Analysis of semi-log-optimal investment strategies

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Abstract: We introduce a sequential investment strategy, called semi-log-optimal strategy. This strategy is related to the log-optimal portfolio approach, where instead of logarithmic objective function its Taylor series approximation is used. The asymptotic rate of growth is analyzed under the only assumption that the market is stationary and ergodic. The performance of the strategy is compared to the optimal asymptotic rate of growth provided by the log-optimal strategy. Bound on the deviation of the performances is shown. The advantage of our semi-log-optimal portfolio approach is that it performs very close to the log-optimal strategy, meanwhile it allows a simpler and more "standardized" computation of the corresponding portfolio vector.

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1 Introduction

The purpose of this paper is to investigate sequential investment strategies for financial markets. Investment strategies are allowed to use information collected from the past of the market and determine, at the beginning of a trading period, a portfolio, that is, a way to distribute their current capital among the available assets. The goal of the investor is to maximize his wealth on the long run without knowing the underlying distribution generating the stock prices. The only assumption we use in our mathematical analysis is that the daily price relatives form a stationary and ergodic process. Under this assumption the asymptotic rate of growth has a well-defined maximum which can be achieved in full knowledge of the distribution of the entire process, see Algoet and Cover [3]. In this paper new strategy, called semi-log-optimal strategy, is proposed which guarantees an almost optimal asymptotic growth rate of capital for all stationary and ergodic markets, and uses only the conditional first and second moments of the market vectors, therefore it has small computational complexity.

To make the analysis feasible, some simplifying assumptions are used that need to be taken into account. First of all, we assume that assets are arbitrarily divisible and all assets are available in unbounded quantities at the current price at any given trading period. We also ignore transaction costs in the mathematical analysis. Another key assumption is that the behavior of the market is not affected by the actions of the investor using the strategy under investigation. This assumption is realistic when the investor handles small amounts of capital compared to the total trading volume on the market. Under this hypothesis, testing the methods on past stock-market data is meaningful.

The rest of the paper is organized as follows. In Section 2 the mathematical model is described, and related results are surveyed briefly. In Section 3 the new semi-log-optimal sequential investment strategies is introduced and the performance of the strategy is compared to the optimal asymptotic rate of growth provided by the log-optimal strategy. Bounds on the deviation of the performance is shown.

2 Setup, the log-optimal strategy

The model of stock market investigated in this paper is the one considered, among others, by Breiman [5], Algoet and Cover [3]. Consider a market of d assets. A *market vector* $\mathbf{x} = (x^{(1)}, \dots x^{(d)}) \in R_+^d$ is a vector of d nonnegative numbers representing price relatives for a given trading period. That is, the j-th component $x^{(j)} \geq 0$ of $\mathbf{x}$ expresses the ratio of the closing and opening prices of asset j. In other words, $x^{(j)}$ is the factor by which capital invested in the j-th asset grows during the trading period.

The investor is allowed to diversify his capital at the beginning of each trading period according to a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})$. The j-th component $b^{(j)}$ of $\mathbf{b}$ denotes the proportion of the investor's capital invested in asset j. Throughout the paper we assume that the portfolio vector $\mathbf{b}$ has nonnegative components with $\sum_{j=1}^d b^{(j)} = 1$. The fact that $\sum_{j=1}^d b^{(j)} = 1$ means that the investment strategy is self financing and consumption of capital is excluded. The non-negativity of the components of $\mathbf{b}$ means that short selling and buying stocks on margin are not permitted. Let S_0 denote the investor's initial capital. Then at the end of the trading period the investor's wealth becomes

$$S_1 = S_0 \sum_{j=1}^d b^{(j)} x^{(j)} = S_0 < \mathbf{b}, \mathbf{x} >,$$

where $< \cdot, \cdot >$ denotes inner product.

The evolution of the market in time is represented by a sequence of market vectors $\mathbf{x}_1, \mathbf{x}_2, \dots \in R_+^d$, where the j-th component $x_i^{(j)}$ of $\mathbf{x}_i$ denotes the amount obtained after investing a unit capital in the j-th asset on the i-th trading period. For $j \leq i$ we abbreviate by $\mathbf{x}_j^i$ the array of market vectors $(\mathbf{x}_j, \dots, \mathbf{x}_i)$ and denote by Δ_d the simplex of all vectors $\mathbf{b} \in R_+^d$ with nonnegative components summing up to one. An *investment strategy* is a sequence $\mathbf{B}$ of functions

$$\mathbf{b}_i : \left(R_+^d\right)^{i-1} \to \Delta_d \ , \qquad i = 1, 2, \dots$$

so that $\mathbf{b}_i(\mathbf{x}_1^{i-1})$ denotes the portfolio vector chosen by the investor on the i-th trading period, upon observing the past behavior of the market. We write $\mathbf{b}(\mathbf{x}_1^{i-1}) = \mathbf{b}_i(\mathbf{x}_1^{i-1})$ to ease the notation.

Starting with an initial wealth S_0, after n trading periods, the investment strategy $\mathbf{B}$ achieves the wealth

$$S_n = S_0 \prod_{i=1}^{n} < \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_i >= S_0 e^{\sum_{i=1}^{n} \log < \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_i >} = S_0 e^{n W_n(\mathbf{B})}.$$

where $W_n(\mathbf{B})$ denotes the *average growth rate*

$$W_n(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^{n} \log < \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_i > .$$

In this paper we assume that the market vectors are realizations of a random process, and describe a statistical model. Our view is completely nonparametric in that the only assumption we use is that the market is stationary and ergodic, allowing arbitrarily complex distributions. More precisely, assume that $\mathbf{x}_1, \mathbf{x}_2, \ldots$ are realizations of the random vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots$ drawn from the vector-valued stationary and ergodic process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$. The sequential investment problem, under these conditions, have been considered by, e.g., Breiman [5], Algoet and Cover [3], Algoet [1, 2], Györfi and Schäfer [8], Györfi, Lugosi, Udina [7]. The fundamental limits, determined in [3], [1, 2], reveal that the so-called *log-optimum portfolio* $\mathbf{B}^* = \{\mathbf{b}^*(\cdot)\}$ is the best possible choice. More precisely, on trading period n let $\mathbf{b}^*(\cdot)$ be such that

$$\mathbf{b}^*(\mathbf{X}_1^{n-1}) = arg \max_{\mathbf{b}(\cdot)} E \left\{ \log < \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n > \middle| \, \mathbf{X}_1^{n-1} \right\}.$$

If $S_n^* = S_n(\mathbf{B}^*)$ denotes the capital achieved by a log-optimum portfolio strategy $\mathbf{B}^*$, after n trading periods, then for any other investment strategy $\mathbf{B}$ with capital $S_n = S_n(\mathbf{B})$ and for any stationary and ergodic process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{S_n}{S_n^*} \le 0 \quad \text{almost surely}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log S_n^* = W^* \quad \text{almost surely,}$$

where

$$W^* = E \left\{ \log < \mathbf{b}^*(\mathbf{X}_{-\infty}^{-1}), \mathbf{X_0} > \right\}$$

is the maximal possible growth rate of any investment strategy.

Thus, (almost surely) no investment strategy can have a faster rate of growth than a log-optimal portfolio. Of course, to determine a log-optimal portfolio, full knowledge of the (infinite-dimensional) distribution of the process is required. Strategies achieving the same rate of growth without knowing the distribution are called *universally consistent*, i.e., an investment strategy $\mathbf{B}$ is called universally consistent with respect to a class of stationary and ergodic processes $\{\mathbf{X}_n\}_{-\infty}^{\infty}$, if for each process in the class,

$$\lim_{n\to\infty} \frac{1}{n} \log S_n(\mathbf{B}) = W^* \quad \text{almost surely.}$$

The surprising fact that there exists a strategy, universally consistent with respect to the class of all stationary and ergodic processes with $E|\log X^{(j)}| < \infty$ for all $j = 1, \dots, d$, was first proved by Algoet [1] and by Györfi and Schäfer [8].

3 The semi-log-optimal strategy

Put

$$h(x) = (x-1) - \frac{1}{2}(x-1)^2,$$

which is the second order Taylor expansion of the function $\log x$ at $x = 1$. On the n-th trading period the semi-log-optimal portfolio selection is defined by

$$\bar{\mathbf{b}}(\mathbf{X}_1^{n-1}) = arg \max_{\mathbf{b}(\cdot)} E\left\{ h\left(< \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n >\right) \middle| \mathbf{X}_1^{n-1} \right\}.$$

and $\bar{S}_n = S_n(\bar{\mathbf{B}})$.

Theorem 3.1. *For any stationary and ergodic process $\{\mathbf{X_n}\}_{-\infty}^{\infty}$, for which $1-a \leq X_n^j \leq 1+c$, $0.4 > a > 0$, $c > 0$ we get*

$$W^* \geq \liminf_n \frac{1}{n} \log \bar{S}_n \geq W^* - \frac{5}{6} E[\max_i E(|X_0^{(i)} - 1|^3 | \boldsymbol{X}_{-\infty}^{-1})].$$

with probability 1.

The practical importance of Theorem 3.1 can be grasped by the following consideration. On financial markets, where assets are traded on daily base, especially on stock markets, limits are set up for the maximal daily percentage change in traded assets. If this maximum is reached, e.g. the price drops sharply within a day, then the trading of the actual asset is suspended for the rest of the day. For instance, if we assume a very practical limit of 10%, then our result says, that our semi-log-optimal strategy performs within $5/6 \cdot 0.1^3 \simeq 0.083\%$ to the one of the log-optimal strategy.

In the proof of the theorem we apply the following lemma:

Lemma 3.2. *For any stationary and ergodic process $\{\mathbf{X_n}\}_{-\infty}^{\infty}$, for which $1-a \leq X_n^{(j)} \leq 1+c$, $1 > a > 0$, $c > 0$ we get*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \max_j E\left(|X_i^{(j)} - 1|^3 | \boldsymbol{X}_1^{i-1}\right) = E\left[\max_j E\left(|X_0^{(j)} - 1|^3 | \boldsymbol{X}_{-\infty}^{-1}\right)\right]$$

with probability 1.

Proof. Introduce notation

$$\overline{w}_n = \max_j E\left(|X_0^{(j)} - 1|^3 | \mathbf{X}_{-n+1}^{-1}\right)$$

where $n = 1, 2 \ldots$. Put

$$g(\mathbf{X}_n) = (|X_n^{(1)} - 1|^3, ..., |X_n^{(d)} - 1|^3).$$

Note that

$$\max_{\mathbf{b}(.)} E[< \mathbf{b}(\mathbf{X}_{-n+1}^{-1}), g(\mathbf{X}_0) > | \mathbf{X}_{-n+1}^{-1}] = \overline{w}_n \tag{1}$$

First we show that $\{\overline{w}_n\}$ is a sub-martingale. Random variable $\overline{w}_n$ is measurable with respect to $\mathbf{X}_{-n+1}^{-1}$. We have to prove, that $E[\overline{w}_{n+1} | \mathbf{X}_{-n+1}^{-1}] \geq \overline{w}_n$. If a portfolio is $\mathbf{X}_{-n+1}^{-1}$-measurable, then it is also $\mathbf{X}_{-n}^{-1}$-measurable, therefore we get

$$\begin{aligned}
\bar{w}_n &= E[\bar{w}_n | \mathbf{X}_{-n+1}^{-1}] \\
&= E[\max_{\mathbf{b}(.)} E[< \mathbf{b}(\mathbf{X}_{-n+1}^{-1}), g(\mathbf{X}_0) > | \mathbf{X}_{-n+1}^{-1}] | \mathbf{X}_{-n+1}^{-1}] \\
&\leq E[\max_{\mathbf{b}(.)} E[< \mathbf{b}(\mathbf{X}_{-n}^{-1}), g(\mathbf{X}_0) > | \mathbf{X}_{-n}^{-1}] | \mathbf{X}_{-n+1}^{-1}] \\
&= E[\bar{w}_{n+1} | \mathbf{X}_{-n+1}^{-1}],
\end{aligned}$$

where in the last equation we applied formula (1). Thus $\overline{w}_n$ is a submartingale and $E|\overline{w}_n|_+ \leq \max\{a^3, c^3\}$, then we can apply convergence theorem of submartingales and we conclude that there exists a random variable $\overline{w}_\infty$ such that

$$\lim_{n\to\infty} \overline{w}_n = \overline{w}_\infty.$$

with probability 1.

We apply Breimann's ergodic theorem [4]: Let $Z = \{Z_i\}_{-\infty}^{\infty}$ be a stationary and ergodic process. For each positive integer i, let T^i denote the operator that shifts any sequence $\{\ldots, z_{-1}, z_0, z_1, \ldots\}$ by i digits to the left. Let $f_1, f_2, \ldots$ be a sequence of real-valued functions such that $\lim_{n\to\infty} f_n(Z) = f(Z)$ almost surely for some function f. Assume that $E \sup_n |f_n(Z)| < \infty$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f_i(T^i Z) = Ef(Z) \qquad \text{almost surely.}$$

Introduce the notation $f_i(\mathbf{X}) = \overline{w}_i(\mathbf{X})$,

$$f_i(T^i\mathbf{X}) = \overline{w}_i(T^i\mathbf{X}) = \max_j E\left(|X_i^{(j)} - 1|^3 | \mathbf{X}_1^{i-1}\right)$$

Furthermore criteria $E \sup_i |f_i(\mathbf{X})| < \infty$ is fulfilled because the returns are bounded. Thus by the Breiman ergodic theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \max_j E\left(|X_i^{(j)} - 1|^3 | \mathbf{X}_1^{i-1}\right) = E\left[\max_j E\left(|X_0^{(j)} - 1|^3 | \mathbf{X}_{-\infty}^{-1}\right)\right]$$

with probability 1. $\square$

Proof of Theorem 3.1. For the second order Taylor expansion of the function $\log z$ at $z = 1$ we get the following bounds

$$\log z \geq h(z) - \frac{1}{2}|z-1|^3$$

and

$$\log z \leq h(z) + \frac{1}{3}|z-1|^3,$$

where $0.6 < z$. In addition, by taking into account the definition of semi optimal portfolio $\bar{b}(\mathbf{X}_1^{n-1})$ we get,

$$\begin{aligned}
& E(\log < \bar{\mathbf{b}}(\mathbf{X}_1^{n-1}), \mathbf{X}_n > |\mathbf{X}_1^{n-1}) + \frac{1}{2}E(| < \bar{\mathbf{b}}(\mathbf{X}_1^{n-1}), \mathbf{X}_n > -1|^3|\mathbf{X}_1^{n-1}) \\
\geq\ & E(h(< \bar{\mathbf{b}}(\mathbf{X}_1^{n-1}), \mathbf{X}_n >)|\mathbf{X}_1^{n-1}) \\
\geq\ & E(h(< \mathbf{b}_n^*, \mathbf{X}_n >)|\mathbf{X}_1^{n-1}) \\
\geq\ & E(\log < \mathbf{b}^*(\mathbf{X_1^{n-1}}), \mathbf{X}_n > |\mathbf{X_1^{n-1}}) - \mathbf{\frac{1}{3}E}(| < \mathbf{b}^*(\mathbf{X_1^{n-1}}), \mathbf{X_n} > \mathbf{-1}|^\mathbf{3}|\mathbf{X_1^{n-1}}).
\end{aligned} \tag{2}$$

We derive simple bounds for formulae $E(| < \bar{\mathbf{b}}(\mathbf{X}_1^{n-1}), \mathbf{X}_n > -1|^3|\mathbf{X}_1^{n-1})$ and $E(| < \mathbf{b}^*(\mathbf{X}_1^{n-1}), \mathbf{X}_n > -1|^3|\mathbf{X}_1^{n-1})$. Considering the portfolio vector as a discrete probability distribution and taking into account that function $|z-1|^3$ is convex, we can apply the Jensen's inequality

$$\begin{aligned}
| < \bar{\mathbf{b}}(\mathbf{X}_1^{n-1}), \mathbf{X}_n > -1|^3 &= |\sum_{i=1}^{d} \bar{b}^{(i)}(\mathbf{X}_1^{n-1})(X_n^{(i)} - 1)|^3 \\
&\leq \sum_{i=1}^{d} \bar{b}^{(i)}(\mathbf{X}_1^{n-1})|X_n^{(i)} - 1|^3.
\end{aligned}$$

Taking the conditional expectation at both sides of the last inequality and then by straightforward manipulations we get

$$\begin{aligned}
E(| < \bar{\mathbf{b}}(\mathbf{X}_1^{n-1}), \mathbf{X}_n > -1|^3|\mathbf{X}_1^{n-1}) &\leq \sum_{i=1}^{d} \bar{b}^{(i)}(\mathbf{X}_1^{n-1}) E(|X_n^{(i)} - 1|^3|\mathbf{X}_1^{n-1}) \\
&\leq \max_i E(|X_n^{(i)} - 1|^3|\mathbf{X}_1^{n-1}).
\end{aligned} \tag{3}$$

Similarly

$$E(| < \mathbf{b}^*(\mathbf{X}_1^{n-1}), \mathbf{X}_n > -1|^3|\mathbf{X}_1^{n-1}) \leq \max_i E(|X_n^{(i)} - 1|^3|\mathbf{X}_1^{n-1}) \tag{4}$$

holds for the log-optimal portfolio strategy. According to inequalities (2), (3) and (4) we get

$$\begin{aligned} & E(\log < \bar{\mathbf{b}}(\mathbf{X}_1^{n-1}), \mathbf{X}_n > |\mathbf{X}_1^{n-1}) \\ \geq \quad & E(\log < \mathbf{b}^*(\mathbf{X}_1^{n-1}), \mathbf{X}_n > |\mathbf{X}_1^{n-1}) - \frac{5}{6} \max_i E(|X_n^{(i)} - 1|^3 |\mathbf{X}_1^{n-1}). \end{aligned} \tag{5}$$

Consider the following decomposition

$$\frac{1}{n} \log \bar{S}_n = \bar{U}_n + \bar{V}_n,$$

where

$$\bar{U}_n = \frac{1}{n} \sum_{i=1}^{n} [\log < \bar{\mathbf{b}}(\mathbf{X}_1^{i-1}), \mathbf{X}_i > -E[\log < \bar{\mathbf{b}}(\mathbf{X}_1^{i-1}), \mathbf{X}_i > |\mathbf{X}_1^{i-1}]]$$

and

$$\bar{V}_n = \frac{1}{n} \sum_{i=1}^{n} E[\log < \bar{\mathbf{b}}(\mathbf{X}_1^{i-1}), \mathbf{X}_i > |\mathbf{X}_1^{i-1}].$$

It can be shown that $\bar{U}_n \to 0$ a.s., since it is an average of bounded martingale differences. So

$$\liminf_{n\to\infty} \bar{V}_n = \liminf_{n\to\infty} \frac{1}{n} \log \bar{S}_n. \tag{6}$$

Similarly, consider the following decomposition

$$\frac{1}{n} \log S_n^* = U_n^* + V_n^*,$$

where

$$U_n^* = \frac{1}{n} \sum_{i=1}^{n} [\log < \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i > -E[\log < \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i > |\mathbf{X}_1^{i-1}]]$$

and

$$V_n^* = \frac{1}{n} \sum_{i=1}^{n} E[\log < \mathbf{b}^*(\mathbf{X}_1^{i-1}), \mathbf{X}_i > |\mathbf{X}_1^{i-1}].$$

Again, it can be shown that $U_n^* \to 0$ a.s. So

$$\lim_{n\to\infty} V_n^* = \lim_{n\to\infty} \frac{1}{n} \log S_n^*. \tag{7}$$

Taking the arithmetic average on both sides of inequality (5) over trading periods $1, \ldots, n$, then taking the limes inferior of both sides as n goes to infinity and applying equalities (6), (7) as well as Lemma 3.2, we arrive to the following inequality

$$\begin{aligned} \liminf_n \frac{1}{n} \log \bar{S}_n \quad & \geq \quad \lim_{n\to\infty} \frac{1}{n} \log S_n^* - \lim_{n\to\infty} \frac{5}{6n} \sum_{i=1}^{n} \max_j E\left(|X_i^{(j)} - 1|^3 |\mathbf{X}_1^{i-1} \right) \\ & = \quad W^* - \frac{5}{6} E\left[\max_j E\left(|X_0^{(j)} - 1|^3 |\mathbf{X}_{-\infty}^{-1} \right) \right] \end{aligned} \tag{8}$$

□

4 Algorithm for finding the semi-log-optimal portfolio

The advantage of our semi-log-optimal portfolio approach is that it performs very close to the log-optimal strategy, meanwhile it allows a simpler and more "standardized" computation of the corresponding portfolio vector, as its is detailed below. When we calculate the semi-log-optimal portfolio, then instead of searching for the maximum of logarithmic objective function

$$E\left\{\log\left(<\mathbf{b},\mathbf{X}_n>\right)\middle|\,\mathbf{X}_1^{n-1}\right\}$$

we look for the maximum of its quadratic approximation

$$E\left\{h\left(<\mathbf{b},\mathbf{X}_n>\right)\middle|\,\mathbf{X}_1^{n-1}\right\}. \tag{9}$$

over simplex $\mathbf{b}\in\Delta_d$. One important advantage of using a quadratic objective function is that it leads to a known class of mathematical programming problems. The next lemma gives the quadratic objective function (9) in more explicit form.

Lemma 4.1. *Finding the semi-log-optimal portfolio on the n-th trading period is equivalent to finding the solution of the following quadratic programming task*

maximize $g(\boldsymbol{b},\boldsymbol{X}_1^{n-1})$ *in variable* $\boldsymbol{b}$
subject to $\sum_{i=1}^{d} b_i = 1, \boldsymbol{b}\geq 0$,

where the objective function is the following

$$g(\boldsymbol{b},\boldsymbol{X}_1^{n-1}) = 2<\boldsymbol{b},\boldsymbol{m}(\boldsymbol{X}_1^{n-1})> -\frac{1}{2}<\boldsymbol{b},\boldsymbol{K}(\boldsymbol{X}_1^{n-1})\boldsymbol{b}>,$$

where $\boldsymbol{m}(\boldsymbol{X}_1^{n-1}) = E(\boldsymbol{X}_n|\boldsymbol{X}_1^{n-1})$ *and* $\boldsymbol{K}(\boldsymbol{X}_1^{n-1}) = \{K_{i,j}\}$, $K_{i,j} = E(X_n^{(i)}X_n^{(j)}|\boldsymbol{X}_1^{n-1})$.

Proof By straightforward calculation we get

$$\begin{aligned} E\left[h(<\mathbf{b},\mathbf{X}_n>)|\mathbf{X}_1^{n-1}\right] &= E\left[2<\mathbf{b},\mathbf{X}_n> -\tfrac{1}{2}<\mathbf{b},\mathbf{X}_n>^2 -3/2|\mathbf{X}_1^{n-1}\right] \\ &= 2<\mathbf{b},E(\mathbf{X}_n|\mathbf{X}_1^{n-1})> -\tfrac{1}{2}<\mathbf{b},\mathbf{K}(\mathbf{X}_1^{n-1})\mathbf{b}> -3/2. \end{aligned}$$

□

Because $\mathbf{K}$ is positive semi-definite, the objective function is convex, furthermore the constraints are linear functions. It is known from optimization theory that for point $\mathbf{b}$ to be an optimum point it is necessary and sufficient that $\mathbf{b}$ is a Karush-Kuhn-Tucker (KKT) point. The most common method of solving a quadratic programming task is an interior point method, such as LOQO [10].

References

[1] P. Algoet. Universal schemes for prediction, gambling, and portfolio selection. *Annals of Probability*, 20:901–941, 1992.

[2] P. Algoet. The strong law of large numbers for sequential decisions under uncertainity. *IEEE Transactions on Information Theory*, 40:609–634, 1994.

[3] P. Algoet and T. Cover. Asymptotic optimality asymptotic equipartition properties of log-optimum investments. *Annals of Probability*, 16:876–898, 1988.

[4] L. Breiman. The individual ergodic theorem of information theory. *Annals of Mathematical Statistics*, 31:809–811, 1957. Correction. *Annals of Mathematical Statistics*, 31:809–810, 1960.

[5] L. Breiman. Optimal gambling systems for favorable games. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, pages 65–78, Berkeley, 1961. University of California Press.

[6] L. Györfi, M. Kohler, A. Krzyżak, and H Walk. *A Distribution-Free Theory of Nonparametric Regression*. Springer, New York, 2002.

[7] L. Györfi, G. Lugosi, and F. Udina. Nonparametric kernel-based sequential investment strategies. *Mathematical Finance*, 16:337–357, 2006.

[8] L. Györfi and D. Schäfer. Nonparametric prediction. In J. A. K. Suykens, G. Horváth, S. Basu, C. Micchelli, and J. Vandevalle, editors, *Advances in Learning Theory: Methods, Models and Applications*, pages 339–354. IOS Press, NATO Science Series, 2003.

[9] W. F. Stout. *Almost Sure Convergence*. Academic Press, New York, 1974.

[10] R. J. Vanderbei. *LOQO: An Interior Point Code for Quadratic Programming*. Statistics and Operations Research,Princeton University, SQR-94-15 , 1998.

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