.e. $p_H(t) = H(t)$. Of course, any reasonable utility function could be used to define a marginal price, the logarithm is only a special case.

The Minimal Martingale Measure: This is a particular choice of measure, which is often selected because it "disturbs" the model as little as possible. This is to be understood in the sense that a process which is independent of traded assets will have the same distribution under the minimal martingale measure as under the original measure. Assume the semimartingale, S, is special such that it has locally integrable jumps and consequently has the unique decomposition

$$S(t) = S(0) + A(t) + M(t) ,$$

where $A(0) = M(0) = 0$, A is predictable and of finite variation, and M is a local martingale. In this case, one may write

$$dS(t) = \lambda(t)d\langle M\rangle_t + dM(t) ,$$

where λ is the market price of risk process and $\langle M\rangle$ is the predictable projection of the quadratic variation of M. The minimal martingale measure, if it exists, is defined by the density

$$Z(T) = \mathcal{E}(-\lambda \cdot M)_T \ ,$$

where $\mathcal{E}(\cdot)$ is the stochastic exponential. In other words, Z is the solution to the SDE

$$dZ(t) = -\lambda(t)Z(t)dM(t).$$

In financial terms, the minimal martingale measure puts the market price of any unspanned risk, that is, risk factors that cannot be hedged by trading in the market, equal to zero. In the general case Z may not be a martingale, and it may become negative. In such cases the minimal martingale measure is not a true probability measure. If S is continuous, then using the minimal martingale measure provides the same prices as the fair pricing concept. In the general case, when asset prices may exhibit jumps, the two methods for pricing assets are generally different.

Good Deal Bounds: Some authors have proposed to price claims by defining a bound on the market prices of risk that can exist in the market. Choosing a martingale measure in an incomplete market amounts to the choice of a specific market price of risk. As mentioned, the minimal martingale measure is obtained by putting the market price of risk of non-traded risk factors equal to zero. For this reason, the price derived from the minimal martingale measure always lies within the good-deal bounds. Of course, given the assumption that the set of prices within the good deal bound is non-empty. It follows that the fair price lies within the good deal bound in the case of continuous asset prices. In the general case the fair price need not lie within a particular good deal bound.

Another application of fair pricing is found in the Benchmark approach. However, here the motivation was somewhat different as I will describe below.

Notes

The idea of using the GOP for pricing purposes is stated explicitly for the first time in the papers [Bajeux-Besnaino and Portait (1997a,b)] and further argued in [Aurell *et al.* (2000a,b)]. In the latter case, the arguments for using the GOP seem to be subject to the criticism raised by Samuelson, but the method as such is not inconsistent with no arbitrage. Utility based pricing is reviewed in [Davis (1997)] and [Henderson and Hobson (2008)]. The minimal martingale measure is discussed in, for instance, [Schweizer (1995)], and the relationship with the GOP is discussed in [Becherer (2001)] and [Christensen and Larsen (2007)]. Good deal bounds were introduced by [Cochrane and Saá-Requejo (2000)] and extended to a general setting in [Björk and Slinko (2006)]. The later has a discussion of the relationship to the minimal martingale measure.

1.4.2. *A World Without a Risk-neutral Measure*

In this section I consider a complete market. To keep matters as simple as possible, assume there is a risk-free savings account where the short rate, r, is assumed to be constant. Hence,

$$dS^{(0)}(t) = rS^{(0)}(t)dt.$$

There is only one risky asset given by the stochastic differential equation

$$dS^{(1)}(t) = S^{(1)}(t)\left(a(t)dt + b(t)dW(t)\right) \ ,$$

where W is a standard one-dimensional Wiener process. It is assumed that a and b are strictly positive processes such that the solution $S^{(1)}$ is unique and well-defined, but no other assumptions are made. The parameter processes a and b can be general stochastic processes. The market price of risk θ is then well-defined as $\theta(t) = \frac{a(t)-r(t)}{b(t)}$ and consequently this may also be a stochastic process. The usual approach when pricing options and other derivatives is to define the stochastic exponential

$$\Lambda(t) = \exp\left(-\frac{1}{2}\int_0^t \theta^2(s)ds - \int_0^t \theta(s)dW(s)\right).$$

If $\Lambda(t)$ is a martingale, then the Girsanov theorem implies the existence of a measure Q such that

$$\tilde{W}(t) \triangleq W(t) - \int_0^t \theta(s)ds$$

is a standard Wiener process under the measure Q.

However, it is well-known that Λ need not be a martingale. This is remarked in most text-books, see for instance [Karatzas and Shreve (1988)] or [Revuz and Yor (1991)]. The latter contains examples from the class of Bessel processes.

By the Itô formula, Λ satisfies the stochastic differential equation

$$d\Lambda(t) = -\theta(t)\Lambda(t)dW(t)$$

and so is a local martingale, see [Karatzas and Shreve (1988)]. As mentioned in Example 1.9 some additional conditions are required to ensure Λ to be a martingale. The question here is, what happens if the process $\Lambda(t)$ is not a martingale? The answer is given in the theorem below:

Theorem 1.9. *Suppose the process $\Lambda(t)$ is not a true martingale. Then*

(1) If there is a stopping time $\tau \leq T$, such that $\mathbb{P}(\int_0^\tau \theta^2(s)ds = \infty) > 0$, then there is no equivalent martingale measure for the market under any numéraire and the GOP explodes. An attempt to apply risk-neutral pricing or fair pricing will result in Arrow-Debreu prices that are zero for events with positive probability.
(2) If $\int_0^T \theta^2(s)ds < \infty$ almost surely, then the GOP is well-defined and the original measure P is an equivalent martingale measure when using the GOP as numéraire.
(3) If $\int_0^T \theta^2(s)ds < \infty$ almost surely, then $\Lambda(t)$ is a strict supermartingale and there is no risk-neutral measure when using the risk free asset as a numéraire. Moreover, the risk-free asset can be outperformed over some interval $[0, \tau] \subseteq [0, T]$.
(4) The fair price is the price of the cheapest portfolio that replicates the given pay-off.

The theorem shows that fair pricing is well-defined in cases where risk-neutral pricing is not. Although presented here in a very special case the result is in fact true in a very general setting. The result may look puzzling at first because usually the existence of a risk-neutral measure is associated with the absence of arbitrage. However, continuous time models may contain certain types of "arbitrage" arising from the ability to conduct an infinite number of trade. A prime example is the so-called doubling strategy, which involves doubling the investment until the time when a favorable event happens and the investor realizes a profit. Such "arbitrage" strategies are easily ruled out as being inadmissible by Definition 1.4 because they generally require an infinite debt capacity. Hence they are not arbitrage strategies in the sense of Definition 1.3. But imagine a not-so-smart investor, who tries to do the opposite thing. He may end up losing money with certainty by applying a so-called "suicide strategy", which is a strategy that costs money but results in zero terminal wealth. A suicide strategy could, for instance, be a short position in the doubling strategy (if it where admissible). Suicide strategies exist *whenever asset prices are unbounded* and they need not be inadmissible. Hence, they exist in for instance the Black-Scholes model and other popular models of finance. If a primary asset has a built-in suicide strategy, then the asset can be outperformed, by a replicating portfolio without the suicide strategy. This suggests the existence of an arbitrage opportunity, but that is not the case. If an investor attempts to sell the asset and buy a (cheaper) replicating portfolio, the resulting strategy is not necessarily admissible. Indeed, this strategy may suffer large, temporary losses before maturity, at which point of course it becomes strictly positive. It is important to note that whether or not the temporary losses of the portfolio are bounded is strictly dependent on the numéraire. This insight was developed by [Delbaen and Schachermayer (1995b)], who showed that the arbitrage strategy under consideration is *lower bounded* under some numéraire, if and only if that numéraire can be outperformed. Given the existence of a market price of risk, the "arbitrage" strategy is never strictly positive at all times before maturity. If this was the case, then any investor could take an unlimited position in this arbitrage and the GOP would no longer be a well-defined object. The important difference between having a lower bounded and an unbounded arbitrage strategy is exactly that if the strategy is lower bounded, then by the fundamental theorem of asset pricing there can be no equivalent martingale measure. In particular, if the risk-free asset contains a built-in suicide strategy, then there cannot be an EMM when the risk-free asset is applied as numéraire. On the other hand, a different numéraire may still work allowing for consistent pricing of derivatives.

Hence if one chooses a numéraire which happens to contain a built-in suicide strategy, then one cannot have an equivalent martingale measure. This suggests that one needs only take care that the right numéraire is chosen, in order to have a consistent theory for pricing and hedging. Indeed, the GOP is such a numéraire, and it works even when the standard numéraire - the risk free savings account - can be outperformed. Moreover, the existence of a GOP is completely numéraire independent. In other words, the existence of a GOP is the acid test of any model that is to be useful when pricing derivatives.

Obviously, the usefulness of the described approach relies on two things. Firstly, there is only a theoretical advantage in the cases where risk-neutral pricing fails, otherwise the two approaches are completely equivalent, and fair pricing is merely a special case of the change of numéraire technique.

Remark 1.1. In principle, many other numéraires could be used instead of the GOP, as long as they do not contain suicide strategies. For instance, the portfolio choice of a utility maximizing investor with increasing utility function will never contain a suicide strategy. However, for theoretical reasons the GOP is more convenient. Establishing the existence of a portfolio which maximizes the utility of a given investor is non-trivial. Moreover, the existence of a solution may depend on the numéraire selected, whereas the existence of the GOP does not.

In practice, usefulness requires documentation that the risk-free asset can be outperformed, or equivalently that the risk-free asset denominated in units of the GOP is a strict local martingale. This is quite a hard task. The question of whether the risk-free asset can be outperformed is subject to the so-called peso problem: only one sample path is ever observed, so it is quite hard to argue that a portfolio exists which can outperform the savings account *almost surely*. At the end of the day, almost surely means with probability one, not probability 99.999%. From the earlier discussion, it is known that the GOP will outperform the risk-less asset sooner or later, so over very long horizon, the probability that the risk-free asset is outperformed is rather high and it is more than likely that even if one were to have (or construct) a number of observations, they would all suggest that the risk-free asset could be outperformed. A better, and certainly more feasible, approach if one were to document the usefulness of the fair pricing approach is to show that the predictions of models, in which the savings account can be outperformed, are in line with empirical observations. Some arguments have started to appear, for instance in the literature on continuous time "bubbles", cited in the notes.

Notes

In a longer sequence of papers the fair pricing concept was explored as part of the so-called *benchmark approach*, advocated by Eckhard Platen and a number of co-authors, see for instance [Heath and Platen (2002a,b,c, 2003)], [Miller and Platen (2005)], [Platen (2001, 2002, 2004a,b,c,d, 2005a)], [Christensen and Platen (2005)] and [Platen and West (2005)].

The proof of Theorem 1.9 is found in this literature, see in particular [Christensen and Larsen (2007)]. Some calibrations which indicate that models without an EMM could be realistic are presented in [Platen (2004c)] and [Fergusson and Platen (2006)].

Related to this approach is the recent literature on continuous asset price bubbles. Bubbles are said to exist whenever an asset contains a built-in suicide strategy, because in this case, the current price of the asset is higher than the price of a replicating portfolio. References are [Loewenstein and Willard (2000a,b)], [Cassesse (2005)] and [Cox and Hobson (2005)]. In this literature, it is shown that bubbles can be compatible with equilibrium, and that they are in line with observed empirical observations. To some extent they lend further support to the relevance of fair pricing.

1.5. Empirical Studies of the GOP

Empirical studies of the GOP are relatively limited in numbers. Many date back to the seventies and deal with comparisons to the mean-variance model. Here, the empirical studies will be separated into two major groups, answering to broad questions.

- How is the GOP composed? Issues that belong to this group include what mix of assets constitute the GOP and, in particular, whether the GOP equals the market portfolio or any other diversified portfolio.
- How does the GOP perform? Given an estimate of the GOP it is of some practical importance to document its value as an investment strategy.

The conclusions within those areas are reasonably consistent across the literature and the main ones are

- It is statistically difficult to separate the GOP from other portfolios - this conclusion appears in all studies known to the author. It appears that the GOP is riskier than the mean-variance tangency portfolio and the market portfolio, but the hypothesis that the GOP is the market portfolio cannot be formally rejected. It may be well-approximated by a levered position in the market. This is consistent with different estimates of the risk aversion coefficient, γ, of a power utility investor which different authors have estimated to be much higher than one (corresponding to a log-investor). A problem in most studies is the lack of statistical significance and it is hard to find significant proof of the composition. Often, running some optimization program will imply a GOP that only invests in a smaller subset of available assets.
- The studies that use the GOP for investment purposes generally conclude that although it may be subject to large short-term fluctuations, growth maximization performs rather well even on time horizons which are not excessively long. Hence, although the GOP does not maximize expected utility for a non-logarithmic investors, history shows that portfolio managers using the GOP strategy can become rather successful. However, the cases where the GOP is applied for invest-

ment purposes are of a somewhat anecdotal nature. Consequently, the focus of this section will be the first question.

Notes

For some interesting reading on the actual performance of growth optimal portfolios in various connections, see [Thorp (1971, 1998)], [Grauer and Hakansson (1985)], [Hunt (2004, 2005)] and [Ziemba (2005)]. Edward Thorp constructed a hedge-fund, PNP, which successfully applied the GOP strategy to exploit prices in the market out of line with mathematical models, see in particular [Poundstone (2005)]. The reader is referred to the quoted papers in the following subsection, since most of these have results on the performance of the GOP as well. There seems to be very few formal studies, which consider the performance of growth optimal strategies.

1.5.1. *Composition of the GOP*

Discrete Time Models: The method used for empirical testing is replicated, at least in principle, by several authors and so is explained briefly. Assume a discrete time set-up, as described in Section 1.2.1. Hence the market is given as $S = (S^{(0)}(t), S^{(1)}(t), \ldots, S^{(d)}(t))$ with the return between time t and $t+1$ for asset i denoted by $R^i(t)$ as usual. Recall from the myopic properties of the GOP, that the GOP strategy can be found by maximizing the expected growth rate between t and $t+1$

$$\sup_{\delta} \; \mathbb{E}_t \left[\log \left(\frac{S^{(\delta)}(t+1)}{S^{(\delta)}(t)} \right) \right],$$

for each $t \in \{0, \ldots T-1\}$. From Equation (1.5), the first order conditions for this problem are

$$\mathbb{E}_{t-1} \left[\frac{1 + R^i(t)}{1 + R^{\underline{\delta}}(t)} \right] = 1 \tag{1.17}$$

for all $i \in \{0, \ldots, d\}$. The first order conditions provide a testable implication. A test that some given portfolio is the GOP can be carried out by forming samples of the quantity

$$Z^i(t) \triangleq \frac{1 + R^i(t)}{1 + R^{\underline{\delta}}(t)},$$

where $1+R^{\underline{\delta}}(t)$ is the return of candidate GOP portfolio. The test consists of checking whether

$$\bar{Z} \triangleq \frac{1}{T}\sum_{t=1}^{T} Z_t^i$$

is statistically different across assets. However, note that this requires the first order conditions to be satisfied. In theory, GOP deflated assets are supermartingales - in particular they may in discrete time be supermartingales that are not martingales, see Example 1.3. The assumption implicit in the first order condition above is that optimal GOP fractions are assumed in an inner point. A theoretically more correct approach then, would be to test whether these quantities on average are below one. As the variance of returns may differ across assets and independence is unrealistic, an applied approach is to use the Hotelling T^2 statistic to test this hypothesis. This is a generalization of the student-t distribution. To be a valid test, this actually requires normality from the underlying variables, which is clearly unreasonable, since if returns would have support on the entire real axis, a growth optimizer would seek the risk-free asset. The general conclusion from this approach is that it cannot reject the hypothesis that the GOP equals the market portfolio.

Because this approach is somewhat dubious, alternatives have been suggested. An entirely different way of solving the problem is to find the GOP in the market by making more specific distributional assumptions, and calculating the GOP ex ante and study its properties. This allows a comparison between the theoretically calculated GOP and the market portfolio. The evidence is somewhat mixed. [Fama and Macbeth (1974)] compares the mean-variance efficient tangent portfolio to the GOP. Perhaps the most important conclusion of this exercise is that although the β of the historical GOP is large and deviates from one, the growth rate of this portfolio cannot be statistically separated from that of the market portfolio. This is possibly related to the fact that [Fama and Macbeth (1974)] construct the time series of growth rates from relatively short periods and hence the size of growth rates is reasonably small compared to the sample variance which in turn implies small t-stats. Still, it suggests that the GOP could be more highly levered than the tangency portfolio. This does not imply, of course, that the GOP is different from the market. This would be postulating a beta of one to be the beta of the market portfolio and it requires one to believe that the CAPM holds.

Although the cited study finds it difficult to reject the proposition that the market portfolio is a proxy for the GOP, it suggests that the GOP can be more risky than this portfolio. Note that the market portfolio itself has to be proxied. Usually this is done by taking a large index such as S&P 500 as a proxy for the market portfolio. Whether this approximation is reasonable is debatable. Indeed, this result is verified in most available studies. An exception is [Long (1990)], who examines different proxies for the GOP. The suggested proxies are examined using the first order conditions as described above. Although the results of formal statistic tests are absent, the intuition is in line with earlier empirical research. In the article, three proxies are examined

(1) A market portfolio proxy
(2) A levered position in the market portfolio
(3) A Quasi-Maximum-Likelihood estimate

The study concludes, that using a quasi maximum likelihood estimate of the GOP exhibits superior performance. However, using a market portfolio proxy as numéraire will yield a time series of numéraire adjusted returns which have a mean close to zero. A levered position in the market portfolio, on the other hand, will increase the variance of the numéraire adjusted returns and seems to be the worst option.

A general conclusion, when calibrating the market data to a CAPM type model is that the implied relative risk aversion for a representative agent is (much) higher than one, one being the relative risk aversion of an agent with logarithmic utility. This somehow supports the conclusion that the GOP is more risky than the market portfolio.

A few other studies indicate that the GOP could be a rather narrow portfolio selecting only a few stocks. A study which deals more specifically with the question of what assets to include in the GOP was conducted by [Grauer (1981)]. Assuming that returns on assets follow a (discrete) approximate normal distribution, he compares the mix of assets in a mean-variance efficient portfolio and a GOP, with limits on short sales. Out of twenty stock, the GOP and the mean-variance tangency portfolio both appeared to be very *undiversified* - the typical number of stocks picked by the GOP strategy was three. (Similar experimental results can be found in Chapters 2 and 4 of this volume.) Furthermore, there appeared to be a significant difference between the composition of a mean-variance efficient portfolio and the GOP. In a footnote, Grauer suggests that this may be due to the small number of states (which makes hedging simpler) in the distribution of returns. This does not explain the phenomena for the tangency portfolio, which, if CAPM holds, should have the same composition as the market portfolio. It suggested that the lack of diversification is caused by the imposed short sale constraint. Although the reason for this explanation is unclear, it is shown that if the short sale constraint is lifted, then the GOP model takes a position in all 20 stocks. In [Grauer and Hakansson (1985)] a simple model for returns is assumed and the investor's problem is then solved using non-linear programming. It appears that the growth optimal strategy is well approximated by this method and it yields a significantly higher mean geometric return than other strategies investigated in the sample. Analyzing the composition of the GOP provides a somewhat mixed picture of diversification: before 1940, the GOP consists of a highly levered position in Government and Corporate bonds, but only few stocks. Then a switch occurs towards a highly levered position in stocks until the late sixties at which point the GOP almost leaves the stock market and turns to the risk-free asset to become a quite conservative strategy in the last period of the sample which ends in 1982. This last period of conservatism may be due to estimation problems and it is remarked by the authors that by analyzing the ex-post returns it appears that the GOP is *too* conservative. Still, the article is not able to support the claim that the GOP, in general, is a well-diversified portfolio.

Continuous Time Models: Only very few studies have been made in continuous time. With the exception of [Hunt (2004)], who uses a geometric Brownian motion with one driving factor as the model for stock prices. This implies that shocks are perfectly correlated across assets and log returns are normally distributed. Despite the fact that such a model is rejected by the data, a GOP is constructed, and its properties are investigated. The formed GOP strategy in this setting also consists of only a few stocks, but imposing a short sale constraint *increases* the level of diversification in GOP strategy, contrary to the result mentioned above. The study is subject to high parameter uncertainty, and the assumption of one driving factor implies that the GOP strategy is far from unique; in theory it can be formed from any two distinct assets. For this reason, the conclusions about the composition of the GOP might be heavily influenced by the choice of model.

It appears that to answer the question of what mix of assets are required to form the GOP, new studies will have to be made. In particular obtaining closer approximation of real stock dynamics is warranted. This could potentially include jumps and should at least have several underlying uncertainty factors driving returns. The overall problem so far seem to have been a lack of statistical power, but certainly having a realistic underlying model seem to be a natural first step. Furthermore, the standard test in equation (1.5) may be insufficient if the dynamics of GOP deflated assets will be that of a true supermartingale. The test may lead to an acceptance of the hypothesis that a given portfolio is growth-optimal, when the true portfolio is in fact more complex. Hence, tests based on the first order condition should in principle be one-sided.

Notes

Possibly the first study to contain an extensive empirical study of the GOP was [Roll (1973)]. Both [Roll (1973)] and [Fama and Macbeth (1974)] suggest that the market portfolio should approximate the GOP and both use an index as a GOP candidate and both are unable to reject the conclusion that the GOP is well approximated by the market portfolio. [Roll (1973)] use the S&P 500, whereas [Fama and Macbeth (1974)] uses a simple average of returns on common stocks listed on NYSE. Using the first order condition as a test of growth optimality is also done by [Long (1990)] and [Hentschel and Long (2004)]. [Bicksler and Thorp (1973)] assume two different distributions, calculate the implied GOP based on different amounts of leverage and find it to be highly levered. Second, many growth *suboptimal* portfolios are impossible to separate from the true GOP in practice. [Rotando and Thorp (1992)] calibrate S&P 500 data to a (truncated) normal distribution and calculate the GOP formed by the index and risk-less borrowing. This results in a levered position in the index of about 117 percent. [Pulley (1983)] also reaches the conclusion that the GOP is not a very diversified portfolio. However, in Pulley's study, the composition of the GOP and mean-variance based approximations are very similar, see [Pulley (1983)][Table 2]. For general results suggesting the market portfolio to be the result of a representative agent with high risk aversion see for instance the econometric literature related to the equity premium puzzle of [Mehra and Prescott (1985)]. Some experimental evidence is presented in [Gordon *et al.* (1972)], showing that as individuals become more wealthy, their investment strategy would approximate that of the GOP. [Maier *et al.* (1977a)] conduct simulation studies to investigate the composition of the GOP and the performance of various proxies.

1.6. Conclusion

The GOP has fascinated academics and practitioners for decades. Despite the arguments made by respected economists that the growth properties of the GOP are irrelevant as a theoretical foundation for portfolio choice, it appears that it is still viewed as a practically applicable criterion for investment decisions. In this debate it was emphasized that the utility paradigm in comparison suffers from being somewhat more abstract. The arguments that support the choice of the GOP is based on very specific growth properties, and even though the GOP is the choice of a logarithmic investor, this interpretation is often just viewed as coincidental. The fact that over time the GOP will outperform other strategies is an intuitively appealing property, since when the time comes to liquidate the portfolio it only matters how much money it is worth. Still, some misunderstandings seem to persist in this area, and the fallacy pointed out by Samuelson probably should be studied more carefully by would-be applicants of this strategy, before they make their decision. Moreover, the dominance of the GOP may require some patience. Studies show that it will take many years before probability that the GOP will do better than even the risk-free asset becomes high.

In recent years, it is in particular the numéraire property of the GOP which is being researched. This property relates the GOP to pricing kernels and hence makes it applicable for pricing derivatives. Hence, it appears that the GOP may have a role to play as a tool for asset and derivative pricing. The practical applicability and usefulness still needs to be validated empirically, in particular the problem of finding a well-working GOP proxy needs attention. This appears to be an area for further research in the years to come.

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