Chapter 4

LOG-OPTIMAL PORTFOLIO SELECTION STRATEGIES WITH PROPORTIONAL TRANSACTION COSTS

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Abstract

Discrete time growth optimal investment in stock markets with proportional transaction costs is considered. The market process is a sequence of returns, and it is modelled by a first order Markov process. Assuming that the distribution of the market process is known, we show sequential investment strategies such that, in the long run, the growth rate on trajectories is greater than or equal to the growth rate of any other investment strategy with probability 1.

Keywords: portfolio selection, log-optimal investment, proportional transaction cost, dynamic optimization.

AMS Subject Classification (2010): –

1. Introduction

The purpose of this paper is to investigate sequential investment strategies for financial markets such that the strategies are allowed to use information collected from the past of the market and determine, at the beginning of a trading period, a portfolio, that is, a way to distribute their current capital among the available assets. The goal of the investor is to maximize his wealth on the long run. If there is no transaction cost then the only assumption we use in our mathematical analysis is that the daily price relatives form a stationary and ergodic process. Under this assumption the the best strategy (called log-optimum strategy) can be constructed in full knowledge of the distribution of the entire process, see Algoet and Cover [2]. Moreover, Györfi and Schäfer [11], Györfi, Lugosi and Udina [10] and Györfi, Udina and Walk [12] constructed empirical (data driven) growth optimum strategies in case of unknown distributions. The empirical results show that the performance of these empirical investment strategies measured on past NYSE data is solid, and sometimes even spectacular.
The problem of optimal investment with proportional transaction cost has been essentially formulated and studied in continuous time only (cf. Akien, Sulem and Taksar [1], Davis and Norman [5], Eastham and Hastings [6], Korn [18], Morton and Pliska [20], Palczewski and Stettner [21], Pliska and Suzuki [22], Shreve, Soner and Xu [26], Shreve and Soner [27], Taksar, Klass and Assaf [29]).

Papers dealing with growth optimal investment with transaction costs in discrete time setting are seldom. Cover and Iyengar [16] formulated the problem of horse race markets, where in every market period one of the assets has positive pay off and all the others pay nothing. Their model included proportional transaction costs and they used a long run expected average reward criterion. There are results for more general markets as well. Sass and Schäl [24] investigated the numeraire portfolio in context of bond and stock as assets. Iyengar [14], [15] investigated growth optimal investment with several assets assuming independent and identically distributed (i.i.d.) sequence of asset returns. Bobryk and Stettner [4] considered the case of portfolio selection with consumption, when there are two assets, a bank account and a stock. Furthermore, long run expected discounted reward and i.i.d asset returns were assumed. In the case of discrete time and non i.i.d. market process, the most far reaching study was Schäfer [25] who considered the maximization of the long run expected average growth rate with several assets and proportional transaction costs, when the asset returns follow a stationary Markov process. Györfi and Vajda [13] extended the expected growth optimality mentioned above to the almost sure (a.s.) growth optimality.

In this paper we study the problem of discrete time growth optimal investment in stock markets with proportional transactions costs. If the market process is first order Markov process and the distribution of the market process is known, then we show simple sequential investment strategies such that, in the long run, the growth rate on trajectories is greater than or equal to the growth rate of any other investment strategy with probability 1.

2. Mathematical setup: investment with transaction cost

Consider a market consisting of $d$ assets. The evolution of the market in time is represented by a sequence of market vectors $s_1, s_2, \ldots \in \mathbb{R}_+^d$, where

$$s_i = (s_i^{(1)}, \ldots, s_i^{(d)})$$

such that the $j$-th component $s_i^{(j)}$ of $s_i$ denotes the price of the $j$-th asset at the end of the $i$-th trading period. ($s_0^{(j)} = 1$.)

In order to apply the usual prediction techniques for time series analysis one has to transform the sequence $\{s_i\}$ into a sequence of return vectors $\{x_i\}$ as follows:

$$x_i = (x_i^{(1)}, \ldots, x_i^{(d)})$$

such that

$$x_i^{(j)} = \frac{s_i^{(j)}}{s_{i-1}^{(j)}}.$$

Thus, the $j$-th component $x_i^{(j)}$ of the return vector $x_i$ denotes the amount obtained at the end of the $i$-th trading period after investing a unit capital in the $j$-th asset.
The investor is allowed to diversify his capital at the beginning of each trading period according to a portfolio vector \( \mathbf{b} = (b^{(1)}, \ldots, b^{(d)})^T \). The \( j \)-th component \( b^{(j)} \) of \( \mathbf{b} \) denotes the proportion of the investor’s capital invested in asset \( j \). Throughout the paper we assume that the portfolio vector \( \mathbf{b} \) has nonnegative components with \( \sum_{j=1}^{d} b^{(j)} = 1 \). The fact that \( \sum_{j=1}^{d} b^{(j)} = 1 \) means that the investment strategy is self-financing and consumption of capital is excluded. The non-negativity of the components of \( \mathbf{b} \) means that short selling and buying stocks on margin are not permitted. To make the analysis feasible, some simplifying assumptions are used that need to be taken into account. We assume that assets are arbitrarily divisible and all assets are available in unbounded quantities at the current price at any given trading period. We also assume that the behavior of the market is not affected by the actions of the investor using the strategies under investigation.

For \( j \leq i \) we abbreviate by \( x^i_j \) the array of return vectors \( (x_j, \ldots, x_i) \). Denote by \( \Delta_d \) the simplex of all vectors \( \mathbf{b} \in \mathbb{R}^d_+ \) with nonnegative components summing up to one. An investment strategy is a sequence \( \mathbf{B} \) of functions

\[
\mathbf{b}_i : \left( \mathbb{R}^d_+ \right)^{i-1} \to \Delta_d, \quad i = 1, 2, \ldots
\]

so that \( \mathbf{b}_i(x^{i-1}_1) \) denotes the portfolio vector chosen by the investor on the \( i \)-th trading period, upon observing the past behavior of the market. We write \( \mathbf{b}(x^{i-1}_1) = \mathbf{b}_i(x^{i-1}_1) \) to ease the notation.

In this section our presentation of the transaction cost problem utilized the formulation in Kalai and Blum [17] and Schäfer [25] and Györfi and Vajda [13]. Let \( S_n \) denote the gross wealth at the end of trading period \( n \), \( n = 0, 1, 2, \ldots \), where without loss of generality let the investor’s initial capital \( S_0 \) be 1 dollar, while \( N_n \) stands for the net wealth at the end of trading period \( n \). Using the above notations, for the trading period \( n \), the net wealth \( N_{n-1} \) can be invested according to the portfolio \( \mathbf{b}_n \), therefore the gross wealth \( S_n \) at the end of trading period \( n \) is

\[
S_n = N_{n-1} \sum_{j=1}^{d} b^{(j)}_n x^{(j)}_n = N_{n-1} \langle \mathbf{b}_n, \mathbf{x}_n \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes inner product.

At the beginning of a new market day \( n + 1 \), the investor sets up his new portfolio, i.e. buys/sells stocks according to the actual portfolio vector \( \mathbf{b}_{n+1} \). During this rearrangement, he has to pay transaction cost, therefore at the beginning of a new market day \( n + 1 \) the net wealth \( N_n \) in the portfolio \( \mathbf{b}_{n+1} \) is less than \( S_n \).

The rate of proportional transaction costs (commission factors) levied on one asset are denoted by \( 0 < c_s < 1 \) and \( 0 < c_p < 1 \), i.e., the sale of 1 dollar worth of asset \( i \) nets only \( 1 - c_s \) dollars, and similarly we take into account the purchase of an asset such that the purchase of 1 dollar’s worth of asset \( i \) costs an extra \( c_p \) dollars. We consider the special case when the rate of costs are constant over the assets.

Let’s calculate the transaction cost to be paid when select the portfolio \( \mathbf{b}_{n+1} \). Before rearranging the capitals, at the \( j \)-th asset there is \( b^{(j)}_n x^{(j)}_n N_{n-1} \) dollars, while after rearranging we need \( b^{(j)}_{n+1} N_n \) dollars. If \( b^{(j)}_n x^{(j)}_n N_{n-1} \geq b^{(j)}_{n+1} N_n \) then we have to sell and the
transaction cost at the \( j \)-th asset is
\[
c_s \left( b_n^{(j)} x_n^{(j)} N_{n-1} - b_\ast^{(j)} n N_n \right),
\]
otherwise we have to buy and the transaction cost at the \( j \)-th asset is
\[
c_p \left( b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1} \right).
\]

Let \( x^+ \) denote the positive part of \( x \). Thus, the gross wealth \( S_n \) decomposes to the sum of the net wealth and cost the following - self-financing - way
\[
N_n = S_n - \sum_{j=1}^d c_s \left( b_n^{(j)} x_n^{(j)} N_{n-1} - b_\ast^{(j)} n N_n \right)^+ - \sum_{j=1}^d c_p \left( b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1} \right)^+,
\]
or equivalently
\[
S_n = N_n + c_s \sum_{j=1}^d \left( b_n^{(j)} x_n^{(j)} N_{n-1} - b_\ast^{(j)} n N_n \right)^+ + c_p \sum_{j=1}^d \left( b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1} \right)^+.
\]

Dividing both sides by \( S_n \) and introducing ratio
\[
w_n = \frac{N_n}{S_n},
\]
we get
\[
1 = w_n + c_s \sum_{j=1}^d \left( \frac{b_n^{(j)} x_n^{(j)}}{\langle b_n, x_n \rangle} - b_\ast^{(j)} n w_n \right)^+ + c_p \sum_{j=1}^d \left( b_{n+1}^{(j)} N_n - \frac{b_n^{(j)} x_n^{(j)}}{\langle b_n, x_n \rangle} \right)^+.
\]

For given previous return vector \( x_n \) and portfolio vector \( b_n \), there is a portfolio vector \( \tilde{b}_{n+1} = \tilde{b}_{n+1}(b_n, x_n) \) for which there is no trading:
\[
\tilde{b}_{n+1}^{(j)} = \frac{b_n^{(j)} x_n^{(j)}}{\langle b_n, x_n \rangle}
\]
such that there is no transaction cost, i.e., \( w_n = 1 \).
Log-optimal portfolio selection strategies with proportional transaction costs

For arbitrary portfolio vectors \( \mathbf{b}_n, \mathbf{b}_{n+1} \), and return vector \( \mathbf{x}_n \) there exists a unique cost factor \( w_n \in [0,1] \), i.e. the portfolio is self financing. The value of cost factor \( w_n \) at day \( n \) is determined by portfolio vectors \( \mathbf{b}_n \) and \( \mathbf{b}_{n+1} \) as well as by return vector \( \mathbf{x}_n \), i.e.

\[
\begin{align*}
w_n &= w(\mathbf{b}_n, \mathbf{b}_{n+1}, \mathbf{x}_n),
\end{align*}
\]

for some function \( w \). If we want to rearrange our portfolio substantially, then our net wealth decreases more considerably, however, it remains positive. Note also, that the cost does not restrict the set of new portfolio vectors, i.e., the optimization algorithm searches for optimal vector \( \mathbf{b}_{n+1} \) within the whole simplex \( \Delta_d \). The value of the cost factor ranges between

\[
1 - c_s \quad \frac{1}{1 + c_p} \leq w_n \leq 1.
\]

Without loss of generality we consider the special case of \( c_s = c_p =: c \). Then

\[

c_s \left( b^{(j)}_n x^{(j)}_n N_{n-1} - b^{(j)}_{n+1} N_n \right)^+ + c_p \left( b^{(j)}_{n+1} N_n - b^{(j)}_n x^{(j)}_n N_{n-1} \right)^+
\]

\[
= c \left| b^{(j)}_n x^{(j)}_n N_{n-1} - b^{(j)}_{n+1} N_n \right|.
\]

Starting with an initial wealth \( S_0 = 1 \) and \( w_0 = 1 \), wealth \( S_n \) at the closing time of the \( n \)-th market day becomes

\[
\begin{align*}
S_n &= N_{n-1} \langle \mathbf{b}_n, \mathbf{x}_n \rangle \\
&= w_{n-1} S_{n-1} \langle \mathbf{b}_n, \mathbf{x}_n \rangle \\
&= \prod_{i=1}^n \left[ w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \langle \mathbf{b}_i, \mathbf{x}_i \rangle \right].
\end{align*}
\]

Introduce the notation

\[
g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}, \mathbf{x}_i) = \log(w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \langle \mathbf{b}_i, \mathbf{x}_i \rangle),
\]

then the average growth rate becomes

\[
\frac{1}{n} \log S_n = \frac{1}{n} \sum_{i=1}^n \log \left( w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \langle \mathbf{b}_i, \mathbf{x}_i \rangle \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^n g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}, \mathbf{x}_i).
\]

Our aim is to maximize this average growth rate.

In the sequel \( \mathbf{x}_i \) will be random variable and is denoted by \( \mathbf{X}_i \), and we assume the following

Conditions:

(i) \( \{ \mathbf{X}_i \} \) is a homogeneous and first order Markov process,
(ii) the Markov kernel is continuous, which means that for $\mu(B|x)$ being the Markov kernel defined by

$$\mu(B|x) := \mathbb{P}\{X_2 \in B \mid X_1 = x\}$$

we assume that the Markov kernel is continuous in total variation, i.e.,

$$V(x, x') := \sup_{B \in \mathcal{B}} |\mu(B|x) - \mu(B|x')| \to 0$$

if $x' \to x$ such that $\mathcal{B}$ denotes the family of Borel $\sigma$-algebra, further

$$V(x, x') < 1 \text{ for all } x, x',$$

(iii) and there exist $0 < a_1 < 1 < a_2 < \infty$ such that $a_1 \leq X^{(j)} \leq a_2$ for all $j = 1, \ldots, d$.

We note that Conditions (ii) and (iii) imply

$$\max_{x, x'} V(x, x') < 1. \quad (4)$$

For the usual stock market daily data, Condition (iii) is satisfied with $a_1 = 0.7$ and with $a_2 = 1.2$ (cf. Fernholz [8]).

Let’s use the decomposition

$$\frac{1}{n} \log S_n = I_n + J_n, \quad (5)$$

where $I_n$ is

$$I_n = \frac{1}{n} \sum_{i=1}^{n} \{g(b_{i-1}, b_i, X_{i-1}, X_i) - \mathbb{E}\{g(b_{i-1}, b_i, X_{i-1}, X_i) \mid X_{i-1}^{i-1}\}\}$$

and

$$J_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{g(b_{i-1}, b_i, X_{i-1}, X_i) \mid X_{i-1}^{i-1}\}. $$

$I_n$ is an average of martingale differences. Under the condition (iii), the random variable $g(b_{i-1}, b_i, X_{i-1}, X_i)$ is bounded, therefore $I_n$ is an average of bounded martingale differences, which converges to 0 almost surely, since according to the Chow Theorem (cf. Theorem 3.3.1 in Stout [28])

$$\sum_{i=1}^{\infty} \frac{\mathbb{E}\{g(b_{i-1}, b_i, X_{i-1}, X_i)^2\}}{i^2} < \infty$$

implies that

$$I_n \to 0$$

almost surely. Thus, the asymptotic maximization of the average growth rate $\frac{1}{n} \log S_n$ is equivalent to the maximization of $J_n$. 
Under the condition (i), we have that
\[
\mathbb{E}\{g(b_{i-1}, b_i, X_{i-1}, X_i) | X_i^{i-1}\} = \mathbb{E}\{(\log w(b_{i-1}, b_i, X_{i-1}) \langle b_i, X_i \rangle) | X_i^{i-1}\} = \log w(b_{i-1}, b_i, X_{i-1}) + \mathbb{E}\{\log \langle b_i, X_i \rangle | b_i, X_i^{i-1}\}
\]
def\(= v(b_{i-1}, b_i, X_{i-1})\),
therefore the maximization of the average growth rate \(\frac{1}{n} \log S_n\) is asymptotically equivalent to the maximization of
\[
J_n = \frac{1}{n} \sum_{i=1}^{n} v(b_{i-1}, b_i, X_{i-1}).
\]

The terms in the average \(J_n\) have a memory, which transforms the problem into a dynamic programming setup (cf. Merhav, Ordentlich, Seroussi and Weinberger [19]).

3. Growth optimal portfolio selection algorithms

Györfi and Vajda [13] studied the following two optimal portfolio selection rules. Let \(0 < \delta < 1\) denote a discount factor. Let the discounted Bellman equation be defined as follows:
\[
F_\delta(b, x) = \max_{b'} \{ v(b, b', x) + (1 - \delta)\mathbb{E}\{F_\delta(b', X_2) | X_1 = x\} \}.
\]

One can show that this discounted Bellman equation (7) and also the more general Bellman equation (10) below, have a unique solution (cf. Schäfer [25] and the proof of Proposition 1 below). Concerning the Bellman equation (7), the so-called Value Iteration may result in the solution: for fixed \(0 < \delta < 1\), put
\[
F_{\delta,0} = 0
\]
and
\[
F_{\delta,k+1}(b, x) = \max_{b'} \{ v(b, b', x) + (1 - \delta_k)\mathbb{E}\{F_{\delta,k}(b', X_2) | X_1 = x\} \},
\]
for \(k = 0, 1, \ldots\). Then Banach’s fixed point theorem implies that the value iteration converges uniformly to the unique solution.

**Rule 1.** Schäfer [25] introduced the following non-stationary rule. Put
\[
\bar{b}_1 = \{1/d, \ldots, 1/d\}
\]
and
\[
\bar{b}_{i+1} = \arg \max_{b'} \{ v(\bar{b}_i, b', X_i) + (1 - \delta_i)\mathbb{E}\{F_{\delta_i}(b', X_{i+1}) | X_i\} \},
\]
for \(1 \leq i\), where \(0 < \delta_i < 1\) is a discount factor such that \(\delta_i \downarrow 0\). Schäfer [25] proved that for the conditions (i), (ii) (in a weakened form) and (iii) and under some mild conditions
on $\delta_i$’s for Rule 1, the portfolio $\{\tilde{b}_i\}$ with capital $\tilde{S}_n$ is optimal in the sense that for any portfolio strategy $\{b_i\}$ with capital $S_n$,

$$\liminf_{n \to \infty} \left( \frac{1}{n} \mathbb{E}\{ \log \tilde{S}_n \} - \frac{1}{n} \mathbb{E}\{ \log S_n \} \right) \geq 0.$$ 

Győrfi and Vajda [13] extended this optimality in expectation to pathwise optimality such that under the same conditions

$$\liminf_{n \to \infty} \left( \frac{1}{n} \log \tilde{S}_n - \frac{1}{n} \log S_n \right) \geq 0$$

a.s.

**Rule 2.** Győrfi and Vajda [13] introduced a portfolio with stationary (time invariant) recursion. For any integer $1 \leq k$, put

$$b_{i}^{(k)} = \{1/d, \ldots, 1/d\}$$

and

$$b_{i+1}^{(k)} = \arg \max_{b'} \{ v(b_{i}^{(k)}, b', X_i) + (1 - \delta_k) \mathbb{E}\{ F_{b_{i}^{(k)}, X_{i+1}} | X_{i} \} \},$$

for $1 \leq i$. The portfolio $B^{(k)} = \{b_{i}^{(k)}\}$ is called the portfolio of expert $k$ with capital $S_n(B^{(k)})$. Choose an arbitrary probability distribution $q_k > 0$, and introduce the combined portfolio with its capital

$$\tilde{S}_n = \sum_{k=1}^{\infty} q_k S_n(B^{(k)}).$$

Győrfi and Vajda [13] proved that under the above mentioned conditions, for Rule 2,

$$\lim_{n \to \infty} \left( \frac{1}{n} \log \tilde{S}_n - \frac{1}{n} \log \tilde{S}_n \right) = 0$$

a.s. Notice that maybe non of the averaged growth rates $\frac{1}{n} \log \tilde{S}_n$ and $\frac{1}{n} \log \tilde{S}_n$ are convergent to a constant, since we didn’t assume the ergodicity of $\{X_i\}$.

Next we introduce further portfolio selection rules. Introduce the (non-discounted) Bellman equation:

$$W_c^* + F(b, x) = \max_{b'} \left\{ v(b, b', x) + \mathbb{E}\{ F(b', X_2) | X_1 = x \} \right\}.$$  \hspace{1cm} (8)

According to Proposition 1 below such a solution $(W_c^*, F)$ exists, where $W_c^* \in \mathbb{R}$ is unique according to Proposition 2 below.

**Rule 3.** Introduce a stationary rule such that put

$$b_{1}^{*} = \{1/d, \ldots, 1/d\}$$

and

$$b_{i+1}^{*} = \arg \max_{b'} \{ v(b_{i}^{*}, b', X_i) + \mathbb{E}\{ F(b', X_{i+1}) | X_{i} \} \}. \hspace{1cm} (9)$$
Theorem 1. Under the Conditions (i), (ii) and (iii), if $S_n^*$ denotes the wealth at period $n$ using the portfolio $\{b_n^*\}$ then
\[
\lim_{n \to \infty} \frac{1}{n} \log S_n^* = W_c^*
\]
a.s., while if $S_n$ denotes the wealth at period $n$ using any other portfolio $\{b_n\}$ then
\[
\limsup_{n \to \infty} \frac{1}{n} \log S_n \leq W_c^*
\]
a.s.

Remark 1. For i.i.d. market process, Iyengar [14], [15] observed that even in discrete time setup there is no trading with positive probability, i.e.,
\[
\mathbb{P}\{\tilde{b}_{n+1}(b_n^*, X_n) = b_{n+1}^*\} > 0,
\]
where the no-trading portfolio $\tilde{b}_{n+1}$ has been defined by (2). Moreover, one may get an approximately optimal selection rule, if $b_{n+1}^*$ is restricted on an appropriate neighborhood of $\tilde{b}_{n+1}(b_n^*, X_n)$.

Remark 2. The problem is more simple if the market process is i.i.d. Then, on the one hand $v$ has the form
\[
v(b, b', x) = \log w(b, b', x) + \mathbb{E}\{\log \langle b', X_2 \rangle \mid X_1 = x\}
= \log w(b, b', x) + \mathbb{E}\{\log \langle b', X_2 \rangle\},
\]
while the Bellman equation (8) looks like as follows:
\[
W_c^* + F(b, x) = \max_{b'} \{v(b, b', x) + \mathbb{E}\{F(b', X_2) \mid X_1 = x\}\}
= \max_{b'} \{v(b, b', x) + \mathbb{E}\{F(b', X_2)\}\}.
\]
This problem was studied by Iyengar [14], [15]. As to Theorem 1, also conditional expectation in context of $F$ in (9) simplifies to expectation, and its proof shows that the last assumption in Condition (ii) can be omitted. For Theorem 2 the analogue holds.

Remark 3. Use of portfolio $b_n^*$ in Theorem 1 needs a solution of the non-discounted Bellman equation (8). For this, an iteration procedure is given in Lemma 2 below.

Remark 4. In practice, the conditional expectations are unknown and they can be replaced by estimates. It’s an open problem what is the loss in growth rate if we apply estimates in the Bellman equation
\[
W_c^* + F(b, x) = \max_{b'} \{\log w(b, b', x) + \mathbb{E}\{\log \langle b', X_2 \rangle \mid X_1 = x\}\}
+ \mathbb{E}\{F(b', X_2) \mid X_1 = x\}\}. 
\]
A modification of Rule 3 allows to avoid solving the Bellman equation (8), but yields the corresponding convergence result.

**Rule 4.** Choose a sequence $0 < \delta_n < 1$, $n = 1, 2, \ldots$ such that

$$
\delta_n \downarrow 0, \quad \sum_n \delta_n = \infty, \quad \frac{\delta_{n+1}}{\delta_n} \to 1 \quad (n \to \infty),
$$

e.g., $\delta_n = \frac{1}{n+1}$. Set

$$
F'_1 := 0,
$$
and iterate

$$
F'_{n+1} := M_{\delta_n} F'_n - \max_{b,x} (M_{\delta_n} F'_n)(b,x) \quad (n = 1, 2, \ldots)
$$

with

$$
(M_{\delta_n} F'_n)(b,x) := \max_b \left\{ v(b, \tilde{b}, x) + (1 - \delta_n) \mathbb{E}\{F'_n(\tilde{b}, X_2) \mid X_1 = x\} \right\}.
$$

Put

$$
b'_1 = \{1/d, \ldots, 1/d\}
$$

and

$$
b'_{i+1} = \arg\max_{\tilde{b}} \left\{ v(b'_i, \tilde{b}, X_i) + (1 - \delta_i) \mathbb{E}\{F'_i(\tilde{b}, X_{i+1}) \mid X_i\} \right\},
$$

for $1 \leq i$. This non-stationary rule can be interpreted as a combination of the value iteration and Rule 1.

**Theorem 2** Under the Conditions (i), (ii) and (iii), if $S'_n$ denotes the wealth at period $n$ using the portfolio $\{b'_n\}$ then

$$
\lim_{n \to \infty} \frac{1}{n} \log S'_n = W_c^* 
$$

a.s.

Note that according to Theorem 1, if $S_n$ denotes the wealth at period $n$ using any portfolio $\{b_n\}$ then

$$
\limsup_{n \to \infty} \frac{1}{n} \log S_n \leq W_c^* 
$$

a.s.

### 4. Proofs

We split the statement of Theorem 1 into two propositions.

**Proposition 1** Under the Conditions (i), (ii) and (iii) the Bellman equation (8) has a solution $(W_c^*, F)$ such that the function $F$ is bounded and continuous, where

$$
\max_{b,x} F(b, x) = 0.
$$
Thus, by Banach’s fixed point theorem, the Bellman equation

\[(M_{\delta} g)(b, x) := \max_{b'} \left\{ v(b, b', x) + (1 - \delta) \mathbb{E}\{g(b', X_2) \mid X_1 = x\} \right\}. \tag{10}\]

By continuity assumption (ii) this leads to an operator

\[M_{\delta} : C \to C.\]

(See Schäfer [25] p.114.)

The operator \(M_{\delta}\) is continuous, even Lipschitz continuous with Lipschitz constant \(1 - \delta\). Indeed, for \(g, g' \in C\) from the representation

\[(M_{\delta} g')(b, x) \geq v(b, b^*_g(b, x), x) + (1 - \delta) \mathbb{E}\{g'(b^*_g(b, x), X_2) \mid X_1 = x\}\]

and from the corresponding representation of \((M_{\delta} g')(b, x)\) one obtains

\[
|M_{\delta} g - M_{\delta} g'|\|_\infty \leq (1 - \delta) \|g - g'\|_\infty.
\]

for all \((b, x) \in \Delta_d \times [a_1, a_2]^d\), therefore

\[
\|M_{\delta} g - M_{\delta} g'|\|_\infty \leq (1 - \delta) \|g - g'\|_\infty.
\]

Thus, by Banach’s fixed point theorem, the Bellman equation

\[
\lambda + F(b, x) = \max_{b'} \left\{ v(b, b', x) + (1 - \delta) \mathbb{E}\{F(b', X_2) \mid X_1 = x\} \right\}, \tag{11}\]

i.e.,

\[
\lambda + F = M_{\delta} F
\]

with \(\lambda \in \mathbb{R}\), has a unique solution if \(0 < \delta < 1\). (11) corresponds to (7) for \(\lambda = 0\), \(0 < \delta < 1\) with the unique solution denoted by \(F_{\delta}\), and to (8) for \(\lambda = W_{c}^*\) and \(\delta = 0\).

We notice

\[
\sup_{0<\delta<1} \delta \|F_{\delta}\|_\infty \leq \max_{b, b', x} |v(b, b', x)| < \infty,
\]

(see Schäfer [25], Lemma 4.2.3). Similarly to Iyengar [14], put

\[
m_{\delta} := \max_{(b, x)} F_{\delta}(b, x), \tag{12}\]

where we get that

\[
\sup_{0<\delta<1} \delta m_{\delta} < \infty.
\]

Put

\[
W_{c}^* := \lim_{\delta \downarrow 0} \delta m_{\delta}
\]
and
\[ \tilde{F}_\delta(b, x) := F_\delta(b, x) - m_\delta. \]  \hfill (13)

Thus,
\[ \max_{(b, x)} \tilde{F}_\delta(b, x) = 0. \]  \hfill (14)

\( \tilde{F}_\delta \) satisfies the Bellman equation (11) with \( \lambda = \delta m_\delta \), therefore
\[ \delta m_\delta + \tilde{F}_\delta = M_\delta \tilde{F}_\delta = M_0 \tilde{F}_\delta + (M_\delta \tilde{F}_\delta - M_0 \tilde{F}_\delta) \]  \hfill (15)

It is easy to check that
\[ \| M_\delta \tilde{F}_\delta - M_0 \tilde{F}_\delta \|_\infty \leq \delta \| \tilde{F}_\delta \|_\infty. \]  \hfill (16)

By Lemma 1 below
\[ \sup_{0 < \delta < 1} \| \tilde{F}_\delta \|_\infty < \infty. \]  \hfill (17)

Now we choose a sequence \( \delta_n \) with \( \delta_n \downarrow 0 \) such that
\[ \delta_n m_{\delta_n} \to W^*_c. \]  \hfill (18)

Lemma 1 further states that
\[ \sup_{0 < \delta < 1} | \tilde{F}_\delta(b, \bar{x}) - \tilde{F}_\delta(b, x) | \to 0 \]

(even uniformly with respect to \( (b, x) \)), because of compactness of \( \Delta_d \times [a_1, a_2]^d \) when \( (b, \bar{x}) \to (b, x) \), i.e., there is equicontinuity for \( \{ \tilde{F}_\delta \} \), which together with (17) implies that there exist a subsequence \( \delta_{n_l} \) and a function \( \tilde{F} \in C \) such that \( \tilde{F}_{\delta_{n_l}} \) converges in \( C \) to \( \tilde{F} \) (cf. Ascoli-Arzelà theorem, [31]). Thus, by continuity of \( M_0 \), we get the convergence of \( M_0 \tilde{F}_{\delta_{n_l}} \) in \( C \) to \( M_0 \tilde{F} \). Therefore
\[ W^*_c + \tilde{F} = M_0 \tilde{F}, \]

i.e., \( \tilde{F} \in C \) solves the Bellman equation (8). \( \tilde{F} \) is continuous on a compact set, therefore it is bounded, where
\[ \max_{b, x} \tilde{F}(b, x) = 0. \]

**Lemma 1.** If \( F_\delta \) denotes the solution of the discounted Bellman equation (7) then (17) holds and it implies that
\[ \sup_{0 < \delta < 1} | F_\delta(\bar{b}, \bar{x}) - F_\delta(b, x) | \to 0 \]  \hfill (19)

when \( (b, \bar{x}) \to (b, x) \).

**Proof.** We use the decomposition
\[ F_\delta(\bar{b}, \bar{x}) - F_\delta(b, x) = F_\delta(\bar{b}, \bar{x}) - F_\delta(b, \bar{x}) + F_\delta(b, \bar{x}) - F_\delta(b, x). \]
Concerning the first term in this decomposition we assumed that $F_{\delta}$ the solution of the discounted Bellman equation (7), therefore

\[
F_{\delta}(\bar{b}, \bar{x}) - F_{\delta}(b, x)
= \max_{b'} \{ v(\bar{b}, b', x) + (1 - \delta)E\{F_{\delta}(b', X_2) \mid X_1 = \bar{x} \} \} \\
- \max_{b''} \{ v(b, b'', x) + (1 - \delta)E\{F_{\delta}(b'', X_2) \mid X_1 = x \} \} \\
\leq \max_{b'} \{ v(\bar{b}, b', \bar{x}) + (1 - \delta)E\{F_{\delta}(b', X_2) \mid X_1 = \bar{x} \} \} \\
- (v(b, b', \bar{x}) + (1 - \delta)E\{F_{\delta}(b', X_2) \mid X_1 = \bar{x} \}) \\
= \max_{b'} \{ v(\bar{b}, b', \bar{x}) - v(b, b', \bar{x}) \},
\]

therefore

\[
\sup_{0 < \delta < 1} |F_{\delta}(\bar{b}, \bar{x}) - F_{\delta}(b, x)| \leq \max_{b'} |v(\bar{b}, b', \bar{x}) - v(b, b', \bar{x})| \rightarrow 0 \tag{20}
\]

when $(\bar{b}, \bar{x}) \rightarrow (b, x)$. Concerning the second term in this decomposition, we get that

\[
F_{\delta}(b, x) - F_{\delta}(b', x) = \max_{b'} \{ v(b, b', x) + (1 - \delta)E\{F_{\delta}(b', X_2) \mid X_1 = x \} \} \\
- \max_{b''} \{ v(b, b'', x) + (1 - \delta)E\{F_{\delta}(b'', X_2) \mid X_1 = x \} \} \\
\leq \max_{b'} \{ v(b, b', x) + (1 - \delta)E\{F_{\delta}(b', X_2) \mid X_1 = x \} \} \\
- (v(b, b', x) + (1 - \delta)E\{F_{\delta}(b', X_2) \mid X_1 = x \}) \\
\leq \max_{b'} \{ v(b, b', x) - v(b, b', x) \} \\
+(1 - \delta) \max \{ E\{F_{\delta}(b', X_2) \mid X_1 = \bar{x} \} \} \\
- E\{F_{\delta}(b', X_2) \mid X_1 = x \} \}
\]

Moreover

\[
E\{F_{\delta}(b', X_2) \mid X_1 = \bar{x} \} - \bar{F}_{\delta}(b', X_2) | X_1 = x \} \\
= E\{\tilde{F}_{\delta}(b', X_2) \mid X_1 = \bar{x} \} - E\{\tilde{F}_{\delta}(b', X_2) \mid X_1 = x \} \\
\leq \|\tilde{F}_{\delta}\|_{\infty} V(x, \bar{x}),
\]

where the function $V$ has been defined for Condition (ii). Thus,

\[
\sup_{0 < \delta < 1} |F_{\delta}(b, \bar{x}) - F_{\delta}(b, x)| \leq \max_{b'} |v(b, b', \bar{x}) - v(b, b', x)| + \sup_{0 < \delta < 1} \|\tilde{F}_{\delta}\|_{\infty} V(x, \bar{x}). \tag{21}
\]

The inequalities in (20) and (21) and boundedness of $g$ and thus also of $v$ (by Condition (iii)) yield

\[
\sup_{0 < \delta < 1} |F_{\delta}(\bar{b}, \bar{x}) - F_{\delta}(b, x)| \leq \text{const} + \sup_{0 < \delta < 1} \|\tilde{F}_{\delta}\|_{\infty} V(x, \bar{x}).
\]
for some \( \text{const} < \infty \). Noticing

\[
\sup_{(b, x), (\bar{b}, \bar{x})} |F_\delta(\bar{b}, \bar{x}) - F_\delta(b, x)| = \sup_{(b, x), (\bar{b}, \bar{x})} |\tilde{F}_\delta(\bar{b}, \bar{x}) - \tilde{F}_\delta(b, x)| = \|\tilde{F}_\delta\|_\infty
\]

(by (13) and (14)), we then obtain

\[
\sup_{0 < \delta < 1} \|\tilde{F}_\delta\|_\infty \leq \text{const} + \sup_{0 < \delta < 1} \|\tilde{F}_\delta\|_\infty \max_{x, \bar{x}} V(x, \bar{x})
\]

and thus (17) by (4). Condition (ii) and (17) yield that the right hand side of (21) converges to 0 when \((\bar{b}, \bar{x}) \to (b, x)\). Then (20) and (21) imply (19).

\[\blacksquare\]

Proposition 2 Assume that the Bellman equation (8) has a solution \((W^*_c, F)\) such that the function \(F\) is bounded. If \(S^*_n\) denotes the wealth at period \(n\) using the portfolio \(\{b^*_n\}\) then

\[
\lim_{n \to \infty} \frac{1}{n} \log S^*_n = W^*_c
\]
a.s., while if \(S_n\) denotes the wealth at period \(n\) using any other portfolio \(\{b_n\}\) then

\[
\limsup_{n \to \infty} \frac{1}{n} \log S_n \leq W^*_c
\]
a.s. These statements imply that \(W^*_c\) in the Bellman equation (8) is unique.

Proof. We have to show that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g(b^*_i, b^*_{i+1}, X_i, X_{i+1}) = W^*_c
\]
a.s. and

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g(b_i, b_{i+1}, X_i, X_{i+1}) \leq W^*_c
\]
a.s. Because of the martingale difference argument in Section 2, these two limit relations are equivalent to

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} v(b^*_i, b^*_{i+1}, X_i) = W^*_c
\]
a.s. and

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} v(b_i, b_{i+1}, X_i) \leq W^*_c
\]
a.s. (8) and (9) imply that

\[
W^*_c + F(b^*_i, X_i) = v(b^*_i, b^*_{i+1}, X_i) + \mathbb{E}\{F(b^*_{i+1}, X_{i+1}) \mid b^*_{i+1}, X_i\}, \tag{22}
\]
while for any portfolio \(\{b_i\}\),

\[
W^*_c + F(b_i, X_i) \geq v(b_i, b_{i+1}, X_i) + \mathbb{E}\{F(b_{i+1}, X_{i+1}) \mid b_{i+1}, X_i\}. \tag{23}
\]
Because of (22), we get that
\[
\frac{1}{n} \sum_{i=1}^{n} v(b^*_i, b^*_i, X_i) = W^*_c + \frac{1}{n} \sum_{i=1}^{n} (F(b^*_i, X_i) - \mathbb{E}\{F(b^*_i, X_{i+1}) \mid X_i\})
\]
\[
= W^*_c + \frac{1}{n} \sum_{i=1}^{n} F(b^*_i, X_i) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{F(b^*_i, X_{i+1}) \mid X_i\}
\]
\[
= W^*_c + \frac{1}{n} \sum_{i=1}^{n} (F(b^*_i, X_i) - \mathbb{E}\{F(b^*_i, X_{i+1}) \mid X_i\})
\]
\[
+ \frac{1}{n} (F(b^*_1, X_1) - \mathbb{E}\{F(b^*_1, X_{n+1}) \mid X_1\}).
\]

By the condition of Theorem 1, the function \(F\) is bounded, therefore the Chow theorem can be applied for martingale differences, and so
\[
\frac{1}{n} \sum_{i=1}^{n} v(b^*_i, b^*_i, X_i) \rightarrow W^*_c
\]
a.s. Similarly, because of (23), we get that
\[
\frac{1}{n} \sum_{i=1}^{n} v(b_i, b_{i+1}, X_i)
\]
\[
\leq W^*_c + \frac{1}{n} \sum_{i=1}^{n} (F(b_i, X_i) - \mathbb{E}\{F(b_{i+1}, X_{i+1}) \mid b_{i+1}, X_i\})
\]
\[
= W^*_c + \frac{1}{n} \sum_{i=1}^{n} F(b_i, X_i) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{F(b_{i+1}, X_{i+1}) \mid X_i\}
\]
\[
\rightarrow W^*_c
\]
a.s.
\]

**Corollary 1** Assume the conditions of Proposition 1 and let \(m_\delta\) defined by (12). Then
\[
\delta m_\delta \rightarrow W^*_c \text{ as } \delta \downarrow 0.
\]

**Proof.** Since in the proof of Proposition 1 \(\lim \sup_{\delta \downarrow 0} \delta m_\delta\) can be replaced by \(\lim \inf_{\delta \downarrow 0} \delta m_\delta\), uniqueness of \(W^*_c\) yields existence of \(\lim_{\delta \downarrow 0} \delta m_\delta = W^*_c\). For each sequence \(\delta_n \downarrow 0\) a subsequence \(\delta_n\) exists such that \(\tilde{F}_{\delta_n}\) converges in \(C\) to some solution \(F\) of (8). This proves the second assertion.
Lemma 2 Assume Conditions (i), (ii) and (iii). Let \( \delta_n \) and \( F'_n \) be as in Rule 4. Then \( F'_n \) converges in \( C \) to a set of solutions \( F \) of the Bellman equation (8), further

\[
w_n := \max_{\mathbf{b}, \mathbf{x}} (M_{\delta_n} F'_n)(\mathbf{b}, \mathbf{x}) \rightarrow W^*_c \text{ as } n \rightarrow \infty.
\]

Proof. We can write

\[
F'_{n+1} = M_{\delta_n} F'_n - w_n
\]

with the continuous operator \( M_{\delta_n} : C \rightarrow C \) according to (10). It holds

\[
|F'_{n+1}(\mathbf{b}, \mathbf{x}) - F'_{n+1}(\mathbf{b}, \mathbf{x})| = |(M_{\delta_n} F'_n)(\mathbf{b}, \mathbf{x}) - (M_{\delta_n} F'_n)(\mathbf{b}, \mathbf{x})| \\
\leq |(M_{\delta_n} F'_n)(\mathbf{b}, \mathbf{x}) - (M_{\delta_n} F'_n)(\mathbf{b}, \mathbf{x})| \\
+ |(M_{\delta_n} F'_n)(\mathbf{b}, \mathbf{x}) - (M_{\delta_n} F'_n)(\mathbf{b}, \mathbf{x})| \\
\leq \max_{\mathbf{b}' \mathbf{x}} |v(\mathbf{b}, \mathbf{b}', \mathbf{x}) - v(\mathbf{b}, \mathbf{b}', \mathbf{x})| \\
+ \max_{\mathbf{b}' \mathbf{x}} |v(\mathbf{b}, \mathbf{b}', \mathbf{x}) - v(\mathbf{b}, \mathbf{b}', \mathbf{x})| \\
+ \max_{\mathbf{x}, \mathbf{x}'} V(\mathbf{x}, \mathbf{x}') \| F'_n \|_{\infty},
\]

where the inequalities are obtained as in the proof of Lemma 1. Noticing

\[
\max_{\mathbf{b}, \mathbf{x}} F'_n(\mathbf{b}, \mathbf{x}) = 0
\]

and thus

\[
\max_{(\mathbf{b}, \mathbf{x}), (\mathbf{b}, \mathbf{x})} |F'_n(\mathbf{b}, \mathbf{x}) - F'_n(\mathbf{b}, \mathbf{x})| = \| F'_n \|_\infty,
\]

moreover, the boundedness of \( v \) implies that

\[
\| F'_n \|_\infty \leq \text{const} + \max_{\mathbf{x}, \mathbf{x}'} V(\mathbf{x}, \mathbf{x}') \| F'_n \|_\infty
\]

with \( \text{const} < \infty \). Then, by induction,

\[
\| F'_n \|_\infty \leq \frac{\text{const}}{1 - \max_{\mathbf{x}, \mathbf{x}'} V(\mathbf{x}, \mathbf{x})} =: K < \infty.
\]

It can be easily checked that

\[
\| M_{\delta_{n+1}} F'_{n+1} - M_{\delta_n} F'_{n+1} \|_\infty \leq (\delta_n - \delta_{n+1}) \| F'_{n+1} \|_\infty.
\]

According to the proof of Proposition 1, the operator \( M_{\delta_n} \) is Lipschitz continuous with Lipschitz constant \( 1 - \delta_n \). Then

\[
\| F'_{n+2} - F'_{n+1} \|_\infty \\
= \| M_{\delta_{n+1}} F'_{n+1} - M_{\delta_n} F'_{n} \|_\infty \\
\leq \| M_{\delta_{n+1}} F'_{n+1} - M_{\delta_n} F'_{n} \|_\infty + \| M_{\delta_{n+1}} F'_{n+1} - M_{\delta_n} F'_{n} \|_\infty \\
\leq (1 - \delta_n) \| F'_{n+1} - F'_{n} \|_\infty + \left( 1 - \frac{\delta_{n+1}}{\delta_n} \right) \delta_n K.
\]
By the condition on $\delta_n$, we then obtain
\[
\|F'_{n+1} - F'_n\|_\infty \to 0 \text{ as } n \to \infty,
\tag{28}
\]
(cf. Lemma 1(c) in Walk and Zsidó [30]). Now let $(\delta_{n_k})$ be an arbitrary subsequence of $(\delta_n)$. From (25) and (26) and Condition (ii) we obtain
\[
\sup_i |F'_i(\bar{b}, \bar{x}) - F'_i(b, x)| \to 0
\]
when $(\bar{b}, \bar{x}) \to (b, x)$, even uniformly with respect to $(b, x)$. This together with (26) yields existence of a subsequence $(\delta_{n_{k\ell}})$ and of a function $\bar{F} \in C$ (bounded, where $\max_{b, x} \bar{F}(b, x) = 0$) such that
\[
\|F'_{n_{k\ell}} - \bar{F}\|_\infty \to 0 \text{ as } \ell \to \infty.
\tag{29}
\]
Thus, by continuity of $M_0$,
\[
\|M_0F'_{n_{k\ell}} - M_0\bar{F}\|_\infty \to 0 \text{ as } \ell \to \infty.
\tag{30}
\]
By (24),
\[
F'_{n_{k\ell}} + (F'_{n_{k\ell}+1} - F'_{n_{k\ell}}) = M_0F'_{n_{k\ell}} + (M_{\delta_{n_{k\ell}}}F'_{n_{k\ell}} - M_0F'_{n_{k\ell}}) - w_{n_{k\ell}}.
\]
(28) implies that
\[
\|F'_{n_{k\ell}+1} - F'_{n_{k\ell}}\|_\infty \to 0.
\]
By (16) and (26),
\[
\|M_{\delta_{n_{k\ell}}}F'_{n_{k\ell}} - M_0F'_{n_{k\ell}}\|_\infty \leq \delta_{n_{k\ell}}K \to 0.
\]
This together with (29) and (30) yields convergence of $(w_{n_{k\ell}})$ and
\[
\lim_{\ell} w_{n_{k\ell}} + \bar{F} = M_0\bar{F}.
\]
This means that $\bar{F}$ solves the Bellman equation (8) such that $\lim_{\ell} w_{n_{k\ell}} = W^*_c$ (unique by Proposition 2). This convergence results yield the assertion. \hfill \blacksquare

**Proof of Theorem 2.** According to Proposition 2 and its proof it is enough to show
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} v(b'_i, b'_{i+1}, X_i) = W^*_c
\tag{31}
\]
a.s. Rule 4 yields
\[
\begin{align*}
& w_n + F'_{n+1}(b'_n, X_n) \\
& = v(b'_n, b'_{n+1}, X_n) + (1 - \delta_n)\mathbb{E}\{F'_n(b'_{n+1}, X_{n+1}) | b'_{n+1}, X_n\},
\end{align*}
\]
where
\[
w_n = \max_{b, x}(M_{\delta_n}F'_n)(b, x).
\]
Then

\[
\frac{1}{n} \sum_{i=1}^{n} v(b'_i, b'_{i+1}, X_i) = \frac{1}{n} \sum_{i=1}^{n} w_i + \frac{1}{n} \sum_{i=1}^{n} (F'_{i+1}(b'_i, X_i) - (1 - \delta_i) \mathbb{E}\{F'_i(b'_{i+1}, X_{i+1}) \mid b'_{i+1}, X_i\}) \\
\quad - \left\{ \frac{1}{n} \sum_{i=1}^{n} (F'_i(b'_i, X_i) - F'_i(b'_{i+1}, X_{i+1})) \right\} \\
+ \left[ \frac{1}{n} \sum_{i=1}^{n} \delta_i \mathbb{E}\{F'_i(b'_{i+1}, X_{i+1}) \mid X_i\} \right]
\]

\[
= A_n + B_n + C_n.
\]

By Lemma 2, \(A_n \to W^*_c\). By (26) and Chow’s theorem \(B_n \to 0\) a.s. Further

\[
|C_n| \leq \frac{1}{n} \left| \sum_{i=1}^{n-1} (F'_{i+2}(b'_i, X_i) - F'_i(b'_i, X_i)) \right| + \frac{1}{n} |F'_2(b'_1, X_1)| + \frac{1}{n} |F'_n(b'_n+1, X_{n+1})| + \frac{1}{n} \sum_{i=1}^{n} \delta_i K
\]

\[
\to 0
\]

by (26) and (28) and \(\delta_n \to 0\). This (31) is obtained.

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**References**


