Nonparametric Estimation for Financial Investment under Log-Utility

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DEDICATED TO

MY PARENTS
to whom I owe so much

PROFESSOR PAUL GLENDINNING
without him I might never have found my way
to mathematical finance and economics

CONTENTS

ABBREVIATIONS

$\lvert\cdot\rvert$	absolute value of a number, cardinality of a set
$<\cdot,\cdot>$	Euclidean scalar product
$\lVert\cdot\rVert_\infty$	supremum norm
$\lVert\cdot\rVert_q$	q-norm (on $\mathbb{R}^d$ or L_q)
$\lVert\cdot\rVert$	other norm
$\mathbb{N}$	positive integers $1, 2, 3, ...$
$\mathbb{N}_0$	nonnegative integers $0, 1, 2, 3, ...$
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	real numbers > 0
$\mathbb{R}_0^+$	real numbers ≥ 0
$\lfloor x \rfloor$	integer part of x
$\lfloor x \rfloor_N$	the smallest $kN (k \in \mathbb{N})$ such that $kN \geq x \geq 0$.
$\lceil x \rceil$	x rounded toward infinity
$.^T$	transpose of a vector or matrix
$\mathrm{spr}(\cdot)$	spectrum of a matrix
exp	exponential to the base e
log	logarithm to the base e
lb	logarithm to the base 2
$a_n = o(b_n)$	Landau symbol for: $a_n/b_n \to 0$
$a_n = O(b_n)$	Landau symbol for: a_n/b_n is a bounded sequence
A^C	complement of the set A
$\bar{A}$	closure of the set A
$conv(A)$	convex hull of the set A

$\mathbf{1}_A$	characteristic function of the set A
$\mathrm{diam}(A)$	Euclidean diameter $\sup_{a,b\in A}\|a-b\|_\infty$ of the set A
$\rho(x,A)$	Euclidean distance $\inf_{a\in A}\|x-a\|_\infty$ from x to the set A
$H(A,B)$	Hausdorff distance $\max\{\sup_{a\in A}\rho(a,B),\sup_{b\in B}\rho(b,A)\}$ between the sets A and B
$\mathcal{B}(S)$	Borelian $\sigma-$algebra on the topological space S
$f(x)_{\mid x=y}$	f evaluated in y
f^+, f_+	positive part of f, i.e., $\max\{f,0\}$
f^-, f_-	negative part of f, i.e., $\max\{-f,0\}$
$\mathrm{supp}\ f$	support $\overline{\{x: f(x)>0\}}$ of the function f
$\arg\max f$	solution of a maximization problem (in some contexts set-valued, i.e. $\{x: f(x)=\sup_y f(y)\}$, in others a measurably selected solution x with $f(x)=\sup_y f(y)$)
$\mathbf{P}$	probability measure
$\mathbf{P}_X$	distribution of X
$\mathbf{P}_{Y\mid X=x}$	conditional distribution of Y given $X=x$
$f_X(\cdot)$	a density of $\mathbf{P}_X$ w.r.t. the Lebesgue measure
$f_{Y\mid X}(\cdot\mid x)$	a density of $\mathbf{P}_{Y\mid X=x}$ w.r.t. the Lebesgue measure
$\mathbf{Q}_1 \ll \mathbf{Q}_2$	$\mathbf{Q}_1$ is absolutely continuous w.r.t. $\mathbf{Q}_2$
$D(\mathbf{Q}_1\|\|\mathbf{Q}_2)$	Kullback-Leibler distance of $\mathbf{Q}_2$ and $\mathbf{Q}_1$
$a.s.$	$\mathbf{P}$-almost surely, with probability one
$\mathbf{P}$-a.a.	$\mathbf{P}$-almost all
$\mathbf{E}$	mathematical expectation
$\mathbf{E}[Y\mid X]$	conditional expectation of Y given X
$\mathbf{E}[Y\mid X=x]$	conditional expectation of Y given $X=x$
Var	Variance
Cov	Covariance
$N(\mu;\Sigma)$	normal distribution with mean μ and variance-covariance matrix Σ
$L_1(\mathbf{P})$	space of Lebesgue integrable functions w.r.t. $\mathbf{P}$
$L_q(\mathbf{P})$	qth order Lebesgue integrable functions w.r.t. $\mathbf{P}$

const.	a suitable constant
GSM	geometrically strongly mixing
hot.	higher order terms of an expansion
i.i.d.	independent, identically distributed
p.a.	per annum
w.r.t.	with respect to
$\square$	end of proof

All non-standard notation is explained when it occurs for the first time. The random variables in this thesis are understood to be defined on a common probability space $(\Omega, \mathcal{A}, \mathbf{P})$. $\mathbb{R}^d$-valued random variables are implicitly assumed to be measurable w.r.t. the Borelian σ-algebra $\mathcal{B}(\mathbb{R}^d)$. If not stated otherwise, measurability of functions $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ means measurability w.r.t. $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^{d'})$.

SUMMARY

In this thesis we aim to plead for the application of nonparametric statistical forecasting and regression estimation methods to financial investment problems.

In six chapters we explore applications of nonparametric techniques to portfolio selection for financial investment. Clearly, this cannot be more than a crude and somewhat arbitrary selection of topics within this vast area, so we decided to concentrate on some typical situations. Our hope is to be able to illustrate the benefits of nonparametric estimation methods in portfolio selection.

Chapter 1
Introduction: investment and nonparametric statistics

Investment is the strategic allocation of resources, typically of monetary resources, in an environment, typically a market of assets, whose future evolution is uncertain. Investment problems arise in a huge variety of contexts beyond the financial one. Resources may also take the form of energy, of data-processing resources, etc. Strategic investment planning helps to run many processes with higher benefit. In this thesis we focus our attention on *financial* investment, which we think is the "prototypical" example of a resource allocation process.

The three ingredients of financial investment are the market, the actions the investor may take and his investment goal (discussed in detail in Sections 1.1-1.3):

– *As to the market:* We assume that there are m assets in our financial market. The jth asset yields a *return* $X_{i,n}$ on an investment of 1 unit of money during market period n (lasting from "time" $n-1$ to n, time being measured, e.g., in days of trading). The ensemble of returns on the nth day of trading is given by

$$X_n = (X_{1,n}, ..., X_{m,n})^T \in \mathbb{R}_+^m.$$

To the investor, the *return process* $\{X_n\}_{n=1}^{\infty}$ appears to be a stochastic process which, in many real markets, is stationary and ergodic (Definition 1.1.1). In some chapters we impose additional (but realistic) conditions on the distribution of the process. The key point is, however, that

> *we use nonparametric models, i.e. models that do not assume a parametric evolution equations such as ARMA, ARCH and GARCH equation to hold.*

These models guarantee highest flexibility in real applications.

– *As to the investment actions:* We are concerned with an investor who neither consumes nor deposits new money into his portfolio. At the beginning of each market period n, our investor uses all his current wealth to acquire a *portfolio* b_n of the stocks. It will be convenient to describe the portfolio b_n by the proportion $b_{j,n}$ of the investor's current wealth invested in asset j $(j = 1, ..., m)$ during market period n. Thus, b_n is chosen at time $n-1$ from the set S of all portfolios, consisting of the vectors (portfolios)

$$b_n = (b_{1,n}, ..., b_{m,n})^T$$

satisfying $b_{j,n} \geq 0$ and $\sum_{j=1}^{m} b_{j,n} = 1$. In some situations the set of investment actions S may be further narrowed down by the occurence of transaction costs.

– *As to the investment goal:* If W_0 is his initial wealth, an investor using the portfolio strategy $\{b_i\}_{i=0}^{n-1}$ manages to accumulate the wealth $W_n = \prod_{i=1}^{n} < b_i, X_i > W_0$ during n market periods ($< \cdot, \cdot >$ is the Euclidean scalar product). Naturally, the investor aims to maximize W_n. It is known from literature that there is no essential conflict between short run (n finite) and long term ($n \to \infty$) investment. In both cases investment according to the *conditional log-optimal portfolio*

$$b_n^* := \arg\max_{b \in S} \mathbf{E}\left[\log < b, X_n > | \, X_{n-1}, ...X_1\right]$$

at time n is optimal, outperforming any other strategy because of

$$\mathbf{E}\frac{W_n}{W_n^*} \leq 1 \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n} \log \frac{W_n}{W_n^*} \leq 0 \quad \text{with probability 1}$$

(Cover and Thomas, 1991, Theorem 15.5.2). Here, W_n^* is the wealth at time n resulting from a series of conditionally log-optimal investments, W_n the wealth from any other non-anticipating portfolio strategy. We argue that

> *this is sufficient reason for the investor to use a logarithmic utility function, i.e. to maximize the expected future logarithmic return given the past return vectors.*

The conditional log-optimal portfolio depends upon the distribution of the return process $\{X_n\}_n$. Realistically, the true distribution of the market returns and hence the log-optimal strategy is not known to the investor. This makes statistics the natural partner of investment. Statistics is needed to solve the key problem,

to find a non-anticipating portfolio selection scheme $\{\hat{b}_n\}_n$ (working with historical return data only, without knowing the true return distribution) such that for any stationary ergodic return process $\{X_n\}_n$, the investor's wealth $\hat{W}_n := \prod_{i=1}^n < \hat{b}_i, X_i >$ grows –on the average– as fast as with the log-optimum strategy $\{b_n^*\}_n$. More formally, $\{\hat{b}_n\}_n$ should give

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{W_n^*}{\hat{W}_n} \leq 0$$

with probability 1.

Such portfolio selection schemes are known to exist (Algoet, 1992). The disadvantage is that they are fairly complicated and, even worse, they require an enormous amount of past return data to yield practically relevant results. It is the aim of this thesis to provide simplified, yet efficient portfolio selection algorithms based on nonparametric forecasting and estimation techniques. Particular emphasis is put on making the algorithms applicable in considerably large classes of markets.

Chapter 2
Portfolio benchmarking: rates and dimensionality

The performance of a portfolio selection rule is usually compared with that of a benchmark portfolio selection rule. Our benchmark is the log-optimal portfolio selection rule, and as we have seen in Chapter 1, this is the optimal rule. An investor will typically find his own rule underperforming. He can only hope that underperformance vanishes sufficiently fast when – with increasing number of market periods – his estimates for the distribution of the return process and hence his idea of the market become more and more complete. Now, if the investor evaluates the historical returns $X_1, ..., X_n$ leading to the portfolio choice $\hat{b}_{n+1}$ at time n, he will achieve a return $\hat{R}_n =< \hat{b}_{n+1}, X_{n+1} >$ on his investment during the next market period. This should be compared with the return $R_n^* =< b_{n+1}^*, X_{n+1} >$ of the conditional log-optimal portfolio.

From our log-utility point of view we suggest to measure underperformance of $\hat{b}_{n+1}$ in terms of the positivity of $\mathbf{E} \log \frac{R_n^*}{\hat{R}_n}$. The smaller this expectation becomes, the better is the selection rule $\hat{b}_{n+1}$.

Assuming that the return data arises from a process of independent and identically distributed (i.i.d.) random variables, it is important to know at what rate the underperformance $\mathbf{E} \log \frac{R_n^*}{\hat{R}_n}$ vanishes for typical portfolio selection rules. Using notions from information theory we prove a lower bound on this rate in Section 2.1. Even in the simplest of all markets, a market with only finitely many possible return outcomes,

no empirical portfolio selection rule can make underperformance vanish in every market faster than $\frac{1}{n}$ tends to 0, i.e. there is always a market for which the inequality $\mathbf{E} \log \frac{R_n^*}{\hat{R}_n} \geq const. \cdot \frac{1}{n}$ holds (Theorem 2.1.1).

There are empirical portfolio selection rules that achieve this rate. In particular, the *empirical log-optimal portfolio*

$$\hat{b}_{n+1} := \arg\max_{b \in S} \frac{1}{n} \sum_{i=1}^{n} \log < b, X_i > \tag{0.0.1}$$

proves to be rate optimal in as far as

the empirical log-optimal portfolio selection rule (0.0.1) attains the lower bound for the rate at which underperformance vanishes, whatever the number of stocks in the market (Theorem 2.1.3).

Loosely speaking, it compensates for wrong investment decisions as fast as possible. Interesting enough, the findings are largely unaffected by the number of stocks in the market, which is a rather untypical feature in nonparametric estimation (Theorem 2.1.4 shows that this phenomenon perseveres in more complicated market settings).

This is why we discuss the effects of "dimensionality" on the portfolio selection process in more detail in Section 2.2. We argue that a reduction of the whole stock market to some pre-selected stocks is inevitable, e.g., because of computational restrictions. In other words, the investor can only handle a smallish subset of all stocks in the market for investment strategy planning. These stocks have to be selected in the planning phase, even before investment starts. Hence, criteria for the pre-selection of stocks from the market are needed. A common way to do this is to pick the stocks whose chart promises high growth rates. It will turn out, however, that this is fallacious:

> *any selection algorithm that assesses the single stocks seperately, e.g. on the basis of single stock expected returns, is sure to pick the "bad" stocks in some realistic market (Theorem 2.2.1).*

This is a somewhat negative result, but it warns us that reasonable selection schemes have to include further information about the market. We will show that the variance-covariance structure of the stock returns provides sufficient information in many markets (more precisely, in markets with log-normal returns). Section 2.3 illustrates the results with simulations and examples, demonstrating their practical relevance.

Chapter 3
Predicted stock returns and portfolio selection

Having gained the insight that variance-covariance information about the market (inter-stock correlations as well as temporal correlations) are integral to successful investment decisions, we move on to particular investment strategies. In Section 3.1 we consider a strategy which is particularly popular among investors.

The strategy works in two steps, with the past logarithmic returns $Y_n, Y_{n-1}, ..., Y_0$ ($Y_i := \log X_i$) as input data for the investment decision at time n:

1. Produce forecasts of the market future. It is established that forecasts should be based on conditional expectations of future log-returns given

the observed past, i.e. on

$$\hat{Y}_{n+1} := \mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...].$$

2. Invest in those stocks whose forecast $\hat{Y}_{n+1}$ promises to beat a riskless investment in a bond with return rate r, i.e. invest in a stock iff

$$\exp(\hat{Y}_{n+1}) \geq r.$$

We will call this strategy a “greedy strategy”, because it tries to single out the best possible stocks only. As we shall see, this provides us with a natural strategy which can be applied in markets with low log-return variance (Section 3.1).

The major problem in implementing the greedy strategy is the fact that the forecasts $\hat{Y}_{n+1}$ can only be calculated if the distribution of the return process is known to the investor. Hence, we need to derive an estimate $\hat{E}(Y_n, ..., Y_0)$ for the conditional expectation $\hat{Y}_{n+1} = \mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]$ from the market observations $Y_n, ..., Y_0$. It is known from literature that no such forecaster can be *strongly consistent* in the sense of

$$\lim_{n\to\infty} \left| \hat{E}(Y_n, ..., Y_0) - \mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...] \right| = 0 \tag{0.0.2}$$

with probability 1 for any stationary and ergodic process $\{Y_n\}_n$ (Bailey, 1976). This result is discouraging, but it does not rule out the existence of strongly consistent forecasting rules for log-return processes as they arise in real financial markets. In particular, Gaussian log-return processes have been proven to be a good approximation for real log-return processes, but so far no answer has been found to the question whether there exist forecasters that are strongly consistent in any stationary and ergodic Gaussian process. In Section 3.2 we prove that the answer is indeed affirmative. Under weak extra conditions on the Wold coefficients of the process

we present a forecaster $\hat{E}(Y_n, ..., Y_0)$ for stationary and ergodic Gaussian processes which satisfies the strong consistency relation (0.0.2) and which is remarkably easy to compute (Lemma 3.2.1 and Corollary 3.2.3).

This results provides us with the necessary tools to implement the greedy strategy in Gaussian log-return processes. However, the algorithm is of interest very much in its own right, forecasting problems for Gaussian processes arising in many areas.

Section 3.3 proves the convergence properties of the algorithm. Application examples with simulated and real data in Section 3.4 are promising –when the algorithm is run as a mere forecasting algorithm as well as when the algorithm is run as a subroutine for the greedy strategy.

Chapter 4
A Markov model with transaction costs: probabilistic view

In simple markets where returns arise as i.i.d. data, the investor should invest in a constant log-optimal portfolio strategy. This requires him not to change the *proportion* of wealth held in each stock during the investment process. The proportions remain constant, however, the prices of the assets change relatively to each other during each market period, so that the *actual quantities* of the single stocks in the portfolio vary from market period to market period. Thus, a large number of transactions are needed to follow a constant log-optimal strategy. In practice, this is a huge drawback: Much of the wealth accumulated by a log-optimal strategy has to be spent to settle *transaction costs* such as brokerage fees, administrative and telecommunication expenses. The conclusion for the investor must be to adapt his strategy to meet two requirements: to make as few costly transactions as possible, but to make as many as necessary to boost his wealth. The aim of Chapters 4 and 5 is to investigate how these two conflicting requirements can be balanced in one strategy.

To this end we shall assume that the returns arise from a d-stage Markov process. In Chapter 4 the distribution of the return process is known, an unrealistic assumption which we will drop in Chapter 5. Section 4.1 generalizes the market model from Chapter 1 to include transaction costs proportional to the total value of the purchased shares. Not surprisingly, the investor can only afford a limited range of portfolio choices in presence of transaction costs, and as we shall see,

> *in d-stage Markovian return processes it suffices to consider strategies based on portfolio selection functions, i.e. portfolio selection schemes of the form $b_i = c(b_{i-1}, X_{i-d}, ..., X_{i-1})$ with an appropriate function c (Definition 4.1.2).*

Hence, the next portfolio is a function of the last portfolio and the last d observed return vectors. The investor aims to maximize his expected mean logarithmic return as before by choosing an optimal selection function c.

In Section 4.2 we tackle the problem how to obtain an optimal selection function c – if the distribution of the return process were known. The main result demonstrates that

> *an optimal portfolio selection function c can be obtained from a solution of the Bellman equation (Theorem 4.2.1, equation 4.2.2).*

The Bellman equation is known from the theory of dynamic programming, but fundamental differences between classical dynamical programming and the portfolio selection problem will become evident. Further properties of solutions of the Bellman equation will be derived in Section 4.3, results that will be needed for the arguments in Chapter 5.

Chapter 5
A Markov model with transaction costs: statistical view

The Bellman equation considered in Chapter 4 heavily depends upon the distribution of the return process $\{X_n\}_n$ through a peculiar conditional expectation. Hence, the results of Chapter 4 are valid only under the assumption that the investor knows the distribution of the stock return process. Of course, in practice this is illusory. At best, the investor has an estimate of the return distribution at his disposal. This, in turn, allows him to produce an estimate of the conditional expectation in question and hence gives him an approximate Bellman equation involving the observed empirical return data. Using nonparametric regression estimation techniques

> *we will show in Section 5.1 how a natural empirical counterpart of the Bellman equation from Chapter 4 can be found (equation 5.1.2).*

With similar techniques as in Chapter 4 we will establish that this empirical equation can be solved under realistic conditions.

> *This will lead us to a strategy that merely relies on observational data but has the same optimality properties as the (theoretical) optimal portfolio selection rule in presence of transaction costs (Theorems 5.1.1 and 5.1.2).*

For this, we will fall back on generalizations of existing *uniform consistency results* in regression estimation, which will be provided in Section 5.2. In particular, if $\{X_n\}_n$ is a stationary geometrically strongly mixing process and g is taken from a class $\mathcal{G}$ of Lipschitz continuous functions we estimate the conditional expectation

$$R(g,b,x) := \mathbf{E}[g(X_1,b)|X_0 = x] \qquad (b \in S)$$

by a kernel regression estimator $R_n(g,b,x)$. Depending on the smoothness of a density of X_0 (which we assume to exist) we determine the rate of convergence of

$$\sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x\in\mathcal{X},b\in S} |R_n(g,b,x) - R(g,b,x)| \to 0 \quad (n \to \infty),$$

i.e. of the expected uniform estimation error, uniformly in $\mathcal{G}$ (Corollary 5.2.2). This result is of interest in other areas of nonparametric statistics as well.

Finally, Section 5.3 is devoted to the proof of optimality and combines the results from Chapter 4 with uniform consistent regression estimation techniques.

Chapter 6
Portfolio selection functions in stationary return processes

Considering the fact that the investor may have reason to believe that the historical return data does not follow a d-stage Markov process in some cases, we should move on to even more general market models than in the previous chapters. Ignoring transaction costs, we consider a market whose returns are merely stationary and ergodic. It is natural for the investor to take his investment decisions on the basis of recently observed returns, say on the basis of the returns during the last $d \in \mathbb{N}$ market periods (d fixed). This leads us to the notion of log-optimal portfolio selection functions.

We make this more concrete in Section 6.1, where we take our familiar log-utility approach again. The investor tries to find a *log-optimal portfolio selection function*, i.e. a measurable function

$$b^* : \mathbb{R}_+^{dm} \longrightarrow S$$

such that ($< \cdot, \cdot >$ denoting the Euclidean scalar product)

$$\mathbf{E}\left(\log < b^*(X_0, ..., X_{d-1}), X_d >\right) \geq \mathbf{E}\left(\log < f(X_0, ..., X_{d-1}), X_d >\right)$$

for all measurable $f : \mathbb{R}_+^{dm} \longrightarrow S$. For the $(n+1)$st day of trading, b^* advises the investor to acquire the portfolio $b^*(X_{n-d+1}, ..., X_n)$.

Clearly, the concept of log-optimal portfolio selection functions does not reach the same degree of generality as the concept of a conditional log-optimal portfolio (where d is such that the whole observed past is included in the portfolio decision). In spite of being a simplification, this approach nevertheless gives us several advantages over the log-optimal strategy as far as computation, estimation and interpretation are concerned.

With log-optimal portfolio selection functions we face the same problem as with log-optimal portfolios. Both can only be calculated if the true distribution of the return process happens to be known. A practitioner, however, needs to have an estimation procedure that evaluates observed past return data to approximate the true log-optimal device.

In Section 6.2 we therefore develop an algorithm to produce estimates $\hat{b}_n$ of a log-optimal portfolio selection function b^* from past return data.

We require very mild conditions beyond stationarity and ergodicity. More precisely, we assume that the return process $\{X_n\}_{n=0}^{\infty}$ is an $[a,b]^m$-valued stationary and ergodic process ($0 < a \leq b < \infty$ need not be known) and that a Lipschitz condition on the conditional return ratio $\mathbf{E}[X_d / < s, X_d > | X_{d-1} = x_{d-1}, ..., X_0 = x_0]$ holds. The Lipschitz constant L is taken as a known market constant.

Using a stochastic gradient algorithm and combining it with nonparametric regression estimators,

we establish the strong convergence of the estimates $\hat{b}_n$ to the true log-optimal portfolio selection function b^*, avoiding the usual mixing conditions (Theorem 6.2.2).

What is even more important in practical applications:

Selecting portfolios on the basis of the estimated log-optimal portfolio selection functions yields optimal growth of wealth among all other strategies that take their investment decisions on the basis of the last d observations.

Indeed, let $\hat{S}_n$ be the wealth accumulated during n market periods when on the $(i+1)$st day of trading the portfolio $\hat{b}_i(X_{i-d+1}, ..., X_i)$ is selected using the most recent estimate $\hat{b}_i$ of a log-optimal portfolio selection function. Then, if S_n is the wealth accumulated during the same period using any other portfolio selection function of the last d observed return vectors,

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{S_n}{\hat{S}_n} \leq 0$$

with probability 1 (Corollary 6.2.3).

After an appropriate modification, the algorithms and the results remain valid even if the market constant L is unknown in real market applications (Theorem 6.2.4). Section 6.3 proves the findings, and the chapter is rounded off with several realistic examples in Section 6.4.

Chapters 2, 3 and 6 can be read independently from each other, they are self-contained. Chapters 4 and 5 are closely linked, however. Notation that goes beyond common mathematical style is explained where it occurs for the first time. We also refer the reader to the list of abbreviations at the beginning of the thesis. The calculations and plots for the examples were generated using Matlab 4.0 and 6.0.0.88, Minitab 11.2 and R 1.1.1 with historical stock quotes (daily closing prices) from the New York Stock Exchange provided by www.wallstreetcity.com.

ZUSAMMENFASSUNG

Diese Arbeit soll ein Plädoyer sein für die Anwendung nichtparametrischer statistischer Vorhersage- und Schätzmethoden auf Probleme, wie sie bei der Planung von Finanzanlagen und Investitionen auftreten.

In sechs Kapiteln werden verschiedene Anwendungsmöglichkeiten nichtparametrischer Techniken bei der Portfolioauswahl an Finanzmärkten analysiert. Dies kann natürlich nur einen groben und zugegebenermaßen willkürlichen Ausschnitt aus diesem weiten Gebiet widerspiegeln –wir hoffen jedoch, dadurch die Vorzüge nichtparametrischer Schätzmethoden bei der Portfolioauswahl aufzeigen zu können.

Kapitel 1
Einführung: Investment und nichtparametrische Statistik

Investment ist der strategisch geplante Einsatz von Ressourcen (üblicherweise von finanziellen Ressourcen) in einer Umgebung (üblicherweise in einem Finanzmarkt), deren zukünftige Entwicklung zufälligen Fluktuationen unterliegt. Investitionsprobleme treten in einer Vielzahl von Gebieten auch über den finanziellen Kontext hinaus auf. Dabei können Ressourcen u. A. die Form von Energie, von Datenverarbeitungskapazitäten, etc. annehmen. Die strategische Planung von Investitionen hilft, viele Prozesse mit höherem Nutzen zu betreiben. Diese Arbeit konzentriert sich auf *finanzielle* Investitionen, welche gleichsam den "Prototyp" für verschiedenste Prozesse bilden, bei denen Systemressourcen gewinnbringend einzusetzen sind.

Bei Investitionen finanzieller Natur spielen drei Komponenten eine Rolle: der Markt, die Handlungsmöglichkeiten des Investors und sein Investitionsziel. Diese Bausteine werden in den Abschnitten 1.1-1.3 im Detail diskutiert.

– *Zum Markt:* Wir gehen von einem Finanzmarkt mit m Anlagemöglichkeiten (Aktien, festverzinsliche Wertpapiere, ...) aus. Die i. Anlagemöglichkeit erzielt in der Marktperiode n eine *Rendite* $X_{i,n}$ auf eine Investition von

einer Geldeinheit. Die n. Marktperiode dauere vom "Zeitpunkt" $n-1$ bis zum Zeitpunkt n, wobei die Zeit z.B. in Handelstagen gemessen wird. Die Renditen der einzelnen Anlagemöglichkeiten am n. Handelstag werden im Renditevektor

$$X_n = (X_{1,n}, ..., X_{m,n})^T \in \mathbb{R}_+^m$$

zusammengefasst. In den Augen des Investors ist $\{X_n\}_{n=1}^\infty$ ein stochastischer Prozess, welcher in vielen realen Märkten stationär und ergodisch ist (Definition 1.1.1). In manchen Kapiteln dieser Arbeit werden (realistische) Zusatzannahmen über die Verteilung des Prozesses getroffen. Entscheidend ist dabei jedoch,

> *dass wir nichtparametrische Modelle betrachten –Modelle also, die nicht von der Existenz einer parametrischen Entwicklungsgleichung ausgehen, wie sie z.B. ARMA-, ARCH- und GARCH-Prozesse besitzen.*

Diese Modelle garantieren höchste Flexibilität bei der Anwendung in realen Finanzmärkten.

– *Zu den Handlungsmöglichkeiten:* Wir betrachten einen Investor, der weder Teile seines Vermögens auf persönlichen Konsum verwendet, noch seinem Portfolio im Verlauf des Investitionsprozesses neues Geld zufließen lässt. Am Beginn jeder Marktperiode n verwendet der Investor sein gesamtes Vermögen darauf, ein *Aktienportfolio* b_n zu erwerben. Ein solches Portfolio b_n wird durch die Anteile $b_{j,n}$ am aktuellen Gesamtvermögen des Investors beschrieben, welche in der n. Marktperiode in die Anlegemöglichkeit $j = 1, ..., m$ investiert werden. Die Wahl von b_n erfolgt dann aus der Menge S aller Portfolios, welche aus den Vektoren (Portfolios)

$$b_n = (b_{1,n}, ..., b_{m,n})^T$$

besteht, für die $b_{j,n} \geq 0$ und $\sum_{j=1}^m b_{j,n} = 1$. In manchen Situationen wird S weiter durch das Auftreten von Transaktionskosten eingeschränkt.

– *Zum Investitionsziel:* W_0 sei das anfängliche Vermögen des Investors. Verwendet er die Portfoliostrategie $\{b_i\}_{i=0}^{n-1}$, wird er nach n Marktperioden über das Vermögen $W_n = \prod_{i=1}^n < b_i, X_i > W_0$ verfügen ($< \cdot, \cdot >$ bezeichnet das Euklidische Skalarprodukt). Ziel des Investors ist es, für W_n

einen möglichst großen Wert zu erzielen. Aus der Literatur ist bekannt, dass dabei kein grundlegender Konflikt zwischen nahen (n endlich) und fernen ($n \to \infty$) Investitionshorizonten besteht. In beiden Fällen ist eine Investition zum Zeitpunkt n gemäß dem *bedingt log-optimalen Portfolio*

$$b_n^* := \arg\max_{b \in S} \mathbf{E}\left[\log < b, X_n > | X_{n-1}, ...X_1\right]$$

optimal. Es übertrifft jede andere Strategie indem

$$\mathbf{E}\frac{W_n}{W_n^*} \leq 1 \quad \text{und} \quad \limsup_{n\to\infty} \frac{1}{n} \log \frac{W_n}{W_n^*} \leq 0 \quad \text{mit Wahrscheinlichkeit 1}$$

(Cover and Thomas, 1991, Theorem 15.5.2). W_n^* ist dabei das Vermögen zum Zeitpunkt n, das der Investor durch eine Serie von bedingt log-optimalen Investitionen erzielt, W_n das Vermögen mit einer beliebigen anderen Portfoliostrategie, die nicht über mehr Information verfügt als aus vergangenen Marktbeobachtungen ableitbar (eine sogenannte “kausale” Strategie).

Dies sollte für den Investor Grund genug sein, eine logarithmische Nutzenfunktion zu verwenden, d.h. mit dem Wissen um die in der Vergangenheit beobachteten Renditevektoren die Maximierung der erwarteten zukünftigen logarithmierten Rendite zu betreiben.

Das bedingt log-optimale Portfolio leitet sich aus der Verteilung des Renditeprozesses $\{X_n\}_n$ ab. In der Realität ist die wahre Verteilung der Renditen und damit auch die bedingt log-optimale Strategie dem Investor nicht bekannt. An diesem Punkt bedarf die Finanzplanung der Statistik als Partner. Die Statistik dient dem Investor zur Lösung des Problems,

eine Methode zu finden, die nur anhand historischer Renditedaten und ohne Kenntnis der wahren Renditeverteilung eine optimale kausale Portfoliostrategie $\{\hat{b}_n\}_n$ *erzeugt. Optimalität wird hier in dem Sinn verwendet, dass die Strategie für jeden stationären und ergodischen Renditeprozess* $\{X_n\}_n$ *das Vermögen* $\hat{W}_n := \prod_{i=1}^n < \hat{b}_i, X_i >$ *des Investors im Mittel genauso schnell wachsen lässt wie die log-optimalen Strategie* $\{b_n^*\}_n$*. Formal ausgedrückt soll* $\{\hat{b}_n\}_n$ *garantieren, dass mit Wahrscheinlichkeit 1*

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{W_n^*}{\hat{W}_n} \leq 0.$$

Es ist bekannt, dass solche Methoden existieren (Algoet, 1992). Diese bringen jedoch den Nachteil mit sich, höchst komplex zu sein und zur Erzeugung praktisch verwertbarer Ergebnisse eine Unmenge historischer Daten zu benötigen. Ein Ziel dieser Arbeit ist es, vereinfachte, aber effiziente Algorithmen zur Portfolioauswahl zu entwickeln, die auf nichtparametrischen Vorhersage- und Schätzverfahren basieren. Die Algorithmen sollen so gestaltet sein, dass sie für möglichst große Klassen von Märkten anwendbar sind.

Kapitel 2
Der Vergleich von Portfolios: Konvergenzraten und Dimension

Die Güte einer Methode zur Portfolioauswahl wird in der Regel durch den Vergleich mit einer Referenzstrategie beurteilt. Unsere Referenzstrategie ist die log-optimale Portfolioauswahl, die –wie wir in Kapitel 1 gesehen haben– eine optimale Verhaltensregel darstellt. Dem Investor wird es nicht gelingen, letztere zu übertreffen. Natürlich wird er hoffen, dass der Mangel an Leistungsfähigkeit seiner eigenen Strategie im Verlauf des Investitionsprozesses verschwindet, wenn nämlich seine Schätzungen für die Verteilung des Renditeprozesses mit zunehmender Menge verfügbarer historischer Daten immer besser werden. Wählt der Investor zum Zeitpunkt n anhand der Beobachtungen $X_1, ..., X_n$ sein Portfolio, wird er in der nächsten Marktperiode eine Rendite von $\hat{R}_n =< \hat{b}_{n+1}, X_{n+1} >$ erwirtschaften, während die log-optimale Strategie $R_n^* =< b_{n+1}^*, X_{n+1} >$ liefert. Der Vergleich beider Werte ermöglicht die Einschätzung, um wieviel $\hat{b}_{n+1}$ der log-optimalen Strategie b_{n+1}^* unterlegen ist.

> *Vom Standpunkt einer logarithmischen Nutzenfunktion ist es daher angebracht, die Unterlegenheit der Strategie $\hat{b}_{n+1}$ an der Positivität der erwarteten Differenz der log-Renditen, an $\mathbf{E} \log \frac{R_n^*}{\hat{R}_n}$ zu messen. Je kleiner dieser Wert, desto besser ist die Strategie $\hat{b}_{n+1}$.*

Zur Beurteilung der Qualität der Strategie $\hat{b}_{n+1}$ ist also insbesondere zu analysieren, mit welcher Geschwindigkeit $\mathbf{E} \log \frac{R_n^*}{\hat{R}_n}$ gegen Null strebt. Dabei wird davon ausgegangen, dass die Renditen in einem Prozess von unabhängigen, identisch verteilten Zufallsvariablen auftreten. Unter Verwendung von Konzepten der Informationstheorie wird in Abschnitt 2.1 eine untere Schranke für diese Konvergenzgeschwindigkeit abgeleitet. Diese besagt, dass selbst im einfachsten aller Märkte, einem Markt mit nur endlich vielen möglichen Renditekonstellationen gilt:

Es gibt keine Portfolioauswahlregel, die ihre Unterlegenheit im Vergleich zur log-optimalen Strategie in jedem Markt schneller kompensiert als $\frac{1}{n}$ gegen Null strebt, d.h. es gibt stets einen Markt, für den $\mathbf{E} \log \frac{R_n^*}{\hat{R}_n} \geq const. \cdot \frac{1}{n}$ *(Theorem 2.1.1).*

Es gibt jedoch Portfolioauswahlregeln, die diese Rate erreichen. Insbesondere das *empirisch log-optimale Portfolio*

$$\hat{b}_{n+1} := \arg\max_{b \in S} \frac{1}{n} \sum_{i=1}^{n} \log < b, X_i > \tag{0.0.3}$$

erweist sich hier als günstig:

Das empirische log-optimale Portfolio (0.0.3) erreicht die untere Schranke für die Konvergenzrate von $\mathbf{E} \log \frac{R_n^*}{\hat{R}_n}$ *(Theorem 2.1.3).*

Etwas leger ausgedrückt könnte man sagen, dass das empirisch log-optimale Portfolio seine Defizite mit optimaler Geschwindigkeit wettzumachen vermag. Die Ergebnisse gelten weitestgehend unabhängig von der Anzahl der Aktien am betrachteten Markt. Dies ist untypisch für nichtparametrische Schätzverfahren und bedarf daher genauerer Diskussion (Theorem 2.1.4 zeigt, dass dieses Phänomen auch in komplizierter gearteten Märkten auftritt).

Aus diesem Grund schließen wir in Abschnitt 2.2 eine detailliertere Diskussion der Auswirkungen der Dimension des Marktes auf die Portfolioauswahl an. Beschränkte rechnerische Kapazitäten werden den Investor bei seiner Investitionsplanung dazu zwingen, sich auf eine kleinere Teilmenge aller Aktien am Markt zu beschränken. Diese Teilmenge muss bereits in der Planungsphase, also vor dem eigentlichen Investitionsprozess ausgewählt werden. Es werden Kriterien für diese Vorauswahl benötigt. Üblicherweise würde man vorgehen, indem man einzelne Aktien auswählt, deren Chart hohe Wachstumspotentiale versprechen. Es wird gezeigt werden, dass dieser Weg mit substantiellen Unzulänglichkeiten behaftet ist:

Jedes Auswahlverfahren, das die einzelnen Aktien getrennt, z.B. anhand ihrer erwarteten logarithmierten Rendite, beurteilt, wird mit Sicherheit in einem realistischen Markt die falsche Auswahl treffen (Theorem 2.2.1).

Dieses negative Resultat zeigt, dass Portfolioauswahlverfahren über die einzelnen erwarteten log-Renditen hinausgehende Information benötigen. Die Varianz-Kovarianz-Struktur der Renditen wird in Märkten mit log-normal verteilten Renditen hinreichend viel Information vermitteln. In Abschnitt 2.3 werden die Resultate anhand von Simulationen und realen Beispielen illustriert und ihre praktische Relevanz aufgezeigt.

Kapitel 3
Renditevorhersagen und Portfolioauswahl

Mit der Erkenntnis, dass erfolgreiche Portfolioauswahl Information über die Varianz-Kovarianz-Struktur der Aktien am Markt bedarf (es spielen sowohl zeitliche Korrelationen als auch Korrelationen zwischen den einzelnen Aktien eine Rolle), wird in Abschnitt 3.1 eine Investmentstrategie vorgestellt, die sich unter den Investoren großer Beliebtheit erfreut.

Die Strategie ist zweistufig und verwendet dabei die historischen log-Renditen $Y_n, Y_{n-1}, ..., Y_0$ ($Y_i := \log X_i$) als Eingangsdaten für die Investitionsentscheidung zur Zeit n:

1. Erstelle eine Schätzung für die Zukunft des Marktes. Es wird gezeigt werden, dass Vorhersagen für den Markt auf bedingten Erwartungen für zukünftige log-Renditen bei gegebener Vergangenheit basieren sollten, d.h. auf
$$\hat{Y}_{n+1} := \mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...].$$
2. Investiere ausschließlich in die Aktien, deren Vorhersagen $\hat{Y}_{n+1}$ eine bessere Rendite verheißen als ein festverzinsliches Wertpapier mit Rendite r. In eine Aktie wird also investiert genau dann, wenn
$$\exp(\hat{Y}_{n+1}) \geq r.$$

Wir nennen diese Strategie eine Strategie für den “gierigen Investor”, da sie darauf ausgerichtet ist, nur die bestmöglichen Anlagemöglichkeiten herauszupicken. Die Einfachheit der Strategie besticht, und in Märkten mit geringer Varianz der log-Renditen führt sie zu sinnvollen Ergebnissen (Abschnitt 3.1).

Bei der Implementierung der Strategie sieht sich der Investor der Schwierigkeit gegenüber, dass die Vorhersagewerte $\hat{Y}_{n+1}$ nur unter Kenntnis der wahren Verteilung des Prozesses berechnet werden können. Daher wird man sich auf die

Berechnung einer Schätzung $\hat{E}(Y_n, ..., Y_0)$ für den bedingten Erwartungswert $\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]$ aus den Marktbeobachtungen $Y_n, ..., Y_0$ beschränken müssen. Aus der Literatur ist bekannt, dass keine auf solche Weise gewonnene Schätzung *stark konsistent* sein kann in dem Sinne, dass

$$\lim_{n\to\infty} \left| \hat{E}(Y_n, ..., Y_0) - \mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...] \right| = 0 \tag{0.0.4}$$

mit Wahrscheinlichkeit 1 für jeden stationären und ergodischen Prozess $\{Y_n\}_n$ gilt (Bailey, 1976). Dieses Resultat ist einerseits entmutigend, andererseits schließt es nicht aus, dass stark konsistente Vorhersagemechanismen für logarithmierte Renditeprozesse existieren, wie sie in realen Finanzmärkten auftreten. Dabei ist insbesondere an Gaußsche log-Renditeprozesse zu denken, die eine gute Approximation für reale log-Renditeprozesse liefern. Bis jetzt jedoch war die Frage unbeantwortet, ob für stationäre und ergodische Gaußsche Prozesse stark konsistente Vorhersagealgorithmen existieren. Abschnitt 3.2 wird nun eine positive Antwort darauf geben können. Unter schwachen Zusatzvoraussetzungen an die Wold-Koeffizienten des Prozesses

> *wird ein Vorhersagealgorithmus $\hat{E}(Y_n, ..., Y_0)$ für stationäre und ergodische Gaußsche Prozesse entwickelt, der stark konsistent gemäß (0.0.4) ist und der bemerkenswert einfach zu implementieren ist (Corollary 3.2.3).*

Diese Ergebnisse geben uns die Subroutinen an die Hand, um die Strategie für den "gierigen" Investor in Gaußschen log-Renditeprozessen umzusetzen. Der Algorithmus selbst ist jedoch auch unabhängig von seiner hier gegebenen Anwendung von Interesse, treten Vorhersageprobleme für Gaußsche Prozesse doch in einer Vielzahl von Gebieten auf.

Der Beweis der Konvergenzeigenschaften wird in Abschnitt 3.3 geführt. Anwendungsbeispiele mit realen und simulierten Daten schließen sich in Abschnitt 3.4 an und zeigen vielversprechende Ergebnisse, wenn der Algorithmus zur reinen Vorhersage, aber auch als Subroutine für die "gierige" Strategie dient.

Kapitel 4
Ein Markov-Modell mit Transaktionskosten: stochastische Aspekte

In den einfachsten Märkten, in denen die Renditen als unabhängige, identisch verteilte Zufallsvariablen auftreten, sollte in ein zeitlich konstantes log-optimales Portfolio investiert werden. Bei Verwendung eines zeitlich konstanten Portfolios verwendet man auf jede Aktie einen gleichbleibenden Anteil des aktuellen Gesamtvermögens. Der Anteil bleibt somit derselbe, bedingt durch die Änderung der Aktienpreise zueinander ändert sich jedoch die tatsächliche Anzahl an gehaltenen Aktien von Marktperiode zu Marktperiode. Zur Durchführung einer log-optimalen Strategie wird somit eine große Anzahl an Transaktionen notwendig. In der Realität stellt dies einen nicht zu unterschätzenden Nachteil dar. Was immer an Vermögen anwächst, ein Großteil der Gewinne wird zur Begleichung von *Transaktionskosten* wie Maklerprovisionen, Verwaltungs- und Kommunikationskosten wieder abfließen. Folglich muss der Investor seine Strategie diesen Gegebenheiten anpassen: Er muss so wenige kostenintensive Transaktionen wie möglich machen, aber doch so viele, um ein gutes Wertwachstum zu erzielen. Kapitel 4 und 5 widmen sich der Frage, wie diese beiden Anforderungen in einer Strategie miteinander vereinbart werden können.

Zu diesem Zweck nehmen wir an, dass die Renditen sich gemäß einem d-stufigen Markovschen Prozess entwickeln. In Kapitel 4 arbeiten wir unter der Prämisse, dass die Verteilung des Renditeprozesses bekannt ist, eine unrealistische Annahme, die wir in Kapitel 5 fallen lassen werden. Zunächst wird in Abschnitt 4.1 das Marktmodell aus Kapitel 1 um Transaktionskosten erweitert, die proportional zum Volumen gekaufter Aktien anfallen. Es ist nicht überraschend, dass sich der Investor in einer solchen Situation nur eine eingeschränkte Menge von Portfoliozusammenstellungen leisten kann, ohne bankrott zu gehen. Es wird deutlich werden,

> *dass es in d-stufigen Markovschen Renditeprozessen ausreicht, Strategien zu betrachten, die auf Portfolioauswahlfunktionen beruhen, d.h. Strategien der Form $b_i = \phi(b_{i-1}, X_{i-d}, ..., X_{i-1})$ mit einer geeigneten Funktion ϕ (Definition 4.1.2).*

Das nächste zu wählende Portfolio ist somit eine Funktion des letzten gewählten Portfolios und der letzten d am Markt beobachteten Renditevektoren. Wie zuvor strebt der Investor danach, sein zu erwartendes logarithmiertes Vermögenswachstum zu maximieren, hier nun indem er eine optimale Portfolioauswahlfunktion ϕ wählt.

Abschnitt 4.2 legt dar, wie eine optimale Auswahlfunktion c konstruiert werden kann – alles unter der Prämisse, dass die wahre Verteilung der Renditen bekannt wäre. Das Hauptresultat wird zeigen,

> *dass eine optimale Portfolioauswahlfunktion ϕ aus einer Lösung der Bellman-Gleichung konstruiert werden kann (Theorem 4.2.1, Gleichung 4.2.2).*

Die Bellman-Gleichung ist aus der Theorie der dynamischen Optimierung wohlbekannt, dennoch werden sich fundamentale Unterschiede zwischen klassischer dynamischer Optimierung und dem Portfolioauswahl-Problem zeigen. Zur Vorbereitung auf Kapitel 5 werden in Abschnitt 4.3 schließlich weitere analytische Eigenschaften der Lösung der Bellman-Gleichung abgeleitet.

Kapitel 5
Ein Markov-Modell mit Transaktionskosten: statistische Aspekte

Die Bellman-Gleichung, wie sie in Kapitel 4 aufgestellt wurde, hängt entscheidend von der Verteilung des Renditeprozesses $\{X_n\}_n$ ab. Diese Abhängigkeit besteht in Form eines zu evaluierenden bedingten Erwartungswertes. Aus diesem Grund sind die Ergebnisse von Kapitel 4 nur unter der Prämisse gültig, dass der Investor die wahre Verteilung des Renditeprozesses kennt, was in der Praxis natürlich illusorisch ist. Bestenfalls verfügt der Investor über eine Schätzung der Verteilung der Renditen. Diese ermöglicht es ihm, eine Schätzung für bewussten bedingten Erwartungswert zu berechnen, welche ihm dann eine Näherung der Bellman-Gleichung liefert. Mit Hilfe von Techniken aus der nichtparametrischen Regressionsschätzung

> *wird in Abschnitt 5.1 gezeigt, dass zur Bellman-Gleichung aus Kapitel 4 eine natürliche empirische Entsprechung basierend auf Marktbeobachtungen existiert (Gleichung 5.1.2).*

Ähnliche Schlussweisen wie in Kapitel 4 werden es uns ermöglichen, diese empirische Bellman-Gleichung unter realistischen Bedingungen zu lösen.

> *Das wird zu einer Strategie führen, die ausschließlich auf historischen Renditen basiert, dabei jedoch dieselben Optimalitätseigenschaften wie die (theoretisch) optimale Portfolioauswahlstrategie unter Transaktionskosten hat (Theoreme 5.1.1 und 5.1.2).*

In den Betrachtungen von Kapitel 5 werden wir auf Verallgemeinerungen von bekannten Resultaten über die *gleichmäßige Konvergenz von Regressionsschätzern* zurückgreifen. Diese Verallgemeinerungen werden in Abschnitt 5.2 hergeleitet. Ist z.B. $\{X_n\}_n$ ein stationärer Prozess, welcher die geometrischen

Mischungseigenschaft hat, und ist g aus einer Klasse $\mathcal{G}$ lipschitzstetiger Funktionen gewählt, schätzen wir den bedingten Erwartungswert

$$R(g,b,x) := \mathbf{E}[g(X_1,b)|X_0 = x] \qquad (b \in S)$$

durch einen Kernschätzer $R_n(g,b,x)$. In Abhängigkeit von der Glattheit einer Dichte von X_0 (wir nehmen an, dass eine solche existiert) wird die Konvergenzgeschwindigkeit in der Limesrelation

$$\sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x\in\mathcal{X},b\in S} |R_n(g,b,x) - R(g,b,x)| \to 0 \quad (n \to \infty)$$

bestimmt. Dabei wird der zu erwartende maximale Schätzfehler gleichmäßig in der Klasse $\mathcal{G}$ betrachtet (Corollary 5.2.2). Das erhaltene Resultat ist nicht nur im Hinblick auf unsere Anwendung von Interesse, sondern auch darüber hinaus als unabhängiges Resultat in der nichtparametrischen Regressionsschätzung.

Abschnitt 5.3 schließlich widmet sich dem Beweis der Optimalitätseigenschaften des Algorithmus und kombiniert dabei die Ergebnisse aus Kapitel 4 mit den Ergebnissen zur gleichmäßig konsistenten Regressionsschätzung.

Kapitel 6
Portfolioauswahlfunktionen in stationären Renditeprozessen

An realen Finanzmärkten beobachtet man unter Umständen eine Abweichung des Renditeprozesses $\{X_n\}_n$ von einem d-stufigen Markov-Prozess. Deshalb werden in diesem Kapitel noch allgemeinere Marktmodelle zu betrachten sein. Transaktionskosten werden dabei ignoriert, dafür aber Renditeprozesse betrachtet, für die im Wesentlichen nur Stationarität und Ergodizität vorausgesetzt wird. Für den Investor ist es naheliegend, seine Investitionsentscheidungen anhand der letzten d am Markt beobachteten Renditevektoren (d fest) zu treffen. Dies führt zum Konzept von log-optimalen Portfolioauswahlfunktionen.

Dieses Konzept wird in Abschnitt 6.1 eingeführt. Der Investor verwendet wieder eine logarithmische Nutzenfunktion und versucht daher, eine *log-optimale Portfolioauswahlfunktion* zu ermitteln, d.h. eine messbare Funktion

$$b^* : \mathbb{R}_+^{dm} \longrightarrow S,$$

so dass ($< \cdot, \cdot >$ bezeichnet das euklidische Skalarprodukt)

$$\mathbf{E}\left(\log < b^*(X_0,...,X_{d-1}), X_d >\right) \geq \mathbf{E}\left(\log < f(X_0,...,X_{d-1}), X_d >\right)$$

für alle messbaren Funktionen $f : \mathbb{R}_+^{dm} \longrightarrow S$. Für die $(n+1)$. Marktperiode legt b^* dem Investor nahe, das Portfolio $b^*(X_{n-d+1}, ..., X_n)$ zu erwerben.

Das Konzept log-optimaler Portfolioauswahlfunktionen bleibt in seiner Allgemeinheit hinter dem Konzept des bedingt log-optimalen Portfolios zurück (dieses wählt den Parameter d so, dass die ganze Vergangenheit des Prozesses in die Portfolioauswahl einbezogen wird). Obwohl es sich in diesem Sinn um eine Vereinfachung handelt, vereinigen log-optimale Portfolioauswahlfunktionen im Vergleich zum log-optimalen Portfolio einige Vorteile auf sich, insbesondere was Berechnung, Schätzung und Interpretation angeht.

Bei log-optimalen Portfolioauswahlfunktionen sieht sich der Investor demselben Problem gegenüber wie bei der Verwendung log-optimaler Portfolios. Beide können nur berechnet werden, wenn die wahre Verteilung des Renditeprozesses bekannt sein sollte. In der Praxis ist dies nicht der Fall, und man benötigt wieder eine Schätzprozedur, die eine log-optimale Portfolioauswahlfunktion anhand in der Vergangenheit beobachteten Renditedaten annähert.

In Abschnitt 6.2 wird deshalb ein Algorithmus entwickelt, der Schätzungen $\hat{b}_n$ für eine log-optimale Portfolioauswahlfunktion b^* aus historischen Renditedaten berechnet.

Über Stationarität und Ergodizität hinaus werden dabei sehr milde Zusatzvoraussetzungen getroffen, konkret wird davon ausgegangen, dass der Renditeprozess $\{X_n\}_{n=0}^{\infty}$ ein $[a,b]^m$-wertiger stationärer und ergodischer stochastischer Prozess ist ($0 < a \leq b < \infty$ brauchen nicht bekannt zu sein) und dass eine Lipschitzbedingung für den bedingten Renditequotienten $\mathbf{E}[X_d / < s, X_d > |X_{d-1} = x_{d-1}, ..., X_0 = x_0]$ gilt. Die Lipschitzkonstante L sei dabei eine bekannte Marktkonstante.

Mit Hilfe eines stochastischen Gradientenverfahrens und Methoden der nichtparametrischen Regressionsschätzung wird gezeigt,

dass die Schätzungen $\hat{b}_n$ mit Wahrscheinlichkeit 1 gegen die wahre log-optimale Portfolioauswahlfunktion b^* konvergieren, wobei die in der Literatur typischerweise vorausgesetzten Mixing-Bedingungen vermieden werden (Theorem 6.2.2).

In der praktischen Anwendung spielt das folgende Resultat eine noch wichtigere Rolle:

Eine Portfolioauswahl anhand der geschätzten log-optimalen Portfolioauswahlfunktionen liefert ein optimales Vermögenswachstum unter allen Strategien, die ihre Investitionsentscheidungen anhand der letzten d am Markt beobachteten Renditen treffen.

Sei $\hat{S}_n$ das Vermögen, das man nach n Marktperioden erzielt hat, wenn man am $(i+1)$. Handelstag die aktuelle Schätzung $\hat{b}_i$ verwendet, um das Portfolio $\hat{b}_i(X_{i-d+1}, ..., X_i)$ zu wählen. Wenn S_n das Vermögen angibt, das man in derselben Zeit mit einer beliebigen anderen Auswahlstrategie basierend auf den jeweils letzten d beobachteten Renditen erwirtschaftet, so ist

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{S_n}{\hat{S}_n} \leq 0$$

mit Wahrscheinlichkeit 1 (Corollary 6.2.3).

Nach einer geeigneten Modifikation behalten die Algorithmen und die Resultate ihre Gültigkeit, selbst wenn –wie in der Anwendungspraxis– die Marktkonstante L dem Investor unbekannt ist (Theorem 6.2.4). Abschnitt 6.3 beweist die Resultate, und das Kapitel wird mit mehreren realistischen Beispielen in Abschnitt 6.4 abgerundet.

Die Kapitel 2, 3 und 6 können unabhängig voneinander gelesen werden, sie sind in sich abgeschlossen. Kapitel 4 und 5 sind jedoch eng verzahnt. Notationen, die über die mathematische Standardnotation hinausgehen, werden bei ihrem ersten Auftreten erklärt. Der Leser sei auch auf das Abkürzungsverzeichnis am Anfang dieser Arbeit verwiesen. Berechnungen und Schaubilder für die Beispiele wurden mit Matlab 4.0 und 6.0.0.88, Minitab 11.2 sowie R 1.1.1 erzeugt, wobei die historischen Kursnotierungen (tägliche Schlusskurse) der New Yorker Börse von www.wallstreetcity.com Verwendung fanden.

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CHAPTER 1

Introduction: investment and nonparametric statistics

Investment is the strategic allocation of resources, typically of monetary resources, in an environment, typically a market of assets, whose future evolution is uncertain. This definition leaves much room for subjective interpretation. In particular, the following points have to be made more precise:

– *What market is under consideration?* This involves specifying and standardizing the assets traded in the market (e.g. stocks, bonds, options, futures, currencies, gold, oil, ...) as well as setting up a reference system for pricing the assets (e.g. closing or opening prices at the New York Stock Exchange, world market price for raw materials, ...).

– *What actions and instruments may be applied by the investor?* Possible actions may be restricted by exogenous terms and regulations of trade (e.g. transaction costs, brokerage fees, trading limitations) or personal preferences (e.g. to rule out borrowing money or short positions in stocks).

– *What investment goal is pursued by the investor?* Traditionally, the goal is the maximisation of a personal utility function of the returns on the allocated resources. The market being chancy, individual risk aversion preferences may enter the form of the utility function, or restrictions are imposed on the set of possible investment actions.

Thus, "investment" becomes a highly subjective term, including investment as it is understood in this thesis. In the following we set up the specific investment scenario as we shall consider it in this thesis. We believe this scenario is broadly accepted as the typical setting for investment analysis, although we do not deny that particular investment situations require further adaptation and modification. It should also be pointed out that, as future asset prices are subject to random fluctuations, "investment" is a good deal about decision taking under uncertainty, which makes mathematical statistics the natural partner of investment (an observation that may be attributed to the groundbreaking work of Bachelier, 1900, who used statistics to compare his theoretical model with real market data). An economist will find the economic side of this thesis to be lacking. There are excellent books on investment science from a more economic point of view (e.g. Francis, 1980; Luenberger, 1998), but most of them are lacking in statistical depth. This thesis is about investment from a decisively *statistical* point of view – we can therefore only superficially touch upon economic issues.

1.1 The market model

We consider a market in which m assets (which we will think of as stocks and bonds) are traded. Taking a macroeconomic point of view, the prices of the assets (stock quotes, bond values) are generated under the authority of the market as a whole, i.e. by the large ensemble of investors. We assume that for the individual investor there is no way to influence the prices by launching specific investment actions or distributing insider or side information of whatever kind. In this situation, let $P_{1,n}, ..., P_{m,n} > 0$ be the prices of the assets $1, ..., m$ at the beginning of market period n (market period n lasts from "time" $n-1$ to n, time being measured, e.g., in days of trading). To the "powerless" individual investor described above, the asset prices present themselves as a random process on a common probability space $(\Omega, \mathcal{A}, \mathbf{P})$.

The return of an investment of 1 unit of money in asset i at time $n-1$ yields a return

$$X_{i,n} := \frac{P_{i,n}}{P_{i,n-1}}$$

during the subsequent market period. We collect the returns of the single assets in a **return vector**

$$X_n := (X_{1,n}, ..., X_{m,n})^T.$$

We will often work with the log-returns

$$\log X_n := (\log X_{1,n}, ..., \log X_{m,n})^T.$$

The **return process** $\{X_n\}_{n=1}^{\infty}$ and the **log-return process** $\{\log X_n\}_{n=1}^{\infty}$ are stochastic processes on $(\Omega, \mathcal{A}, \mathbf{P})$.

In most of our investigations we will assume that the return process $\{X_n\}_n$ is stationary and ergodic in the sense of the following definition (Stout, 1974, Sec. 3.5; Shiryayev, 1984, V §3):

Definition 1.1.1. *Let $\{X_n\}_{n=1}^{\infty}$ be an $\mathbb{R}^m$-valued stochastic process on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$.*

1. *$\{X_n\}_{n=1}^{\infty}$ is called* **stationary**, *if*
$$\mathbf{P}_{(X_i,\ldots,X_j)} = \mathbf{P}_{(X_{i+t},\ldots,X_{j+t})}$$
for all integers i, j, t with $i \leq j$.
2. *$A \in \mathcal{A}$ is called an* **invariant event** *of $\{X_n\}_{n=1}^{\infty}$, if there exists a $B \in \mathcal{B}((\mathbb{R}^m)^{\infty})$ such that*
$$A = \{X_i, X_{i+1}, \ldots\}^{-1}(B)$$
for all $i \in \mathbb{N}$.
3. *A stationary process $\{X_n\}_{n=1}^{\infty}$ is called* **ergodic**, *if the probability of any invariant event of $\{X_i\}_{i=1}^{\infty}$ is either 0 or 1.*

Stationarity preserves the stochastic regime over time, ergodicity is the setting where time averages along trajectories of the process converge almost surely to expected values under the process distribution:

Theorem 1.1.2. *(Birkhoff Ergodic Theorem, Stout, 1974, Sec. 3.5) Let $\{X_n\}_{n=1}^{\infty}$ be an $\mathbb{R}^m$-valued stationary and ergodic stochastic process on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with $\mathbf{E}|X_1| < \infty$. Then*
$$\frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow \mathbf{E}X_1$$
P-*almost surely (***P**-a.s.*), i.e. for all $\omega \in \Omega$ from a set of probability 1.*

Stationarity and ergodicity are the basic assumptions for most statistical investigations. The stationarity of stock returns is a thoroughly investigated field, both by economists (e.g. Francis, 1980, A24-1) and statisticians (e.g. Franke et al., 2001, Sec. 10.6). It is natural to assume that there is short term stationarity in most stock returns, some authors (Francis, 1980) even claim that return data may be treated as stationary if the time horizon comprises at least one complete business cycle. There is no conclusive answer that proves or disproves stationarity for the majority of stock markets, and it seems as though this has to be decided from case to case. We accept stationarity as a working hypothesis, accounting for the fact that it is common practice to assess and compare the performance of statistical methods in the stationary setting.

Not much is known about the ergodic properties of stock quotes or stock returns, neither from the theoretical economist's point of view, nor from empirical studies. There are indications that the ergodic properties of a market depend very much upon the flow of information in the market and on the microeconomic price generation (Donowitz and El-Gamal, 1997). These are difficult to assess, and so the typical approach has become to derive algorithms under ergodic hypotheses and then let the success of the algorithm justify the hypotheses.

Throughout this thesis we consider **nonparametric models** for $\{X_n\}_n$, i.e. models that do not require a parametrized evolution equation (in contrast to MA, AR, ARMA, ARIMA, ARCH and GARCH models, cf. Brockwell and Davis, 1991, Franke et al., 2001). The nonparametric approach guarantees highest flexibility in modelling, skipping model parameters which otherwise require extensive diagnostic model testing. To be more precise, the following models will be investigated in this thesis:

1. $\{X_n\}_n$ is a sequence of independent identically distributed (i.i.d.) random variables (e.g. with finitely many outcomes) – Chapter 2.1.

2. The conditional distribution of X_{n+1} given $X_n, ..., X_1$ (which we will denote by $\mathbf{P}_{X_{n+1}|X_n,...,X_1}$) is log-normally distributed (i.e. $\mathbf{P}_{\log X_{n+1}|X_n,...,X_1}$ has a normal distribution) – Chapter 2.2.

3. $\{\log X_n\}_n$ is a stationary Gaussian time series (i.e. $(\log X_{n+k}^T, ..., \log X_n^T))$ follows a multivariate normal distribution which depends upon k but not upon n) – Chapter 3.

4. $\{X_n\}_n$ is a Markov process of order d (i.e., we assume $\mathbf{P}_{X_{n+1}|X_n,...,X_1} = \mathbf{P}_{X_{n+1}|X_n,...,X_{n-d+1}}$) – Chapters 4 and 5.

5. $\{X_n\}_n$ is a stationary and ergodic time series – Chapter 6.

Each of these models has been found useful for describing asset return data in real financial markets. Model 1 is the Cox-Ross-Rubinstein model (Cox et al., 1979; Francis, 1980, A24-1 and A24-2; Luenberger, 1998, Ch. 11; Franke et al., 2001, Ch. 7). Models 2 and 3 are models with log-normal returns (Francis, 1980, A24-1; Luenberger, 1998, Ch. 11) which arise, e.g., from a discretisation of the Black-Scholes model (Luenberger, 1998, Ch. 11; Korn and Korn, 1999, Kap. II). In contrast to the classical Black-Scholes model we allow for autocorrelated log-returns in Chapter 3 (i.e. $\mathbf{Cov}(\log X_n, \log X_{n+k}) \neq 0$ for some $k > 0$). In practice, autocorrelation of the log-returns manifests itself for small time lags k (Franke et al., 2001, Ch. 10) as well as large k (*long range dependence*, Ding et al., 1993; Peters, 1997). Many studies have indicated that the logarithms of stock returns slightly depart from a Gaussian distribution (e.g. by *heavy tails*, Mittnik and Rachev, 1993; McCulloch, 1996; Franke et al., 2001, Ch. 10 and the references there). It is therefore advisable to drop the assumption of log-normality of the stock returns wherever possible. This is done in models 4 and 5, model 4 capturing the autocorrelation of stock returns by the Markov property.

We will assume that the asset returns correspond to one of the models 1-5. However, we do not assume the exact form of the true return distribution to be known to the investor (with the exception of Chapter 4). Hence, the investor has to apply statistical estimation and forecasting techniques for strategy planning. Clearly, nonparametric models require nonparametric statistical methods and arguments are usually more involved than in the parametric setting (for an introduction to nonparametric estimation as we will use it see Györfi et al. 1989, 2002). Unfortunately, nonparametric methods are not yet common in econometrics and financial mathematics (Pagan and Ullah, 1999, and Franke et al., 2001, being two of the few notable exceptions). In this thesis we aim to demonstrate what powerful impetus nonparamtric statistical estimation may give to investment strategy planning.

1.2 Portfolios and investment strategies

Having chosen a market model, we turn to the actions that may be taken by the investor. Throughout the investment process, the investor holds varying portfolios of the m assets. Taking a discrete time trading point of view, we assume that the investor is only allowed to rebalance his portfolio at the beginning but not in the course of each market period. The portfolio held at the beginning of market period n (i.e. from time $n-1$ to n) can be given by the quantities $q_{1,n-1}, ..., q_{m,n-1}$ of the single assets owned by the investor ($q_{i,n-1} < 0$ corresponds to borrowed assets, so-called short positions). The investor then enters the nth market period with a portfolio value of

$$W_{n-1}^{+} := \sum_{i=1}^{m} P_{i,n-1} q_{i,n-1}.$$

The remaining value at the end of the market period is

$$W_n^{-} := \sum_{i=1}^{m} P_{i,n} q_{i,n-1} = \sum_{i=1}^{m} X_{i,n} P_{i,n-1} q_{i,n-1}.$$

Hence, if $W_{n-1}^{+} \neq 0$, the portfolio achieved a return of

$$\frac{W_n^{-}}{W_{n-1}^{+}} = \sum_{i=1}^{m} X_{i,n} b_{i,n} \tag{1.2.1}$$

with

$$b_{i,n} := \frac{P_{i,n-1} q_{i,n-1}}{\sum_{j=1}^{m} P_{j,n-1} q_{j,n-1}}.$$

Note that $\sum_{i=1}^{m} b_{i,n} = 1$, and we will find it more convenient to denote a portfolio by the **portfolio vector**

$$b_n := (b_{1,n}, ..., b_{m,n})^T$$

rather than listing $q_{1,n-1}, ..., q_{m,n-1}$. If the investor is allowed to consume an amount c_n before changing his portfolio b_n for b_{n+1} and entering market period $n+1$, then W_n^{+} is given by

$$W_n^{+} = W_n^{-} - c_n. \tag{1.2.2}$$

(1.2.1) and (1.2.2) are the equations governing general discrete time investment.

Throughout this thesis we are concerned with an investor who neither consumes nor deposits new money into his portfolio but reinvests his current portfolio

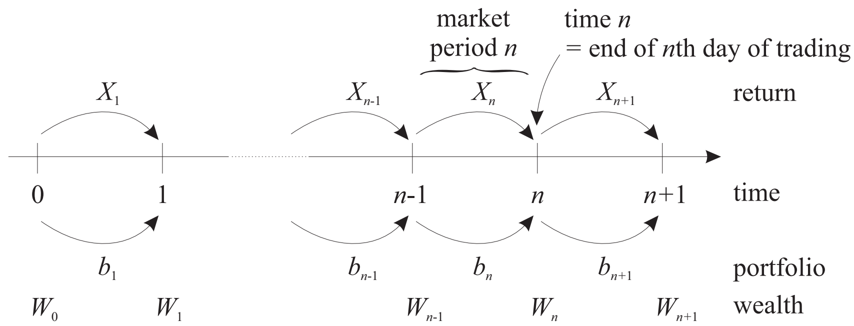


Figure 1.1: Setting for the return and portfolio processes.

value in each market period. Hence, $c_n = 0$ for all n and (1.2.1) and (1.2.2) boil down to

$$W_n := W_n^+ = W_n^- = W_0 \prod_{i=1}^{n} < b_i, X_i >, \tag{1.2.3}$$

where W_n is the current wealth of the investor at time n. Moreover, the factor $< b_n, X_n >:= b_n^T X_n$ can be interpreted as the **portfolio return** during the nth market period.

Moreover, we assume that the investor never enters short positions, i.e. $b_{i,n} \geq 0$. Then $b_{i,n}$ is the proportion of the current wealth W_n invested in asset i at time $n-1$. The portfolio vector b_n chosen at time n is a member of the simplex

$$S := \left\{ (s_1, ..., s_m)^T \,\middle|\, \sum_{i=1}^{m} s_i = 1, s_i \geq 0 \right\}.$$

The choice of b_n depends on the information I_n which the investor can access at time n and which he deems relevant. Thus $b_n = b_n(I_n)$ where I_n typically comprises a number of past observed asset returns (a substring of $X_1, ..., X_{n-1}$), in some cases additional side or insider information about the market and external economic factors. For specific choices of I_n, this is the setting for Chapters 2, 3 and 6 (Figure 1.1).

In reality, the range of portfolio choices is further narrowed by the occurrence of **transaction costs**. Each transaction in a real market (purchase, sale of assets) generates costs (brokerage fees, commission, administrative and communication expenses). The total amount of these fees is withdrawn from the investor's wealth. Thus, the range of portfolios the investor may choose is restricted to those portfolios whose acquisition generates no more transaction costs than the investor's current wealth. Then, roughly speaking, the investor is caught between making as few costly transactions as possible on the one hand and making as many transactions as necessary to boost his wealth on the other hand. No wonder that strategic planning under transaction costs requires much deeper arguments and has received considerable attention in literature for both, discrete and continuous time models (see e.g. Blum and Kalai, 1999; Bobryk and Stettner, 1999; Cadenillas, 2000; Bielecki and Pliska, 2000). We shall return to a typical case of transaction costs in more detail in Chapters 4 and 5.

1.3 Pleading for logarithmic utility

As can be seen from (1.2.3), invested money grows multiplicatively, as a product of daily returns. Suppose the investor wants to maximize the expected value of his terminal wealth. If the daily returns $\{X_i\}_i$ are stationary, maximization of the single expected daily returns is not appropriate. It does not capture autocorrelation in the returns, since, in general,

$$\mathbf{E}\prod_{i=1}^{n} < b_i, X_i > \neq \prod_{i=1}^{n} \mathbf{E} < b_i, X_i > .$$

The expectation is rather determined by the expectation of the *logarithmic* daily returns, since by Taylor expansion (*hot.* denoting terms of order 2 and higher)

$$\mathbf{E}\prod_{i=1}^{n} < b_i, X_i >= 1+\mathbf{E}\log\prod_{i=1}^{n} < b_i, X_i > +hot. = 1+\sum_{i=1}^{n}\mathbf{E}\log < b_i, X_i > +hot.$$

It is widely accepted that for returns below 10% and high frequency data (e.g. daily returns) the logarithmic approximation is convincing (Franke et al., 2001, Sec. 10.1). This leads to the notion of **log-optimal portfolios**, i.e. portfolios that maximize the expected logarithmic utility of the investors's growth of wealth. The log-optimal portfolio of a process $\{X_n\}_n$ of independent and identically distributed (i.i.d.) returns X_n is defined as

$$b^* := \arg\max_{b\in S} \mathbf{E}(\log < b, X_1 >). \tag{1.3.1}$$

Log-optimal portfolios have been suggested first by Kelly (1956), Latané (1959) and Breiman (1961) as diversification strategy for investment in a speculative market given by a process $\{X_n\}_{n=1}^{\infty}$ of i.i.d. return vectors. Since then, numerous investigations, notably by Cover (e.g. Cover, 1980, 1984; Cover and Thomas, 1991) and Algoet (e.g. Algoet and Cover, 1988) have explored the theoretical aspects of this strategy, establishing that investment in log-optimal portfolios yields optimal asymptotic growth rates for the invested wealth. An introduction, various results and sources of reference can be found in Cover and Thomas (1991, Chapter 15). There, for stationary and ergodic return processes $\{X_n\}_n$, (1.3.1) is generalized by the **conditional log-optimal portfolio** (for the nth investment step)

$$b_n^* := \arg\max_{b \in S} \mathbf{E}[\log < b, X_n > | X_{n-1}, ..., X_1]$$

in stationary ergodic return processes (conditioning being void for $n = 1$). The conditional log-optimal portfolio is the log-optimal portfolio under the conditional distribution $\mathbf{P}_{X_n | X_{n-1},...,X_1}$ and hence a random variable. The log-optimal investment strategy $b_1^*, b_2^*, ...$ is a member of the class of **non-anticipating strategies**, i.e. sequences of S-valued random variables $b_1, b_2, ...$ with the property that each b_n is measurable w.r.t. the σ-algebra generated by $X_1, ..., X_{n-1}$ (hence the strategy requires no more information than available at time n). The technical aspects of conditional log-optimal portfolios (we will often drop "conditional" for brevity) are well explored:

Existence and uniqueness of the log-optimal portfolio has been investigated in Österreicher and Vajda (1993) and Vajda and Österreicher (1994), correcting a wrong criterion used in Algoet and Cover (1988). The main result is

Theorem 1.3.1. *(Vajda and Österreicher, 1994) Let $X = (X_1, ..., X_m)$ be a stock market return vector with distribution $\mathbf{P}_X$. Then there exists a log-optimal portfolio $b^* \in S$ with $|\mathbf{E} \log < b^*, X > | < \infty$ if and only if*

$$\mathbf{E}\left|\log \sum_{i=1}^{m} X_i\right| < \infty.$$

b^* is unique if $\mathbf{P}_X$ is not confined to a hyperplane in $\mathbb{R}^m$ containing the diagonal $\{(d, ..., d) \in \mathbb{R}^m | d \in \mathbb{R}\}$.

A good algorithm for the *calculation* of a log-optimal portfolio from the (known) distribution $\mathbf{P}_X$ of the return vector X was given by Cover (1984).

Theorem 1.3.2. *(Cover, 1984) Assume the support of $\mathbf{P}_X$ is of full dimension in $[0,\infty)^m$ and choose some $b^0 \in S$ with non-zero entries. Then the recursively generated portfolio vectors $b^k = (b_1^k, ..., b_m^k)$ with*

$$b_i^{k+1} = b_i^k \cdot \mathbf{E}\frac{X_i}{< b^k, X >}$$

converge to the log-optimal portfolio b^* as $k \to \infty$.

This is closely linked with the following, the *Kuhn-Tucker conditions* for a log-optimal portfolio.

Theorem 1.3.3. *(Cover and Thomas, 1991, Theorem 15.2.1) A portfolio vector $b^* = (b_1^*, ..., b_m^*) \in S$ is a log-optimal portfolio for the return vector $X = (X_1, ..., X_m)$ if and only if it satisfies the conditions*

$$\mathbf{E}\frac{X_i}{< b^*, X >} \quad \begin{cases} = 1 & \text{if} \quad b_i^* > 0, \\ \leq 1 & \text{if} \quad b_i^* = 0. \end{cases}$$

The *superiority of investment according to conditionally log-optimal portfolios* rests upon the following theorem (Algoet and Cover, 1988; Cover and Thomas, 1991, Theorem 15.5.2).

Theorem 1.3.4. *(Algoet and Cover, 1988) Assume the return vectors $\{X_i\}_{i=1}^{\infty}$ form a stationary and ergodic process. Let $S_n^* := \prod_{i=1}^n < b_i^*, X_i >$ be the wealth at time n resulting from a series of conditionally log-optimal investements, S_n the wealth from any other non-anticipating portfolio strategy (both starting with 1 unit of money). Then*

$$\mathbf{E}\frac{S_n}{S_n^*} \leq 1 \quad \text{for all } n \text{ and} \quad \limsup_{n\to\infty} \frac{1}{n} \log \frac{S_n}{S_n^*} \leq 0 \quad \text{with probability 1.} \tag{1.3.2}$$

The second part of (1.3.2) can be interpreted in various ways:

– It proves that, eventually for large n, $S_n < \exp(n\epsilon)S_n^*$ whatever $\epsilon > 0$, which means that no non-anticipating strategy can infinitely often exceed the log-optimal strategy by an amount that grows exponentially fast (i.e. an amount that couldn't be compensated for by investment in a fixed interest rate bank account).

– It proves that the log-optimal portfolio will do at least as well as any other non-anticipating strategy to first order in the exponent of capital growth, i.e. it guarantees $S_n^* = \exp(nW + o(n))$ with highest possible rate W.

From the first part of (1.3.2) Bell and Cover (1988) conclude that

– there is no essential conflict between good short-term and long-run performance. Both are achieved by maximizing the conditional expected log-return.

The log-optimality criterion has not been undisputed, however. In his criticism, Samuelson (1971, also discussed in Markowitz, 1976) considers a market with i.i.d. returns $X_1, X_2, ...$ and compares the expected wealth $\mathbf{E}S_n^*$ from a series of log-optimal investments with the expected wealth $\mathbf{E}S_n^{**}$ from investment in the fixed portfolio

$$b^{**} := \arg\max_{b \in S} \mathbf{E} < b, X_1 >$$

(maximization of expected return). Using the independence and the identical distribution of the returns he finds that

$$\frac{\mathbf{E}S_n^{**}}{\mathbf{E}S_n^*} = \frac{\mathbf{E}\prod_{i=1}^n < b^{**}, X_i >}{\mathbf{E}\prod_{i=1}^n < b_i^*, X_i >} = \left(\frac{max_{b \in S}\mathbf{E} < b, X_1 >}{\mathbf{E} < b_1^*, X_1 >}\right)^n \to \infty \quad (n \to \infty).$$

Hence there are strategies that outperform log-optimal strategies in terms of expected terminal wealth for long run investment. However, when comparing the investor's strategy with a competing strategy we think that the ratio of wealths considered (1.3.2) is more instructive than two seperate expectations for the investor's strategy and the competing strategy. This is a typical example of criticism offered by classical economists who favour the **Markowitz mean-variance approach** to portfolio optimization (Markowitz, 1959; Luenberger, 1998, Ch.6.4ff.). There, the investor seeks to maximize the portfolio performance $\mathbf{E} < b, X >$ under the constraint of not exceeding a certain threshold for the risk $\mathbf{Var} < b, X >$ (or for the value-at-risk, i.e., quantiles of the return distribution, in more modern versions of the mean-variance approach).

We would like to emphasize that it is *not* a question of taste whether or not to use the log-optimal approach. We strongly plead for investment under logarithmic utility because of the following facts:

– In their spirited defence of the log-optimal criterion Algoet and Cover (1988) come to the conclusion that the mean-variance approach lacks generality (e.g. for non-log-normally distributed returns, see Samuelson, 1967, and for multiperiod investment, see Luenberger, 1998, Ch. 8.8).

– It is doubtful whether investment analysis should be founded on expectations (where typically S_n deviates much more from $\mathbf{E}S_n$ than $\log S_n$ from $\mathbf{E}\log S_n$, stabilizing effect of the log-transform). Pathwise results as the second part of (1.3.2) are more instructive than results on averages.

Realistically, the true distribution of market returns and hence the log-optimal strategy is not revealed to the investor. Then the key problem is (as Algoet, 1992, put it):

Find a non-anticipating portfolio selection scheme $\{\hat{b}_n\}_n$ *(a so-called universal portfolio selection scheme) such that for any stationary ergodic market process* $\{X_n\}_n$*, the compounded capital* $\hat{S}_n := \prod_{i=1}^n < \hat{b}_i, X_i >$ *will grow exponentially fast almost surely (i.e. with probability 1) with the same maximum rate as under the log-optimum strategy* $\{b_n^*\}_n$*, that is,* $\lim_{n\to\infty} \log \hat{S}_n/n = \lim_{n\to\infty} \log S_n^*/n$ *almost surely.*

To obtain a universal portfolio selection scheme, under weak conditions on the market one may choose the log-optimal portfolio with respect to some appropriately consistent estimate of $\mathbf{P}_{X_n|X_1,\ldots,X_{n-1}}$ in the nth investment step (more precisely, distribution estimates that almost surely exhibit weak convergence to the true distribution). This was demonstrated by Algoet (1992, Theorem 7). He also provides an appropriate, yet complicated estimation scheme (Algoet, 1992, Theorem 9). Instead, we can also use the more transparent scheme of Morvai et al. (1996). Algoet points out that there are universal portfolio selection schemes that do not require an explicit distribution estimation scheme as a subroutine (Algoet, 1992, Sec. 4.3). But still, all existing algorithms seem to require an enormous amount of past data, making their feasibility in practical situations doubtful (as noted, e.g., in Yakowitz, Györfi et al., 1999). More practicable results have been obtained in the case of independent, identically distributed return vectors. For instance, Morvai (1991, 1992) and Österreicher and Vajda

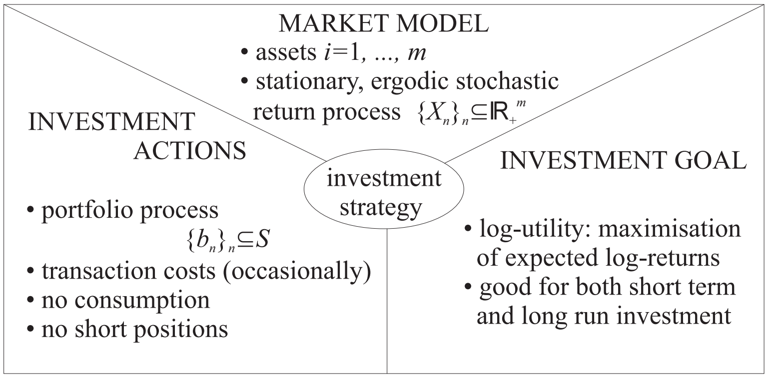


Figure 1.2: Our approach to investment strategy planning.

(1993) propose portfolio strategies which are based on selecting the log-optimal portfolio with respect to the empirical distribution of the data (the so-called **empirical log-optimal portfolio**, more on that in Chapter 2). Those estimators can be computed with reasonable effort. Repeated investment following their strategies asymptotically yields the optimal growth rate of wealth with probability one. However, in merely stationary and ergodic return processes they produce suboptimal results.

This thesis aims to provide simplified, yet efficient portfolio selection algorithms if the log-returns follow a Gaussian process (Chapters 2 and 3), a Markov process (Chapters 4 and 5) or, more general, a stationary and ergodic process (Chapter 6). Our approach is summarized in Figures 1.1 and 1.2.

For the sake of completeness, it should be noted that in recent years, the log-optimality criterion has been generalized in several ways. In particular, researchers tried

– to make the log-optimality criterion risk sensitive, i.e. to introduce devices which allow the investor to adjust the log-optimal strategy to his individual risk aversion level. This may be achieved in two different ways: Either, as in the Markowitz mean-variance model, the investor seeks to maximize the expected log-return under variance constraints (Ye and Li, 1999), or the log-utility is extended by the variance, e.g. when maximizing $-(2/\theta)\log \mathbf{E}\exp(-(\theta/2)\log S_n) = \mathbf{E}\log S_n - (\theta/4)\mathbf{Var}\log S_n + O(\theta^2)$, where $\theta > 0$ is a risk aversion parameter (Bielecki and Pliska, 1999, and Stettner, 1999, for continuous time models; Bielecki, Hernández and Pliska, 1999, for a discrete time model).

- to rid the log-optimality criterion of its stochastic setting, making it applicable to markets with doubtful stochastic properties (Cover, 1991; Cover and Ordentlich, 1996, including side information; Helmbold et al., 1998, investigating algorithmic issues and Blum and Kalai, 1999, for the transaction cost case).

It is left for future research to generalize the results of this thesis to these extended models.

CHAPTER 2

Portfolio benchmarking: rates and dimensionality

Based on market observations, the investor can follow many different empirical portfolio selection rules ("empirical" being synonymous with "based on historical return data"). Not all of these necessarily turn out to be a good choice in view of the investor's goal. Discriminating between "good" and "bad" portfolios requires the investor to compare the performance of his portfolio with the given investment goal. Naturally, "good" empirical portfolio selection rules should approach the investment goal. It is of serious interest to determine how fast the investor approaches his goal as more and more information about the market is gathered. This is the primary task of what we might call "**portfolio benchmarking**". Portfolio benchmarking analyses how well a given portfolio b or a given portfolio selection rule $b = b(X_1, ..., X_n)$ of the past returns $X_1, ..., X_n$ performs with respect to a fixed benchmark, in our case with respect to the expected logarithmic portfolio return in the next market period, $\mathbf{E} \log bX_{n+1}$ (to keep notation simple we write $\log bX_{n+1}$ rather than $\log b^T X_{n+1}$ or $\log < b, X_{n+1} >$ in the sequel). This, of course, requires a standardized way to assess to what extent an empirical portfolio selection rule underperforms in comparison with the log-optimal rule.

In this chapter we analyse how seriously a log-optimal portfolio selection rule based on an estimate for the true return distribution may underperform. To this end, we propose a specific measure of underperformance (cf. (2.1.2)). Establishing a lower bound result on this measure, it will be seen that underperformance cannot vanish at arbitrarily high rate as the investor gathers more and more knowledge about the market (Theorem 2.1.1). All investors are subject to a universally limited rate at which investment rules can succeed in exploiting historical market data.

In fact, the empirical log-optimal portfolio of Chapter 1 turns out to be a selection rule that achieves the optimal rate (Theorem 2.1.3). It is particularly striking that this rate does not depend upon the numbers of stocks included in the portfolio selection process (Theorem 2.1.4). One is tempted to think that arbitrarily large portfolios can be handled successfully without extra precautions. Reasons will be given why this is fallacious and does not obviate the necessity of trying to keep the dimension of the portfolio at reasonably low level by pre-selecting a "good" subset of all possible stocks.

However, the pre-selection of stocks is far from being an easy going thing: As we shall see, there is no way of pre-selecting the stocks on the basis of the performance of the single stocks only (Theorem 2.2.1). To find the optimal portfolio configuration, the investor has to evaluate the log-optimal portfolios of all possible subsets of stocks and compare the resulting expected logarithmic portfolio returns, a huge though necessary computational effort in high dimensions.

2.1 Rates of convergence in i.i.d. models

Suppose the m-dimensional stock return vectors $X_1, X_2, ...$ constitute a sequence of independent, identically distributed (i.i.d.) random variables with distribution $\mathbf{Q} := \mathbf{P}_{X_1}$. $\mathbf{Q}$ is not disclosed to the investor, who, after n market periods, may exploit the observations $X_1, ..., X_n$ to obtain a distribution estimate $\hat{\mathbf{Q}}_n$ of $\mathbf{Q}$. Let F and $\hat{F}_n = \hat{F}_n(\cdot, X_1, ..., X_n)$ denote the cumulative distribution functions associated with $\mathbf{Q}$ and $\hat{\mathbf{Q}}_n$, respectively. We shall restrict our analysis to estimators $\hat{\mathbf{Q}}_n$ whose sensitivity to outliers is such that

$$\Big|\hat{F}_n(x, X_1, ..., X_{i-1}, X_i, X_{i+1}, ..., X_n) \\ -\hat{F}_n(x, X_1, ..., X_{i-1}, X_i', X_{i+1}, ..., X_n)\Big| \leq \frac{c(x, X_i, X_i')}{n} \qquad (2.1.1)$$

for some function $c : \mathbb{R}_+^{3m} \to \mathbb{R}_+$, whatever $i \in \mathbb{N}, x, X_1, ..., X_n, X_1', ...X_n' \in \mathbb{R}_+^m$ may be. Most of the standard distribution estimates share this property, such as the empirical distribution

$$\hat{\mathbf{Q}}_n(A) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_A(X_i) \qquad (A \in \mathcal{B}(\mathbb{R}^m))$$

and kernel estimates

$$\hat{\mathbf{Q}}_n(A) = \int_A \frac{1}{nh_n^d} \sum_{i=1}^n K\left(\frac{x - X_i}{h_n}\right) dx \qquad (A \in \mathcal{B}(\mathbb{R}^m)),$$

h_n being a sequence of nonnegative bandwidths and $K : \mathbb{R}^m \to \mathbb{R}_0^+$ a kernel function.

Having thus learned a "picture" of the market, $\hat{\mathbf{Q}}_n$, the investor allocates his wealth according to the corresponding log-optimal portfolio

$$\hat{b}_{n+1} = \hat{b}_{n+1}(X_1, ..., X_n) = \arg\max_{b \in S} \mathbf{E} \log bY,$$

where the expectation is calculated for $Y \sim \hat{\mathbf{Q}}_n$. This choice yields the random return $\hat{R}_n = \hat{b}_{n+1} X_{n+1}$ during the next market period. In order to determine how well $\hat{b}_n$ reproduces the true log-optimal portfolio $b^* = \arg\max_{b \in S} \mathbf{E} \log bX_1$ with return $R_n^* = b^* X_{n+1}$, we first observe that

$$\begin{aligned}
\mathbf{E} \log R_n^* - \mathbf{E} \log \hat{R}_n &= \mathbf{E} \log b^* X_{n+1} - \mathbf{E}\Big(\mathbf{E}[\log \hat{b}_n(X_1, ..., X_n) X_{n+1} | X_1, ..., X_n] \Big) \\
&\geq \mathbf{E} \log b^* X_{n+1} - \mathbf{E}\Big(\mathbf{E}[\log b^* X_{n+1} | X_1, ..., X_n] \Big) \\
&= \mathbf{E} \log b^* X_{n+1} - \mathbf{E} \log b^* X_{n+1} = 0,
\end{aligned}$$

using the independence of $X_1, ..., X_{n+1}$. Hence

$$\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) := \mathbf{E} \log \frac{R_n^*}{\hat{R}_n} \geq 0 \tag{2.1.2}$$

with equality if $\hat{\mathbf{Q}}_n = \mathbf{Q}$. On the other hand, from Theorem 1.3.4,

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \log \frac{R_i^*}{\hat{R}_i} \geq 0.$$

Taking expectations and using Fatou's lemma we obtain

$$0 \leq \mathbf{E} \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \log \frac{R_i^*}{\hat{R}_i} \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \mathbf{E} \log \frac{R_i^*}{\hat{R}_i}.$$

Therefore, $\{\hat{b}_n\}_n$ is a good portfolio selection rule if

$$\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) = \mathbf{E} \log \frac{R_n^*}{\hat{R}_n} \to 0$$

with high rate as n tends to ∞.

$\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q})$ measures underperformance of $\hat{b}_n$ w.r.t. the benchmark portfolio b^*. In the sequel we will derive asymptotic properties of $\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q})$. The following theorem shows that the limit cannot be achieved at arbitrarily high speed of convergence:

Theorem 2.1.1. *For any sequence of distribution estimates $\hat{\mathbf{Q}}_n$ satisfying (2.1.1), there exists a market distribution $\mathbf{Q}$ and a market constant c for which*

$$\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) \geq \frac{c}{n} \tag{2.1.3}$$

for infinitely many n.

As will be seen in the proof, (2.1.1) is not needed when considering unbiased estimators $\hat{\mathbf{Q}}_n$, i.e., estimators for which $\mathbf{E}\hat{\mathbf{Q}}_n(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{B}(\mathbb{R}^m_+)$.

Proof. Consider a 2 stock market with return vector $(X^{(1)}, X^{(2)}) \in \mathbb{R}^2_+$ and portfolios $(b, 1-b), b \in [0,1]$. We can expand

$$\mathbf{E}\log(bX^{(1)} + (1-b)X^{(2)}) = \mathbf{E}\log((Z-1)b+1) + \mathbf{E}\log X^{(2)} \tag{2.1.4}$$

with the return ratio $Z := X^{(1)}/X^{(2)}$. Thus, in a 2 stock market, the log-optimal portfolio only depends upon the distribution of the return ratio Z. For simplicity, let Z be of the form

$$Z = \begin{cases} A & \text{with probability } p, \\ B & \text{with probability } 1-p \end{cases} \tag{2.1.5}$$

with $p \in (0,1), A, B > 0$ to be chosen later.

We first consider the classical parameter estimation problem of estimating p, which will be linked with the portfolio selection problem at a later stage. $\hat{\mathbf{Q}}_n$ allows the investor to derive an estimate of p,

$$\hat{p}_n = \hat{p}_n(z^n) := \hat{\mathbf{Q}}_n\left(\left\{(x^{(1)}, x^{(2)}) : x^{(1)}/x^{(2)} = A\right\}\right),$$

$z^n \in \{A, B\}^n$ being the observed realisations of the i.i.d. return ratios $Z_1, ..., Z_n$ (independent of Z). If $k(z^n)$ denotes the number of A's in z^n and $B_i^n(p) = \binom{n}{i} p^i (1-p)^{n-i}$ denotes the ith Bernstein polynomial of order n, we can identify

$$f_n(p) := \mathbf{E}\hat{p}_n(Z_1, ..., Z_n) = \sum_{i=0}^{n} \left\{ \binom{n}{i}^{-1} \sum_{z^n : k(z^n) = i} \hat{p}_n(z^n) \right\} B_i^n(p) =: \sum_{i=0}^{n} b_{i,n} B_i^n(p)$$

as a Bézier curve. For reasons to become clear later it is important to study its derivative

$$f_n'(p) = \sum_{i=0}^{n-1} n(b_{i+1,n} - b_{i,n}) B_i^{n-1}(p).$$

Combinatorial arguments given at the end of this proof and relation (2.1.1) yield

$$n|b_{i+1,n} - b_{i,n}| \le const. \tag{2.1.6}$$

independently of i and n. Using $\sum_{i=0}^{n-1} B_i^{n-1}(p) = 1$ we obtain

$$|f_n'(p)| \le const. \tag{2.1.7}$$

for all n and p.

We now choose the true parameter p of the model (2.1.5), which we will denote by p^*:

– If $f_n(p) \not\to p$ as $n \to \infty$ for some $p \in (0,1)$, we fix this p to be the true parameter p^* of the distribution of Z.

– If $f_n(p) \to p$ as $n \to \infty$ for all $p \in (0,1)$, we have

$$\int_{p/2}^{p} f_n'(q)dq = f_n(p) - f_n(p/2) \to p/2 \tag{2.1.8}$$

as $n \to \infty$. From this there exists a $p \in (0,1)$ with $f_n'(p) \not\to 0$ (otherwise (2.1.7) and the Lebesgue dominated convergence theorem lead to a contradiction in (2.1.8). This p is taken to be the true parameter p^* of the distribution of Z.

The mean squared error $MSE(\hat{p}_n) = \mathbf{E}(p^* - \hat{p}_n)_+^2 + \mathbf{E}(p^* - \hat{p}_n)_-^2$ satisfies

$$\mathbf{E}(p^* - \hat{p}_n)_+^2 \ge \frac{1}{2} MSE(\hat{p}_n) \tag{2.1.9}$$

or

$$\mathbf{E}(p^* - \hat{p}_n)_-^2 \ge \frac{1}{2} MSE(\hat{p}_n) \tag{2.1.10}$$

for infinetly many n. In either case the Cramér-Rao lower bound yields (for infinitely many n)

$$\mathbf{E}(p^* - \hat{p}_n)_\pm^2 \ge \frac{1}{2} \left\{ \frac{f_n'(p^*)^2}{I_n(p^*)} + (f_n(p^*) - p^*)^2 \right\}$$

with Fisher information $I_n(p^*) = np^*(1-p^*)$. (Lehmann, 1983, Theorem 6.4). By the choice of p^*, the proof is finished if we can adjust A, B such that

$$\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) \geq const. \cdot \mathbf{E}(p^* - \hat{p}_n)_+^2$$

if (2.1.9) applies or

$$\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) \geq const. \cdot \mathbf{E}(p^* - \hat{p}_n)_-^2$$

if (2.1.10) applies. This is done in the following.

If Z is distributed according to the general form (2.1.5), simple calculations yield that (2.1.4) is maximized by

$$b = b(p) = T\left(-\frac{p(A-B)+B-1}{(A-1)(B-1)}\right)$$

where

$$T(x) = \begin{cases} 1 & \text{if } x > 1, \\ x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x < 0. \end{cases}$$

Suppose (2.1.9) holds for infinitely many n. Set $A := p^*$ and $B := 1+p^*$. Then $b(p^*) = 0$ and $R_n^* = X^{(2)}$. In this case

$$\begin{aligned} \mathbf{E}\left[\log \frac{R_n^*}{\hat{R}_n}\middle| X_1, ..., X_n\right] &= -\mathbf{E}[\log((Z-1)b(\hat{p}_n)+1)|X_1, ..., X_n] \\ &= -\Big(p^* \log((p^*-1)b(\hat{p}_n)+1) + (1-p^*)\log(p^* b(\hat{p}_n)+1)\Big). \end{aligned}$$

More precisely,

$$b(\hat{p}_n) = \begin{cases} 0 & \text{if } \hat{p}_n > p^*, \\ \frac{p^*-\hat{p}_n}{p^*(1-p^*)} & \text{if } p^{*2} \leq \hat{p}_n \leq p^*, \\ 1 & \text{if } \hat{p}_n < p^{*2} \end{cases}$$

and

$$\mathbf{E}\left[\log \frac{R_n^*}{\hat{R}_n}\middle| X_1, ..., X_n\right] = \begin{cases} 0 & \text{if } \hat{p}_n > p^*, \\ D(p^*||\hat{p}_n) & \text{if } p^{*2} \leq \hat{p}_n \leq p^*, \\ D(p^*||p^{*2}) & \text{if } \hat{p}_n < p^{*2} \end{cases}$$

with the Kullback-Leibler distances (relative entropies)

$$\begin{aligned} D(p^*||\hat{p}_n) &= p^* \log \frac{p^*}{\hat{p}_n} + (1-p^*)\log\frac{1-p^*}{1-\hat{p}_n}, \\ D(p^*||p^{*2}) &= -(p^* \log p^* + (1-p^*)\log(1+p^*)). \end{aligned}$$

The L_1-bound on the Kullback-Leibler distance (Cover and Thomas, 1991, Lemma 12.6.1, eq. 12.139) yields

$$D(p^*||\hat{p}_n) \geq \frac{2}{\log 2}(p^* - \hat{p}_n)^2$$

so that

$$\begin{aligned}
\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) &= \mathbf{E}\left(\mathbf{E}\left[\log \frac{R_n^*}{\hat{R}_n}\middle| X_1, ..., X_n\right]\right) \\
&= \mathbf{E}\left(D(p^*||\hat{p}_n)\mathbf{1}_{[p^{*2},p^*]}(\hat{p}_n) + D(p^*||p^{*2})\mathbf{1}_{[0,p^{*2}]}(\hat{p}_n)\right) \\
&\geq \mathbf{E}\left(\frac{2}{\log 2}(p^* - \hat{p}_n)^2\mathbf{1}_{[p^{*2},p^*]}(\hat{p}_n) + D(p^*||p^{*2})\mathbf{1}_{[0,p^{*2}]}(\hat{p}_n)\right) \\
&\geq \min\left\{\frac{2}{\log 2}, \frac{D(p^*||p^{*2})}{p^{*2}}\right\}\mathbf{E}(p^* - \hat{p}_n)_+^2.
\end{aligned}$$

This is what we wanted to show in case (2.1.9) holds. If (2.1.10) applies, we set $A := 2 - p^*$ and $B := 1 - p^*$ and argue similarly.

It remains to prove (2.1.6). For our specific model we can assume

$$X = \begin{cases} (A, 1) & \text{with probability } p, \\ (1, 1/B) & \text{with probability } 1 - p. \end{cases}$$

If the observed return ratios z^n and $z'^n \in \{A, B\}^n$ differ in one digit only, so do the sequences of realisations $x_1, ..., x_n$ and $x_1', ..., x_n'$ of X that generate z^n and z'^n, respectively. Hence, using (2.1.1),

$$\begin{aligned}
\left|\hat{p}_n(z^n) - \hat{p}_n(z'^n)\right| &= \lim_{\epsilon \to 0^+}\Big|\Big(F_n((A+\epsilon, 1+\epsilon), x_1, ..., x_n) - F_n((A, 1), x_1, ..., x_n)\Big) \\
&\quad - \Big(F_n((A+\epsilon, 1+\epsilon), x_1', ..., x_n') - F_n((A, 1), x_1', ..., x_n')\Big)\Big| \\
&\leq \lim_{\epsilon \to 0^+}\Big|F_n((A+\epsilon, 1+\epsilon), x_1, ..., x_n) - F_n((A+\epsilon, 1+\epsilon), x_1', ..., x_n')\Big| \\
&\quad + \Big|F_n((A, 1), x_1, ..., x_n) - F_n((A, 1), x_1', ..., x_n')\Big| \\
&\leq \max_{(x,y)\in\{(A,1),(1,1/B)\}}\left\{\frac{\lim_{\epsilon\to 0^+} c((A+\epsilon, 1+\epsilon), x, y)}{n} + \frac{c((A, 1), x, y)}{n}\right\} =: \frac{c}{n}.
\end{aligned}$$

Let $\mathcal{F}(z^n)$ consist of all elements of $\{A, B\}^n$ which can be generated by changing exactly one of the digits B in z^n to A, and let $\mathcal{G}(z^n)$ consist of all elements of

$\{A, B\}^n$ which can be generated by changing exactly one of the A's in z^n to B. Clearly $|\mathcal{F}(z^n)| = n - k(z^n)$ and $|\mathcal{G}(z^n)| = k(z^n)$. From this

$$\begin{aligned}
b_{i,n} &= \binom{n}{i}^{-1} \sum_{z^n:k(z^n)=i} \hat{p}_n(z^n) = (n-i)^{-1}\binom{n}{i}^{-1} \sum_{z^n:k(z^n)=i} (n-i)\hat{p}_n(z^n) \\
&= (n-i)^{-1}\binom{n}{i}^{-1} \sum_{z^n:k(z^n)=i} \sum_{z'^n \in \mathcal{F}(z^n)} \hat{p}_n(z^n) \\
&= (i+1)^{-1}\binom{n}{i+1}^{-1} \sum_{z'^n:k(z'^n)=i+1} \sum_{z^n \in \mathcal{G}(z'^n)} \hat{p}_n(z'^n) \\
&\quad +(i+1)^{-1}\binom{n}{i+1}^{-1} \sum_{z'^n:k(z'^n)=i+1} \sum_{z^n \in \mathcal{G}(z'^n)} (\hat{p}_n(z^n) - \hat{p}_n(z'^n)) \\
&= \binom{n}{i+1}^{-1} \sum_{z'^n:k(z'^n)=i+1} \hat{p}_n(z'^n) \cdot \left(\frac{1}{i+1} \sum_{z^n \in \mathcal{G}(z'^n)} 1 \right) \\
&\quad +(i+1)^{-1}\binom{n}{i+1}^{-1} \sum_{z'^n:k(z'^n)=i+1} \sum_{z^n \in \mathcal{G}(z'^n)} (\hat{p}_n(z^n) - \hat{p}_n(z'^n)) \\
&= b_{i+1,n} + \left\{ (i+1)^{-1}\binom{n}{i+1}^{-1} \sum_{z'^n:k(z'^n)=i+1} \sum_{z^n \in \mathcal{G}(z'^n)} (\hat{p}_n(z^n) - \hat{p}_n(z'^n)) \right\}.
\end{aligned}$$

The latter bracket $\{...\}$ is an average with constituents bounded from above in absolute value by c/n. Hence $|b_{i+1,n} - b_{i,n}| \le c/n$ and the proof is finished. □

Remark. If the analysis is restricted to unbiased estimators in the sense that $\mathbf{E}\hat{\mathbf{Q}}_n(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{B}(\mathbb{R}_+^m)$, in particular $f_n(p) = p$, then we can choose $p^* = 1/2$ to obtain

$$\mathbf{E}\left[\log \frac{R_n^*}{\hat{R}_n} \middle| X_1, ..., X_n \right] \ge \frac{2}{\log 2}\mathbf{E}(\frac{1}{2} - \hat{p}_n)_+^2 \ge \frac{1}{I_n(1/2)\log 2} = \frac{4}{n \log 2}$$

without having to impose (2.1.1).

It is interesting to note that we can bound $\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q})$ in terms of the Kullback-Leibler distance between $\hat{\mathbf{Q}}_n$ and $\mathbf{Q}$ not only from below but also from above. This was obtained by Barron and Cover (1988, Theorem 1, see also Cover and Thomas, 1991, Theorem 15.4.1):

Theorem 2.1.2. *(Cover and Thomas, 1991) Let* $\mathbf{Q}$ *be the true return distribution and* $\hat{\mathbf{Q}}_n$ *a sequence of distribution estimates, both having densities* q *and* $\hat{q}_n$*, respectively, w.r.t. some common dominating measure. Then*

$$\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) \leq \mathbf{E}D(\mathbf{Q}||\hat{\mathbf{Q}}_n)$$

with the Kullback-Leibler distance

$$D(\mathbf{Q}||\hat{\mathbf{Q}}_n) = \int \log \frac{q(x)}{\hat{q}_n(x)} \mathbf{Q}(dx).$$

Remark. As a consequence of Theorem 2.1.2, consistent distribution estimates, i.e., estimates for which $\mathbf{E}D(\mathbf{Q}||\hat{\mathbf{Q}}_n) \to 0$, generate consistent portfolio selection rules. The results of Györfi et al. (1994), however, show that convergence in Kullback-Leibler distance is a considerably strong requirement. There is no distribution estimate, for example, that is Kullback-Leibler consistent for any return distribution on a non-finite countable set. The results of Algoet and Cover (1988) demonstrate that almost sure weak convergence of $\hat{\mathbf{Q}}_n$ to $\mathbf{Q}$ suffices to obtain consistent portfolio selection rules.

Proof. It suffices to show

$$\mathbf{E}\left[\log \frac{R_n^*}{\hat{R}_n} \middle| X_1, ..., X_n\right] \leq D(\mathbf{Q}||\hat{\mathbf{Q}}_n), \tag{2.1.11}$$

the assertion follows taking expectations. If $D(\mathbf{Q}||\hat{\mathbf{Q}}_n) = \infty$ there is nothing to prove. So, assume $D(\mathbf{Q}||\hat{\mathbf{Q}}_n) < \infty$, which implies absolute continuity of $\mathbf{Q}$ w.r.t. $\hat{\mathbf{Q}}_n$, i.e., $\mathbf{Q} \ll \hat{\mathbf{Q}}_n$. Set $A := \{x : \hat{R}_n = \hat{b}_n x > 0, q(x) > 0, \hat{q}_n(x) > 0\}$. From the Kuhn-Tucker conditions (Theorem 1.3.3) it is clear that $\hat{\mathbf{Q}}_n(\{x : \hat{R}_n > 0\}) = 1$ and, by $\mathbf{Q} \ll \hat{\mathbf{Q}}_n$, also $\mathbf{Q}(\{x : \hat{R}_n > 0\}) = 1$. From this and using $\mathbf{Q} \ll \hat{\mathbf{Q}}_n$ again, $\mathbf{Q}(A) = 1$. Thus

$$\begin{aligned}
\mathbf{E}\left[\log \frac{R_n^*}{\hat{R}_n} \middle| X_1, ..., X_n\right] &= \int_A \log \frac{b^* x}{\hat{b}_n x} \mathbf{Q}(dx) \\
&= \int_A \log \left(\frac{b^* x}{\hat{b}_n x} \cdot \frac{\hat{q}_n(x)}{q(x)} \cdot \frac{q(x)}{\hat{q}_n(x)} \right) \mathbf{Q}(dx)
\end{aligned}$$

$$
\begin{aligned}
&= \int_A \log\left(\frac{b^*x}{\hat{b}_n x}\cdot\frac{\hat{q}_n(x)}{q(x)}\right)\mathbf{Q}(dx) + D(\mathbf{Q}||\hat{\mathbf{Q}}_n) \\
&\leq \log\int_A \frac{b^*x}{\hat{b}_n x}\,\hat{\mathbf{Q}}_n(dx) + D(\mathbf{Q}||\hat{\mathbf{Q}}_n) \leq D(\mathbf{Q}||\hat{\mathbf{Q}}_n),
\end{aligned}
$$

the latter inequality by the Kuhn-Tucker conditions again. □

From this theorem we can easily infer that the lower bound on $\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q})$ given in (2.1.3) is sharp, and that there are estimators attaining the optimal rate of decay, $O(1/n)$. In particular, the log-optimal portfolio based on the empirical distribution does so.

Theorem 2.1.3. *Assume the return distribution* $\mathbf{Q}$ *is supported on a finite set* $\mathcal{M}$*. Let* $\hat{\mathbf{Q}}_n$ *be the empirical distribution on* $\mathcal{M}$ *after* n *observations and*

$$\hat{b}_{n+1} := \arg\max_{b\in S}\frac{1}{n}\sum_{i=1}^{n}\log bX_i$$

the associated log-optimal portfolio, then

$$\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) \leq \frac{|\mathcal{M}|}{n}. \tag{2.1.12}$$

Proof. Let $\Gamma_n := \{\mu|\mu$ empirical distribution of n data on $\mathcal{M}, D(\mu||\mathbf{Q}) > \epsilon\}$. Then according to Sanov's theorem (Cover and Thomas, 1991, Ch. 12)

$$
\begin{aligned}
\mathbf{Q}\Big(D(\mathbf{Q}||\hat{\mathbf{Q}}_n) > \epsilon\Big) &= \mathbf{Q}\left(\hat{\mathbf{Q}}_n \in \Gamma_n\right) \\
&\leq |\mathcal{M}|\exp\left(-n\min_{\mu\in\Gamma_n} D(\mathbf{Q}||\mu)\right) \leq |\mathcal{M}|\exp(-n\epsilon).
\end{aligned}
$$

From this we calculate

$$\mathbf{E}D(\mathbf{Q}||\hat{\mathbf{Q}}_n) = \int_0^\infty \mathbf{Q}\Big(D(\mathbf{Q}||\hat{\mathbf{Q}}_n) > \epsilon\Big)\,d\epsilon \leq |\mathcal{M}|\int_0^\infty \exp(-n\epsilon) = \frac{|\mathcal{M}|}{n}.$$

Application of Theorem 2.1.2 proves the theorem. □

It is an interesting feature of inequality (2.1.12) that the rate itself does not deteriorate when the number m of stocks in the market model grows. This is rather untypical of nonparametric estimation problems. Of course, the number of stocks in the market influences the *constant* in $\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) = O(1/n)$: We have proven that for the empirical log-optimal strategy

$$\mathbf{E} \log \frac{R_i^*}{\hat{R}_i} \sim \frac{c(i)}{i}$$

with $c(i) = O(1)$. To see how $c(i)$ depends on m we allow ourselves some heuristics: Móri (1982) proved that for a (slight) modification of the empirical log-optimal strategy, $n^{(m-1)/2} \hat{S}_n / S_n^*$ converges to a non-degenerate random variable Z,

$$n^{(m-1)/2} \frac{\hat{S}_n}{S_n^*} \to Z \quad \text{in distribution.}$$

This can be rewritten as

$$\sum_{i=1}^{n} \left(\frac{m-1}{2} \cdot \frac{1}{i} - \log \frac{R_i^*}{\hat{R}_i} \right) \to \log Z \quad \text{in distribution.}$$

Taking expectations, the left hand side becomes

$$\sum_{i=1}^{n} \frac{m-1}{2} \cdot \frac{1}{i} - \sum_{i=1}^{n} \mathbf{E} \log \frac{R_i^*}{\hat{R}_i} \sim \sum_{i=1}^{n} \frac{m-1}{2} \cdot \frac{1}{i} - \sum_{i=1}^{n} \frac{c(i)}{i},$$

both sums being of logarithmic growth. To obtain convergence we infer $c(i) \sim \frac{m-1}{2}$, the constant possibly growing linearly with the number of stocks.

Up to a logarithmic factor, the phenomenon of the rate being insensitive to m carries over to more sophisticated settings where the return vector is not necessarily restricted to finitely many outcomes. In particular, we have the following theorem for the empirical log-optimal portfolio.

Theorem 2.1.4. *Assume the return distribution* $\mathbf{Q}$ *is concentrated on a cube* $[A, B]^m$ *with* $0 < A \leq B < \infty$)*. Let*

$$\hat{b}_{n+1} := \arg\max_{b \in S} \frac{1}{n} \sum_{i=1}^{n} \log bX_i$$

be the empirical log-optimal portfolio, then

$$\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) = o\left(\frac{\log^q n}{n^{1/2}} \right)$$

for any $q > \max\{(m-1)/2, 1\}$.

Up to the logarithmic factor, the rate coincides with the classical rate $n^{-1/2}$ of stochastic parameter estimators – regardless of the portfolio dimension m.

Proof. For the proof we can assume $m > 2$: If $m = 2$ we artificially produce a market with 3 assets from the original 2 stocks and a bond returning $A/2$ in each market period. In this setting, we never invest in the bond, i.e., log-optimal investment is the same as in the original 2 stock market. A rate result for the 3 stock market carries over to the 2 stock market.

First we make some preliminary observations on the *covering of the simplex* $S := \{b \in \mathbb{R}^m | b_i \geq 0, \sum_{i=1}^m b_i = 1\}$.

Let $S'_m := \{b \in \mathbb{R}^m | b_i \geq 0, \sum_{i=1}^m b_i \leq 1\}$ and define the mapping $F : S'_{m-1} \to S : (x_1, ..., x_{m-1}) \mapsto (x_1, ..., x_{m-1}, 1 - \sum_{i=1}^{m-1} x_i)$. Fix some $\epsilon > 0$. Clearly, we can cover $S'_{m-1} \subseteq [0,1]^{m-1}$ with $N \leq \lceil 1/\delta \rceil^{m-1}$ $\|\cdot\|_\infty$-balls of radius $\delta := \epsilon/(m-1)$ centered at $c^{(1)}, ..., c^{(N)} \in S'_{m-1}$. For any $x \in S$

$$\begin{aligned}
&\inf_{i=1,...,N} \left\| (x_1, ..., x_m) - F(c^{(i)}) \right\|_\infty \\
&\quad = \inf_{i=1,...,N} \max \left\{ \left\| (x_1, ..., x_{m-1}) - c^{(i)} \right\|_\infty, \left| \sum_{j=1}^{m-1} (c_j^{(i)} - x_j) \right| \right\} \\
&\quad \leq \inf_{i=1,...,N} (m-1) \left\| (x_1, ..., x_{m-1}) - c^{(i)} \right\|_\infty \leq \epsilon.
\end{aligned}$$

It follows that S can be covered by at most $\lceil (m-1)/\epsilon \rceil^{m-1}$ $\|\cdot\|_\infty$-balls of radius ϵ.

Let $X_1, ..., X_n$ denote independent return data and augment this family by a random variable X independent of $X_1, ..., X_n$ with the same distribution as X_1. Introduce the following abbreviations:

$$\begin{aligned}
\Phi_n &:= \max_{b \in S} \frac{1}{n} \sum_{i=1}^n \log bX_i = \frac{1}{n} \sum_{i=1}^n \log \hat{b}_n X_i, \\
L_n &:= \mathbf{E}[\log \hat{b}_n X | X_1, ..., X_n]
\end{aligned}$$

and

$$L^* := \max_{b \in S} \mathbf{E} \log bX = \mathbf{E} \log b^* X.$$

Clearly,

$$L_n \leq \max_{b\in S} \mathbf{E}[\log bX|X_1, ..., X_n] = \max_{b\in S} \mathbf{E}\log bX = L^*.$$

To bound the tail probability of $L^* - L_n \geq 0$ we use the following decomposition

$$\begin{aligned}
\mathbf{P}(L^* - L_n > \epsilon) &= \mathbf{P}(L^* - \Phi_n + \Phi_n - L_n > \epsilon) \\
&\leq \mathbf{P}(L^* - \Phi_n \geq \epsilon - \delta) + \mathbf{P}(\Phi_n - L_n \geq \delta) \\
&=: K_1 + K_2,
\end{aligned}$$

where $\delta > 0$ is chosen later.

Bounding K_1: We start with

$$L^* - \Phi_n = \max_b \mathbf{E}\log bX - \max_b \frac{1}{n}\sum_{i=1}^n \log bX_i \leq \max_b \left| \mathbf{E}\log bX - \frac{1}{n}\sum_{i=1}^n \log bX_i \right|.$$

For $\tau > 0$ to be chosen later, cover S by $N \leq \lceil (m-1)/\tau \rceil^{m-1}$ $\|\cdot\|_\infty$-balls of radius τ centered at $b^{(1)}, ..., b^{(N)}$. Choosing c_1 such that $|\log bX - \log \tilde{b}X| \leq c_1|b - \tilde{b}|$ for all $X \in [A, B]^m$ we obtain

$$\max_b \left| \mathbf{E}\log bX - \frac{1}{n}\sum_{i=1}^n \log bX_i \right| \leq \max_{j=1,...,N} \left| \mathbf{E}\log b^{(j)}X - \frac{1}{n}\sum_{i=1}^n \log b^{(j)}X_i \right| + 2c_1\tau.$$

Under the condition $\epsilon - \delta - 2c_1\tau > 0$, the Hoeffding inequality (Petrov, 1995, 2.6.2, note that $|\log bX| \leq \max\{|\log A|, |\log B|\} =: c_2 < \infty$) yields

$$\begin{aligned}
K_1 &\leq \sum_{j=1}^N \mathbf{P}\left(\left| \mathbf{E}\log b^{(j)}X - \frac{1}{n}\sum_{i=1}^n \log b^{(j)}X_i \right| \geq \epsilon - \delta - 2c_1\tau \right) \\
&\leq 2N\exp\left(-\frac{n(\epsilon - \delta - 2c_1\tau)^2}{2c_2^2} \right) \\
&\leq 2\left\lceil \frac{m-1}{\tau} \right\rceil^{m-1} \exp\left(-\frac{n(\epsilon - \delta - 2c_1\tau)^2}{2c_2^2} \right). \qquad (2.1.13)
\end{aligned}$$

Bounding K_2:

$$\begin{aligned}
\Phi_n - L_n &= \frac{1}{n}\sum_{i=1}^n \log \hat{b}_n X_i - \mathbf{E}[\log \hat{b}_n X|X_1, ..., X_n] \\
&\leq \max_{j=1,...,N} \left(\log b^{(j)}X_i - \mathbf{E}\log b^{(j)}X \right) + 2c_1\tau,
\end{aligned}$$

and under the condition $\delta - 2c_1\tau > 0$ we apply Hoeffding's inequality again to obtain

$$K_2 \leq N \exp\left(-\frac{n(\delta - 2c_1\tau)^2}{2c_2^2}\right) \leq \left\lceil \frac{m-1}{\tau} \right\rceil^{m-1} \exp\left(-\frac{n(\delta - 2c_1\tau)^2}{2c_2^2}\right). \tag{2.1.14}$$

Combining (2.1.13) and (2.1.14) for $\delta := \epsilon/2, \tau := \epsilon/(8c_1)$ yields

$$\mathbf{P}(L^* - L_n > \epsilon) \leq 3 \left\lceil \frac{8c_1(m-1)}{\epsilon} \right\rceil^{m-1} \exp\left(-\frac{n\epsilon^2}{32c_2^2}\right).$$

This in turn implies

$$\begin{aligned}
\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) &= \mathbf{E}(L^* - L_n) = \int_0^\infty \mathbf{P}(L^* - L_n > \epsilon)\, d\epsilon \\
&\leq a_n + 3 \int_{a_n}^\infty \left\lceil \frac{8c_1(m-1)}{\epsilon} \right\rceil^{m-1} \exp\left(-\frac{n\epsilon^2}{32c_2^2}\right)\, d\epsilon \\
&= a_n \left(1 + 24c_1(m-1) \int_{(8c_1(m-1))^{-1}}^\infty \left\lceil \frac{1}{a_n z} \right\rceil^{m-1} \exp\left(-c_3 n a_n^2 z\right)\, dz\right)
\end{aligned}$$

with $c_3 := 2c_1^2(m-1)^2/c_2^2$ and $a_n > 0$ still to be adjusted.

We now have to bound integrals of the latter type: For $a > 0, m > 2$ and $a_n := cn^{-1/2}\log^q n$ (arbitrary $c > 0, q > (m-1)/2$) we find that

$$\int_a^\infty \left\lceil \frac{1}{a_n z} \right\rceil^{m-1} \exp(-c_3 n a_n^2 z)\, dz \leq 2^{m-2} \int_a^\infty \left(\frac{1}{(a_n z)^{m-1}} + 1\right) \exp(-c_3 n a_n^2 z)\, dz$$

where we have used the inequalities $\lceil x \rceil \leq x + 1$ and Jensen's inequality to obtain $(x+1)^{m-1} = 2^{m-1}((x+1)/2)^{m-1} \leq 2^{m-1}(x^{m-1}+1)/2 = 2^{m-2}(x^{m-1}+1)$. Bounding the last integral yields

$$\begin{aligned}
&\int_a^\infty \left\lceil \frac{1}{a_n z} \right\rceil^{m-1} \exp(-c_3 n a_n^2 z)\, dz \\
&\qquad \leq 2^{m-2} \exp(-a c_3 n a_n^2) \frac{1}{a_n^{m-1}} \int_a^\infty \frac{1}{z^{m-1}}\, dz + 2^{m-2} \int_0^\infty \exp(-c_3 n a_n^2 z)\, dz \\
&\qquad = \left(\frac{2}{a}\right)^{m-2} \frac{\exp(-a c_3 n a_n^2)}{(m-2)a_n^{m-1}} + 2^{m-3} \left(\frac{\pi}{c_3}\right)^{1/2} \frac{1}{n^{1/2} a_n}.
\end{aligned}$$

For n sufficiently large

$$\exp(-ac_3 n a_n^2) = \exp(-ac_3 c^2 \log^{2q} n) \le \exp\left(-\frac{m-1}{2}\log n\right) = \left(\frac{1}{n}\right)^{(m-1)/2},$$

and thus

$$\frac{\exp(-ac_3 n a_n^2)}{a_n^{m-1}} \le \left(\frac{1}{c\log^q n}\right)^{m-1} \le \frac{1}{c\log^q n}.$$

By the choice of a_n, $1/(n^{1/2}a_n) = 1/(c\log^q n)$, and we end up with

$$\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) \le a_n\left(1 + \frac{const.}{c\log^q n}\right) = \frac{c\log^q n}{n^{1/2}} + \frac{const.}{n^{1/2}}$$

for all n greater than some integer that depends on m, a, c_3 and c. Hence,

$$\limsup_{n\to\infty} \frac{n^{1/2}}{\log^q n}\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) \le c$$

and from $c > 0$ being arbitrary, we infer

$$\limsup_{n\to\infty} \frac{n^{1/2}}{\log^q n}\Delta(\hat{\mathbf{Q}}_n, \mathbf{Q}) = 0,$$

the assertion for the case $m > 2$. □

2.2 Dimensionality in portfolio selection

Let the investor operate in a market of M stocks with random one day returns $X^{(i)}$ $(i = 1, ..., M)$. Typically, M is large, e.g., $M = 30$ for the DAX (Frankfurt) or Dow Jones IA (New York) stocks, $M = 100$ for the FTSE100 (London). Common wisdom tells us “don’t put all your eggs in one basket”, the economist’s version of this saying (as Samuelson, 1967, put it) goes “diversification pays”. One is tempted to think the more diversified the portfolio, i.e., the more stocks we include in via log-optimal portfolio selection, the better. The results of the last section, where we have seen that the number of stocks does not affect the rate at which empirical log-optimal portfolios approach the optimal performance, make us particularly optimistic. However, we should not forget that there are also several reasons to avoid selection from a huge set of stocks:

1. M very much affects the scale of finite sample underperformance via the constants in the rate results (recall what we infered from Móri, 1982).

2. Standard optimisation methods are computationally demanding in high dimension.

3. If the log-optimal portfolio is calculated with, e.g., Cover's algorithm (Cover, 1984), then at each iteration step an M-dimensional integration has to be carried out, which requires considerable computational effort. Also, Cover's algorithm requires exact knowledge of the M-dimensional return distribution. In practice, such information must be gathered by statistical distribution estimation which faces substantial difficulties for high dimension M (curse of dimensionality, see e.g. Scott, 1992, Chapter 7).

For these reasons, the investor should work with a medium size range of stocks at a time only. In other words, he will have to pre-select $m < M$ stocks from the whole market. These pre-selected stocks are the assets he includes in a log-optimal portfolio. For illustrative purposes, we restrict ourselves to $M = 3$. In this case, the investor may compose a log-optimal portfolio out of 6 possible combinations of 1 or 2 stocks.

$$V^*_{\{n\}} := \mathbf{E} \log X^{(n)}$$

is the maximal expected log-return when the portfolio is composed of stock n only. If two different stocks n and m are considered, the maximal expected log-return is

$$V^*_{\{n,m\}} := \max_{0 \le b \le 1} \mathbf{E} \log \Big((1-b) X^{(n)} + b X^{(m)} \Big).$$

A natural (and in fact a frequently used) way for pre-selection is to start with a first "draught-horse" stock (say stock A) for our portfolio, i.e., a stock such that $V^*_{\{A\}}$ is large. From the two remaining contenders (say stocks B and C) the investor then includes the one with good single performance, e.g. B if $V^*_{\{B\}} > V^*_{\{C\}}$. The hope is to attain the optimum $V^*_{\{A,B\}} = \max_{\{n,m\} \subseteq \{A,B,C\}} V^*_{\{n,m\}}$.

The following result gives conditions under which this method is doomed to failure in the realistic market model of log-normally distributed returns. More precisely, markets with log-normal returns are characterised for which $V^*_{\{1\}} < V^*_{\{2\}} < V^*_{\{3\}}$ and $V^*_{\{2,3\}} < V^*_{\{1,2\}}$ at the same time, the two best single stocks forming a poorer portfolio than the two worst stocks in the market. As a consequence, in order to select the optimal 2 stock combination from the market, the investor has to evaluate all $\binom{M}{m}$ possible choices, a huge computational effort in high dimensions – a effort though that cannot be avoided. This is in contrast to the Markowitz mean-variance approach, where a portfolio built of stock 1 and 2 may be superior to a portfolio built of stock 2 and 3 in terms of risk (i.e., variance of portfolio return), never though in terms of performance (i.e., expected portfolio return).

Theorem 2.2.1. *Consider a given variance-covariance matrix*

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}.$$

Then the condition

$$\sigma_3^2 - 2\sigma_{23} < \sigma_1^2 - 2\sigma_{12} \tag{2.2.1}$$

is necessary and sufficient for a three stock log-normal market

$$X^{(i)} := \exp(Y^{(i)}) \quad (i = 1, 2, 3) \quad \textit{with} \quad \left(Y^{(1)}, Y^{(2)}, Y^{(3)}\right) \sim N((\mu_1, \mu_2, \mu_3)\,; \Sigma)$$

to exist such that

$$\mu_1 < \mu_2 := 0 < \mu_3 \qquad \textit{and} \qquad V^*_{\{2,3\}} < V^*_{\{1,2\}}$$

simultaneously.

The assertion of Theorem 2.2.1 remains valid if a common $\mu \in \mathbb{R}$ is added to μ_1, μ_2 and μ_3.

The theorem implies that single stock performance is of secondary importance in comparison with harmonious teamwork of the stocks. The deeper reason for this is the effect of "volatility pumping" (Luenberger, 1998, Examples 15.2 and 15.3): The specific volatility structure (i.e., covariance structure) in the market may "pump" growth from one stock to others in the portfolio. In our example, if the covariance σ_{12} of stock 1 and stock 2 is sufficiently less than the covariance of stock 1 and stock 3 – preferably sufficiently negative such that whenever 1 plunges, 2 is likely to increase – then more substantial growth can be achieved by balancing stocks 1 and 2 rather than stocks 1 and 3, even though stock 2 might have poorer single performance than stock 3.

Corresponding results for dimension reduction in pattern recognition have been obtained by Touissaint (1971, also discussed in Devroye, Györfi and Lugosi, 1996, Theorem 32.2).

For the proof of the theorem, we need a number of preliminary observations. First consider a 2 stock market with log-normally distributed returns $X^{(i)}$,

$$X^{(i)} := \exp(Y^{(i)}) \quad (i = 1, 2)$$

and

$$\begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}; \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right).$$

Log-optimal investment in log-normal markets has been considered, e.g., by Ohlson, (1972). As noted in Chapter 1, the log-optimal portfolio

$$\left(b^{(1)*}, b^{(2)*}\right) := \arg \min_{b^{(1)}, b^{(2)} \geq 0, b^{(1)} + b^{(2)} = 1} \mathbf{E} \log(b^{(1)} X^{(1)} + b^{(2)} X^{(1)})$$

satisfies the necessary and sufficient Kuhn-Tucker conditions

$$\mathbf{E} \frac{X^{(i)}}{b^{(1)*} X^{(1)} + b^{(2)*} X^{(2)}} \begin{cases} = 1 \\ \leq 1 \end{cases} \quad \text{if} \quad b^{(i)*} \begin{cases} > 0 \\ = 0 \end{cases}$$

(Theorem 1.3.3). In other words

$$(1, 0) \quad \text{log-optimal} \iff \mathbf{E} \frac{X^{(2)}}{X^{(1)}} \leq 1 \tag{2.2.2}$$

$$(0, 1) \quad \text{log-optimal} \iff \mathbf{E} \frac{X^{(1)}}{X^{(2)}} \leq 1 \tag{2.2.3}$$

$$\begin{aligned} (b^{(1)}, b^{(2)}) \quad \text{log-optimal} \iff & \mathbf{E} \frac{X^{(1)}}{b^{(1)} X^{(1)} + b^{(2)} X^{(2)}} \\ & = \mathbf{E} \frac{X^{(2)}}{b^{(1)} X^{(1)} + b^{(2)} X^{(2)}} = 1, \end{aligned} \tag{2.2.4}$$

the latter for $b^{(1)}, b^{(2)} > 0, b^{(1)} + b^{(2)} = 1$.

We rephrase the Kuhn-Tucker conditions in a form that will be more convenient to use. To this end we define $Z := Y^{(2)} - Y^{(1)} \sim N(\mu_2 - \mu_1;\ \sigma_1^2 - 2\sigma_{12} + \sigma_2^2)$ and connect $b^{(1)}, b^{(2)} \neq 0$ to $r \in (0, \infty)$ via $r := b^{(2)}/b^{(1)}, b^{(1)} = 1/(1 + r)$ and

$b^{(2)} = r/(1+r)$. Then we can rewrite the right hand sides of (2.2.2) and (2.2.3) as

$$\mathbf{E} \exp Z \leq 1, \tag{2.2.2'}$$
$$\mathbf{E} \exp(-Z) \leq 1. \tag{2.2.3'}$$

By simple calculations, the right hand side of (2.2.4) is equivalent to the existence of

$$r \in (0, \infty) \quad \text{such that} \quad \mathbf{E} \frac{\exp Z - 1}{1 + r \exp Z} = 0. \tag{2.2.4'}$$

From (2.2.2') to (2.2.4') one can observe the following:

1. The log-optimal portfolio $\left(b^{(1)*}, b^{(2)*}\right)$ only depends upon
$$\mu := \mu_2 - \mu_1 \quad \text{and} \quad \sigma^2 := \sigma_1^2 - 2\sigma_{12} + \sigma_2^2,$$
i.e. $\left(b^{(1)*}, b^{(2)*}\right) = \left(b^{(1)*}(\mu, \sigma^2), b^{(2)*}(\mu, \sigma^2)\right)$.
2. Evaluating $\mathbf{E} \exp Z = \exp(\mu + \sigma^2/2)$ and $\mathbf{E} \exp(-Z) = \exp(-\mu + \sigma^2/2)$ yields:
$$(1, 0) \quad \text{log-optimal} \iff \mu \leq -\frac{\sigma^2}{2}, \tag{2.2.5}$$
$$(0, 1) \quad \text{log-optimal} \iff \mu \geq \frac{\sigma^2}{2}. \tag{2.2.6}$$
3. For $\mu = 0$, $(1/2, 1/2)$ is the log-optimal portfolio, since by symmetry
$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \frac{\exp z - 1}{\exp z + 1} \exp\left(-\frac{z^2}{2\sigma^2}\right) dz = 0.$$

The value of the log-optimal portfolio of stock 1 and stock 2 is

$$V^* := \mathbf{E} \log(b^{(1)*} X^{(1)} + b^{(2)*} X^{(2)}).$$

However, it will be convenient to work with the portfolio improvement

$$\begin{aligned} V_\sigma(\mu) &:= V^* - \mu_1 = \mathbf{E} \log X^{(1)} + \mathbf{E} \log(b^{(1)*} + (1 - b^{(1)*}) \exp Z) - \mu_1 \\ &= \mathbf{E} \log(b^{(1)*}(\mu, \sigma^2) + (1 - b^{(1)*}(\mu, \sigma^2)) \exp Z) \end{aligned} \tag{2.2.7}$$

achieved when including stock 2 in a portfolio of stock 1. $V_\sigma(\mu)$ only depends upon the distribution of

$$Z = Z_{\mu, \sigma^2} \sim N(\mu; \sigma^2),$$

i.e. again on μ and σ^2 only. The following lemma summarizes basic properties of $V_\sigma(\mu)$, similar to the results derived in Ohlson (1972):

Lemma 2.2.2. *1. For any fixed $\sigma \in [0, \infty)$ $V_\sigma(\mu)$ is a continuous function of $\mu \in \mathbb{R}$, strictly increasing on $[-\sigma^2/2, \infty)$.*
2. $V_\sigma(\mu) = 0$ for all $\mu \in (-\infty, -\sigma^2/2]$ and $V_\sigma(\mu) = \mu$ for all $\mu \in [\sigma^2/2, \infty)$.
3. $V_\sigma(0)$ is a nonnegative, strictly increasing continuous function of $\sigma \in [0, \infty)$.

Proof. 1. On the one hand the log-optimal portfolio $\left(b^{(1)*}(\mu, \sigma^2), b^{(2)*}(\mu, \sigma^2)\right)$ is unique (Theorem 1.3.1), on the other hand, a continuous solution, say $\left(b^{(1)}(\mu, \sigma^2), b^{(2)}(\mu, \sigma^2)\right)$, to the maximization problem

$$\mathbf{E}\log(b^{(1)}X^{(1)} + b^{(2)}X^{(2)}) = \mu_1 + \mathbf{E}\log(b^{(1)} + b^{(2)}\exp Z_{\mu,\sigma^2}) \longrightarrow \max_{b^{(1)},b^{(2)}\geq 0, b^{(1)}+b^{(2)}=1} !$$

can be found (Aliprantis and Border, 1999, Theorem 16.31 together with Lemma 16.6). Hence, both coincide and $b^{(1)*}(\mu, \sigma^2)$ is a continuous function of μ. From the equation $V_\sigma(\mu) = \mathbf{E}\log(b^{(1)*} + (1 - b^{(1)*})\exp Z_{\mu,\sigma^2})$ the continuity assertion follows.

Now, let $-\sigma^2/2 < \mu < \nu$. Then

$$\begin{aligned} V &= \mathbf{E}\log(b^{(1)*}(\mu, \sigma^2) + (1 - b^{(1)*}(\mu, \sigma^2))\exp Z_{\mu,\sigma^2}) \\ &< \mathbf{E}\log(b^{(1)*}(\mu, \sigma^2) + (1 - b^{(1)*}(\mu, \sigma^2))\exp Z_{\nu,\sigma^2}) \\ &\leq \mathbf{E}\log(b^{(1)*}(\nu, \sigma^2) + (1 - b^{(1)*}(\nu, \sigma^2))\exp Z_{\nu,\sigma^2}). \end{aligned}$$

The first inequality follows from $-\sigma^2/2 < \mu$, i.e. $b^{(1)*}(\mu, \sigma^2) < 1$, the second inequality holds by definition of $b^{(1)*}$ as a component of the log-optimal portfolio.

2. is a direct consequence of (2.2.5) and (2.2.6). We just check $V_\sigma(\mu) = \mathbf{E}\log X^{(1)} - \mu_1 = \mu_1 - \mu_1 = 0$ for $\left(b^{(1)*}, b^{(2)*}\right) = (1, 0)$ and calculate $V_\sigma(\mu) = \mathbf{E}\log X^{(2)} - \mu_1 = \mu_2 - \mu_1 = \mu$ for $\left(b^{(1)*}, b^{(2)*}\right) = (0, 1)$.

3. As noted above $\mu = 0$ implies $\left(b^{(1)*}, b^{(2)*}\right) = (1/2, 1/2)$. Hence we find that

$$\begin{aligned} V_\sigma(0) &= \log\frac{1}{2} + \mathbf{E}\log(1 + \exp Z_{0,\sigma^2}) \\ &= \log\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\log(1 + \exp(\sigma w))\exp\left(-\frac{w^2}{2}\right)dw. \end{aligned}$$

From this representation we see that $V_\sigma(0)$ is continuous for $\sigma \in (0, \infty)$. Moreover, using the monotone convergence theorem (Williams, 1991, 5.3) we calculate

$$V_0(0) := \lim_{\sigma \to 0^+} V_\sigma(0) = \log \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \log 2 \exp\left(-\frac{w^2}{2}\right) dw = \log \frac{1}{2} + \log 2 = 0.$$

Concerning monotonicity, we remark that in what follows the interchange of differentiation and integration is possible by the standard theorem for integrals depending on a parameter (see e.g. Williams, 1991, A.16.1). Thus for $\sigma > 0$

$$\frac{\partial}{\partial \sigma} V_\sigma(0) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{w}{1 + \exp(-\sigma w)} \exp\left(-\frac{w^2}{2}\right) dw.$$

Finally, since $w/(1 + \exp(-\sigma w)) > w/2$ for all $w \neq 0$,

$$\frac{\partial}{\partial \sigma} V_\sigma(0) > \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{w}{2} \exp\left(-\frac{w^2}{2}\right) dw = 0,$$

and $V_\sigma(0)$ is strictly increasing for $\sigma \geq 0$. □

We are now in the position to finish the

Proof of Theorem 2.2.1. *Sufficiency:* Suppose we are given $\sigma_L^2 := \sigma_3^2 - 2\sigma_{23} + \sigma_2^2 < \sigma_U^2 := \sigma_1^2 - 2\sigma_{12} + \sigma_2^2$. Then, from part 3 (combined with parts 1, 2) of Lemma 2.2.2, we can choose a $W > 0$ with

$$V_{\sigma_L}(0) < W < V_{\sigma_U}(0). \tag{2.2.8}$$

By parts 1 and 2 of the lemma and the intermediate value theorem $\mu_1 := V_{\sigma_U}^{-1}(W)$ and $m := V_{\sigma_L}^{-1}(W)$ are well-defined and – observing the strict monotonicity property of V_{σ_L} and V_{σ_U} as in part 1 of the lemma – we obtain $\mu_1 < 0 < m$ from (2.2.8).

This choice is illustrated in Figure 2.1, where the reduced portfolio values $V_\sigma(\mu)$ were calculated by means of the Cover algorithm (Theorem 1.3.2) and numerical integration (composite trapezoidal rule, Isaacson and Keller, 1994, Sec. 7.5) both with an accuracy of 10^{-7}.

Choose some μ_3 with $0 < \mu_3 < m$. Then, by part 1 again,

$$V_{\sigma_L}(\mu_3) < V_{\sigma_L}(m) = W = V_{\sigma_U}(\mu_1).$$

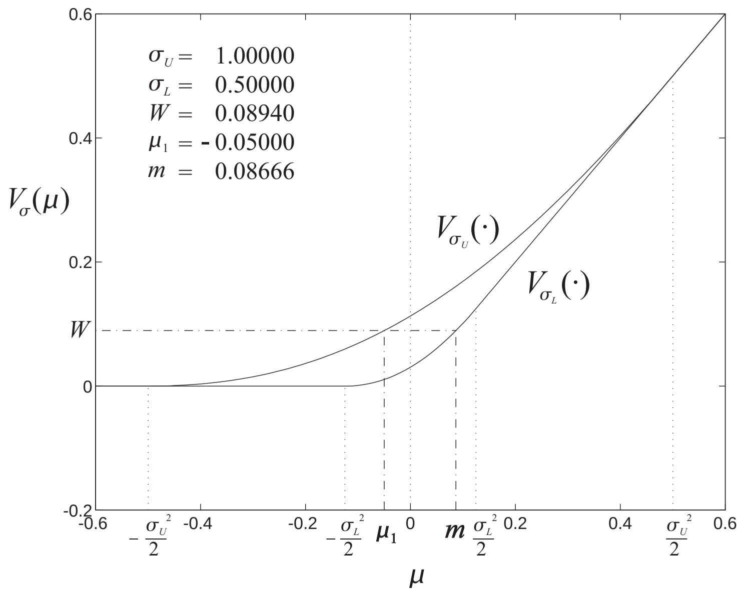


Figure 2.1: An example of the situation as in the proof of Theorem 2.2.1.

Combining this with the definition of the reduced portfolio value $V_\sigma(\mu_i - \mu_j) = V^*_{\{i,j\}} - \mu_j$ in (2.2.7) for a portfolio of stocks $\{i = 1, j = 2\}$ (set $\sigma^2 = \sigma_1^2 - 2\sigma_{12} + \sigma_2^2$) and of stocks $\{j = 2, i = 3\}$ (set $\sigma^2 = \sigma_3^2 - 2\sigma_{23} + \sigma_2^2$), we obtain $V^*_{\{2,3\}} = V_{\sigma_L}(\mu_3) + 0 < V_{\sigma_U}(\mu_1) + 0 = V^*_{\{1,2\}}$.

Necessity: If instead of (2.2.1) we assume $\sigma_3^2 - 2\sigma_{23} \geq \sigma_1^2 - 2\sigma_{12}$, then $\mu_1 < \mu_2 := 0 < \mu_3$ implies $V^*_{\{1,2\}} = V_{\sigma_U}(\mu_1) + 0 \leq V_{\sigma_U}(0) \leq V_{\sigma_L}(0) < V_{\sigma_L}(\mu_3) + 0 = V^*_{\{2,3\}}$ (Lemma 2.2.2, parts 1 and 2 for the first and third inequality, part 3 for the second). □

2.3 Examples

We conclude this chapter with examples of a real market where a situation as in Theorem 2.2.1 is set up by empirical market data. We assume that the distribution of the returns is log-normal with the parameters provided by the standard estimates of mean and variance (thus admittedly oversimplifying things). Figure 2.2 a-c) shows some diagnostic checks run on American Express Co. (AXP) daily log-return data from the closing prices at the New York Stock Exchange 2/1/1998 - 30/11/2000 (data from www.wallstreetcity.com).

Comparing the histogram of the data in Figure 2.2 a) and the normal density with estimated mean and variance, it should be possible to approximately explain the data by assuming a log-normal return distribution. This is supported by the normal probability plots of the log-return data in 2.2 b): lines a,b and c correspond to the normal probability plots from 245 consecutive data each, plot b being moved to the right by 0.05, c by 0.1. 95% confidence bands are shown. The lines a,b and c being roughly parallel there is no alarming sign that the return distribution loses stationarity during the period under investigation. The sample autocovariance function in 2.2 c) (95% confidence bands around zero) suggests approximately uncorrelated (in the Gaussian case independent) day to day data.

Example 2.1: The following 3 stocks are chosen from the range of the Dow Jones Industrial Average.

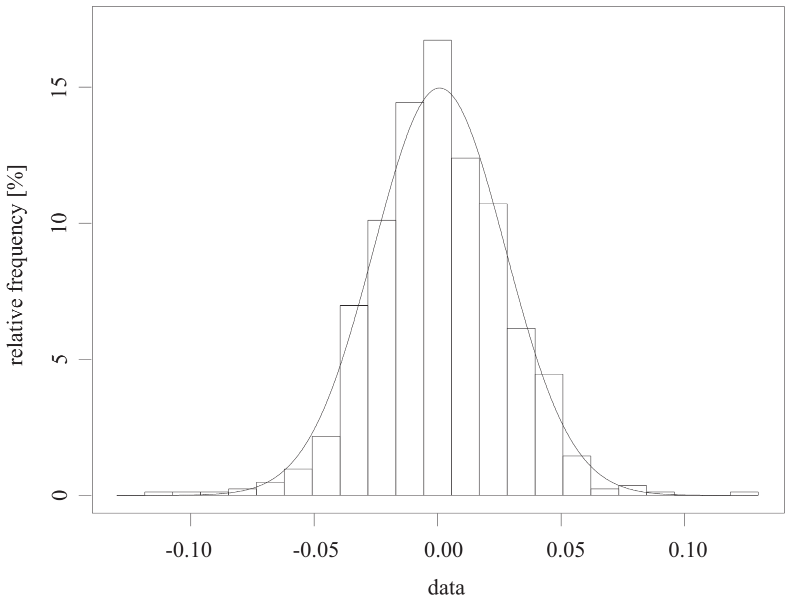
i	stock	estim.: $\mu_i - \mu_2$	$\sigma_i^2 - 2\sigma_{i2} + \sigma_2^2$
1	American Express Co. (AXP)	$-0.4503008 \cdot 10^{-4}$	$5.1832410 \cdot 10^{-4}$
2	Citigroup Inc. (C)	0.0000000	0.0000000
3	United Technologies Corp. (UTX)	$0.1215608 \cdot 10^{-4}$	$8.4458413 \cdot 10^{-4}$

The third column reports estimates based on the empirical mean and variance of the difference $Y^{(i)} - Y^{(2)}$ of the log-returns of stock i and stock 2.

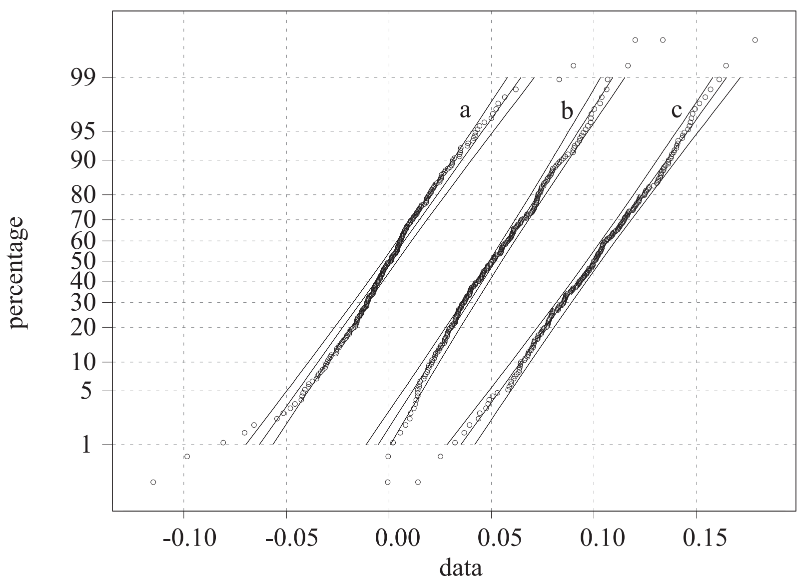
Suppose we want to enhance a portfolio of C by either AXP or UTX. Since $\sigma_1^2 - 2\sigma_{12} + \sigma_2^2 < \sigma_3^2 - 2\sigma_{23} + \sigma_2^2$ we conclude that there is no indication that we should prefer AXP to UTX.

Example 2.2: Next, consider the following stocks from the Dow Jones Transportation Average:

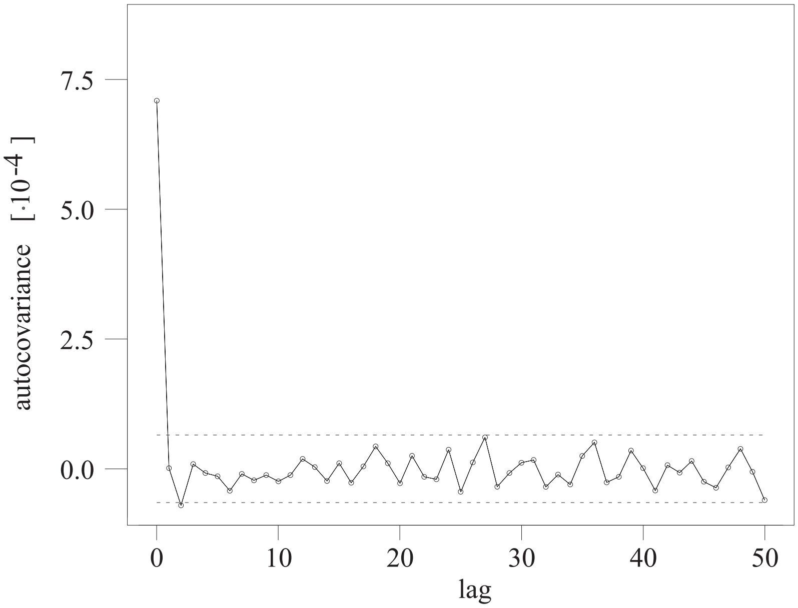
i	stock	estim.: $\mu_i - \mu_2$	$\sigma_i^2 - 2\sigma_{i2} + \sigma_2^2$
1	J.B. Hunt Transp. Serv. (JBHT)	$-0.4207553 \cdot 10^{-4}$	$17.5748915 \cdot 10^{-4}$
2	Yellow Corp. (YELL)	0.0000000	0.0000000
3	Union Pacific Corp. (UNP)	$0.3721663 \cdot 10^{-4}$	$10.7871249 \cdot 10^{-4}$



2.2 a) Histogram and estimated normal density.



2.2 b) Normal probability plot.



2.2 c) Sample autocovariance function.

Figure 2.2: Diagnostic plots for American Express Co. (AXP) log-returns from closing prices NYSE, 2/1/1998-30/11/2000.

If we want to include either JBHT or UNP in a portfolio of YELL shares, we observe that $\sigma_1^2 - 2\sigma_{12} + \sigma_2^2 > \sigma_3^2 - 2\sigma_{23} + \sigma_2^2$. Hence it may be advisable to choose JBHT as an additional stock in spite of its apparently poorer performance. Indeed, calculating the log-optimal portfolio for the alternatives we obtain:

additional stock j	weight b of YELL	$V^*_{\{2,j\}} - V^*_{\{2\}}$	residual value
1 (JBHT)	0.523951	$1.9910 \cdot 10^{-4}$	$-1.6092 \cdot 10^{-10}$
3 (UNP)	0.465490	$1.5407 \cdot 10^{-4}$	$1.3154 \cdot 10^{-10}$

$V^*_{\{2,j\}} - V^*_{\{2\}}$ is the improvement of the portfolio value achieved under inclusion of stock j. As can be seen our suspicion was justified: Choosing JBHT yields (slightly) greater portfolio improvement than UNP. The residual value in the fourth column is $1 - \mathbf{E}X^{(2)}/(bX^{(2)} + (1-b)X_j)$ and indicates that the Kuhn-Tucker condition (2.2.4) for log-optimality of the stated portfolio weight b is satisfied. The values in the third and fourth column were computed with an error of at most 10^{-9} using the composite trapezoidal rule.

CHAPTER 3

Predicted stock returns and portfolio selection

In the last chapter we have seen that portfolio selection on the basis of single-stock performances alone is problematic. Information about the variance-covariance structure of the stock returns in the market is indispensable. Of course, there are several ways to incorporate such information. For example, in a log-normal market, an investor might simply use an estimate of the variance-covariance matrix of the return distribution (conditioned on the past) and run Cover's algorithm (Theorem 1.3.2). Another investor might first try to use market observations to get an idea how the stock returns in the market are correlated and what temporal correlation prevails. Given this knowledge, he then produces forecasts of the stock returns for the next market period and rearranges his portfolio according to these forecasts – typically in a "greedy" way, i.e. trying to pick out the return maximal stock for the next market period.

Clearly, the conditional log-optimal portfolio depends on much more than mere forecasts of the returns in the next market period. Therefore, an investor following the "greedy" strategy is bound to lose out in comparison with the investor using conditional log-optimal portfolios. But still, the greedy strategy is popular among investors (the forecasting bit typically being much of a heuristics) and should therefore be analysed thoroughly. This is the task of this chapter.

In Section 3.1 we formalize the greedy strategy, which gives rise to two questions: How suboptimal can the strategy be in comparison with log-optimal portfolio selection, and, what is a reasonable way to manage the forecasting part? As to the first question, we will see that suboptimality can be bounded in terms of the variance of the logarithmic returns (cf. (3.1.4)). In view of this, the greedy strategy is appealing in markets with sufficiently small stochastic fluctuation – "sufficiently small" depending on how much the investor is prepared to lose out in investment performance.

We will embark on the second question, the forecasting problem, in Section 3.2. Once we have seen that among the many ways of forecasting, so-called "strong" forecasting is the method of choice for the greedy strategy, we will leave the stock market behind and handle the problem in the general framework of Gaussian time series forecasting. Clearly, the prediction of stationary Gaussian time series is of interest very much in its own right, with applications arising in many fields. Based on an approximation argument (Section 3.2.1), a forecasting algorithm will be presented (Section 3.2.2) that – under weak regularity conditions – is strongly consistent for huge classes of Gaussian processes (Theorem 3.2.2). Explicit examples are given, highlighting how general the algorithm is (examples after Corollary 3.2.3). The results are proved in Section 3.3. Simulations and further examples in Section 3.4 conclude the chapter.

3.1 A strategy using predicted log-returns

To avoid unnecessary technicalities, we choose the simplest setting of a market with one stock (with returns $..., X_{-1}, X_0, X_1, X_2, ...$) and one risk-free asset (bond with return r). Is it not hard though to develop analogous techniques for more general markets.

The log-transform has been found to have a stabilizing effect on the return data X_i in so far as $Y_i := \log X_i$ follows a symmetric distribution (around the mean) in many real markets. For this reason, we will use Y_i rather than X_i in the following. Under full knowledge of the process past $Y_n, Y_{n-1}, ...$, the investor is in principle advised to invest the log-optimal proportion

$$b^* := \arg\max_{b\in[0,1]} \mathbf{E}[\log(b\exp(Y_{n+1}) + (1-b)r)|Y_n, Y_{n-1}, ...] \tag{3.1.1}$$

of his wealth in the stock. However, the non-institutional (private) investor typically takes a different stance in two respects:

– His main interest is simply to determine *whether or not* he should invest a given amount of wealth in the specific stock (rather than to determine *what proportion* to invest where).

– He takes the investment decision on the basis of the predicted return of the next market period only.

In particular, he may try to achieve the maximum possible return $\max\{X_{n+1}, r\}$ in the next market period with the following "**greedy strategy**".

1. At each step n he produces an estimate $\hat{Y}_{n+1}$ for the next outcome Y_{n+1} on the basis of the observed $Y_n, ..., Y_1$ (note that it is not possible to observe the process to the infinite past).

2. He invests according to

$$b^*_{approx} := \begin{cases} 1 & \text{if } \exp(\hat{Y}_{n+1}) \geq r, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.2}$$

If the portfolio is not rearranged on a daily basis, but say, on a two month basis, this is a typical example of a buy-and-hold strategy: Once a fixed amount of money is invested according to the investor's belief what the market will look like in two month's time, the portfolio remains unchanged. A similar "greedy" buy-and-hold strategy (where only stocks from the CBS index are picked whose predicted two month return exceeds a certain threshold) has been investigated in a case study described in Franke et al. (2001, Sec. 16.4 and the references there).

Comparing $b^*_{approx} = \arg\max_{b\in[0,1]} \log(b\exp(\hat{Y}_{n+1}) + (1-b)r)$ and (3.1.1), we see that the greedy strategy is very much in the spirit of approximating the log-optimality principle. However, b^* will be a function of $Y_n, Y_{n-1}, ...$ rather than of a single statistic $\hat{Y}_{n+1}$, and the log-optimal portfolio will be diversified (i.e., $b^* \in [0,1]$), not just 0 or 1. As a consequence the investor loses out on investment performance in comparison with the log-optimal portfolio. Two questions arise:

– How should he construct the statistic $\hat{Y}_{n+1}$ in order not to lose out "too much" in comparison with log-optimal portfolio performance?

– What performance loss does the particular $\hat{Y}_{n+1}$ inflict on the investor in the worst possible case?

These are the two problems analysed in this section.

If we want to approximate the log-optimal b^* on the basis of nothing else but $\hat{Y}_{n+1}$, we need to find an approximation g of the target function,

$$\mathbf{E}[\log(b\exp(Y_{n+1}) + (1-b)r)|Y_n, Y_{n-1}, ...] \approx g(b, \hat{Y}_{n+1}, r).$$

Not every g and not every $\hat{Y}_{n+1}$ will be appropriate for this kind of approximation. A Taylor expansion may give us some guideline: Put

$$f_b(y) := \log(b\exp(y) + (1-b)r)$$

and note that

$$0 \le f_b'(y) = \frac{1}{1+(1/b-1)r\exp(-y)} \le 1,$$
$$0 \le f_b''(y) = \frac{(1/b-1)r\exp(-y)}{(1+(1/b-1)r\exp(-y))^2} \le \frac{1}{4}.$$

Now consider the expansion

$$f_b(Y_{n+1}) = f_b(\hat{Y}_{n+1}) + f_b'(\hat{Y}_{n+1})(Y_{n+1}-\hat{Y}_{n+1}) + \frac{1}{2}f_b''(\xi_b)(Y_{n+1}-\hat{Y}_{n+1})^2 \quad (3.1.3)$$

with some random ξ_b from the convex hull of Y_{n+1} and $\hat{Y}_{n+1}$. From (3.1.3) and the $\sigma(Y_n, Y_{n-1}, ...)$-measurability of $\hat{Y}_{n+1}$ we obtain

$$\begin{aligned}\mathbf{E}[f_b(Y_{n+1})|Y_n, Y_{n-1}, ...] \;=\;& f_b(\hat{Y}_{n+1}) + f_b'(\hat{Y}_{n+1})(\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...] - \hat{Y}_{n+1}) \\ & + \frac{1}{2}\mathbf{E}[f_b''(\xi_b)(Y_{n+1}-\hat{Y}_{n+1})^2|Y_n, Y_{n-1}, ...].\end{aligned}$$

As can be seen, the choice

$$\hat{Y}_{n+1} := \mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]$$

not only makes the first order term vanish but also mimimizes the upper bound

$$\left|\frac{1}{2}\mathbf{E}[f_b''(\xi_b)(Y_{n+1}-\hat{Y}_{n+1})^2|Y_n, Y_{n-1}, ...]\right| \le \frac{1}{8}\mathbf{E}[(Y_{n+1}-\hat{Y}_{n+1})^2|Y_n, Y_{n-1}, ...]$$

on the second order term.

Using b^*_{approx} (based on $\hat{Y}_{n+1}$), the investor loses at most

$$\begin{aligned}&\mathbf{E}[f_{b^*}(Y_{n+1}) - f_{b^*_{approx}}(Y_{n+1})|Y_n, Y_{n-1}, ...] \\ &\quad= \mathbf{E}[f_{b^*}(Y_{n+1}) - f_{b^*_{approx}}(\hat{Y}_{n+1})|Y_n, Y_{n-1}, ...] \\ &\qquad\quad +\mathbf{E}[f_{b^*_{approx}}(\hat{Y}_{n+1}) - f_{b^*_{approx}}(Y_{n+1})|Y_n, Y_{n-1}, ...] \\ &\quad\le \mathbf{E}[f_{b^*}(Y_{n+1}) - f_{b^*}(\hat{Y}_{n+1})|Y_n, Y_{n-1}, ...] \\ &\qquad\quad +\mathbf{E}[f_{b^*_{approx}}(\hat{Y}_{n+1}) - f_{b^*_{approx}}(Y_{n+1})|Y_n, Y_{n-1}, ...] \\ &\quad= \frac{1}{2}\mathbf{E}[(f_{b^*}''(\xi_{b^*}) - f_{b^*_{approx}}''(\xi_{b^*_{approx}}))\cdot(Y_{n+1}-\hat{Y}_{n+1})^2|Y_n, Y_{n-1}, ...] \\ &\quad\le \frac{1}{8}\mathbf{E}[(Y_{n+1}-\hat{Y}_{n+1})^2|Y_n, Y_{n-1}, ...] \\ &\quad= \frac{1}{8}\mathbf{Var}[Y_{n+1}|Y_n, Y_{n-1}, ...].\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{E}f_{b^*}(Y_{n+1}) - \mathbf{E}f_{b^*_{approx}}(Y_{n+1}) &\leq \frac{1}{8}\mathbf{E}\mathbf{Var}[Y_{n+1}|Y_n, Y_{n-1}, ...] \\ &= \frac{1}{8}(\mathbf{Var}Y_{n+1} - \mathbf{Var}\hat{Y}_{n+1}) \leq \frac{1}{8}\mathbf{Var}Y_{n+1}, \qquad (3.1.4)\end{aligned}$$

and on the average the investor won't lose more than $\frac{1}{8}\mathbf{Var}Y_{n+1}$. If he is prepared to sacrifice this amount, then the greedy strategy (3.1.2) is possible. Still, this does not obviate the necessity to estimate $\hat{Y}_{n+1} = \mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]$, but as we will see in the next section, for many practically relevant markets this can be done with reasonable effort.

3.2 Prediction of Gaussian log-returns

Prediction problems as in the previous section are one of the core topics of statistical analysis of time series. Using the distinctions of prediction problems in Györfi, Morvai and Yakowitz (1998), the problem of estimating $\mathbf{E}[Y_{n+1}|\mathcal{F}_n]$ for some sub-σ-algebra $\mathcal{F}_n$ of $\sigma(Y_n, Y_{n-1}, ...)$ is a problem of *dynamic forecasting*, i.e., in each market period, the target to be estimated changes ("moving target"). Typical choices for $\mathcal{F}_n$ are $\sigma(Y_n, Y_{n-1}, ...)$, $\sigma(Y_n, ..., Y_0)$ or $\sigma(Y_n, ..., Y_{n-d_n+1})$ for some sequence $d_n \in \mathbb{N}$ with $d_n \to \infty$, depending on what length of the process past should be included. It should be noted that although we consider a *bi-infinite* sequence of random variables $\{Y_i\}_{i=-\infty}^{\infty}$ (square-integrable and defined on a common probability space $(\Omega, \mathcal{A}, \mathbf{P})$), we only observe realisations from "time" $i = 1$ onwards.

The majority of existing algorithms for nonparametric dynamic forecasting (for an introduction see Bosq (1996) or Györfi, Härdle et al. (1998)) rely on mixing conditions which can rarely be verified from observational data. Interest has turned to dynamic forecasting under weak conditions such as mere stationarity and ergodicity, but avoiding mixing conditions. In this context, a forecaster $\hat{E}(Y_n, ..., Y_0)$ for the conditional expectation $\mathbf{E}[Y_{n+1}|\mathcal{F}_n]$ is called **strongly (weakly) universally consistent**, if

$$\lim_{n\to\infty} \left| \hat{E}(Y_n, ..., Y_0) - \mathbf{E}[Y_{n+1}|\mathcal{F}_n] \right| = 0$$

with probability 1 (in the L_1-sense) for any stationary and ergodic process $\{Y_i\}_i$. By stationarity, the computation of weakly consistent estimators ($\mathcal{F}_n :=$

$\sigma(Y_n, Y_{n-1}, ...)$ for the moment) can be reduced to the so-called static forecasting problem (Györfi, Morvai and Yakowitz, 1998), to find $\hat{E}$ such that

$$\lim_{n\to\infty} \mathbf{E}\left|\hat{E}(Y_0, ..., Y_{-n}) - \mathbf{E}[Y_1|Y_0, Y_{-1}, ...]\right| = 0.$$

Based on conditional distribution estimates $\hat{P}(dy|Y_0, ..., Y_{-n})$, Morvai, Yakowitz and Algoet (1997) obtained weakly consistent estimators for the class of bounded stationary ergodic processes, Algoet (1999) for the class of finite-mean stationary ergodic processes. Here again, to obtain convergence with probability 1 instead of L_1-convergence, mixing conditions were needed. Concerning strong universal consistency, we encounter various limitations, one of the most striking derived by Bailey (1976), with Ryabko (1988) sketching an easier intuitional proof formalised by Györfi, Morvai and Yakowitz (1998). Their result states that for any estimator $\hat{E}(Y_n, ..., Y_0)$ of the conditional expectation $\mathbf{E}[Y_{n+1}|Y_n, ..., Y_0]$, there is a stationary ergodic, binary-valued process $\{Y_i\}_i$ such that

$$\mathbf{P}\left(\limsup_{n\to\infty}\left|\hat{E}(Y_n, ..., Y_0) - \mathbf{E}[Y_{n+1}|Y_n, ..., Y_0]\right| \geq \frac{1}{4}\right) \geq \frac{1}{8}.$$

Algoet (1999) used refined techniques to show that there also exists a stationary ergodic, binary-valued sequence $\{Y_i\}_i$ with

$$\mathbf{E}\left(\limsup_{n\to\infty}\left|\hat{E}(Y_n, ..., Y_0) - \mathbf{E}[Y_{n+1}|Y_n, ..., Y_0]\right|\right) \geq \frac{1}{2}.$$

This rules out the existence of strongly universally consistent forecasters for the moving target $\mathbf{E}[Y_{n+1}|Y_n, ..., Y_0]$.

This result is discouraging, but it does not rule out the existence of strongly consistent forecasting rules for log-return processes as they arise in real financial markets. In particular, Gaussian log-return processes have been proven to be a good approximation for real log-return processes. Györfi, Morvai and Yakowitz (1998) note that there has not yet been found an answer to the question whether strongly consistent forecasters for $\mathbf{E}[Y_{n+1}|Y_n, ..., Y_0]$ or even $\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]$ exist in case the process $\{Y_i\}_i$ is known to be Gaussian. The results of this section show that for notably wide classes of Gaussian processes the answer is affirmative. It should be noted, however, that strong universal consistency is by far not the only means of "strong" forecasting. Other methods, so-called "universal" predictors are based on Cesàro-convergence, i.e. the average of the errors (e.g., squared prediction errors, or the squared differences of the estimates and $\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...])$ converges with probability one to the minimal possible value for any (bounded) stationary and ergodic process (Algoet, 1994). Such estimators were obtained by Algoet (1992) and Morvai et al. (1996). Based on Györfi, Lugosi and Morvai (1999), universal predictors for bounded or Gaussian stationary ergodic processes have been constructed by Györfi and Lugosi (2001).

Throughout, we make the following two assumptions:

1. Let $\{Y_n\}_{n=-\infty}^{\infty}$ be a real-valued, purely nondeterministic (i.e., there is no deterministic term in the representation (3.2.2) below), stationary and ergodic Gaussian process with

$$\mathbf{E}Y_n = 0, \quad \mathbf{Var}Y_n = \sigma^2 > 0 \tag{3.2.1}$$

   and autocovariance function $\gamma(k) := \mathbf{E}(Y_{n+k}Y_n)$. The assumption $\mathbf{E}Y_n = 0$ is no restriction, it follows after differencing the original process of log-returns. We denote the differenced process by $\{Y_i\}$ again.

2. From 1. and Wold's decomposition theorem (Hida and Hitsuda, 1993, Theorems 3.2 and 3.3; Shiryayev, 1984, VI §5 Theorem 2), a canonical L_2-representation

$$Y_n = \sum_{j=0}^{\infty} \psi_j \epsilon_{n-j} \tag{3.2.2}$$

   can be found with $\psi_0 = 1$, $\sum_{j=0}^{\infty} |\psi_j|^2 < \infty$ and independent, identically $N(0, \sigma_\epsilon^2)$-distributed innovations ϵ_n. For $z \in \mathbb{C}$, $|z| < 1$, the series $\sum_{j=0}^{\infty} \psi_j z^j$ converges to the transfer function $\psi(z) := \sum_{j=0}^{\infty} \psi_j z^j$, which never vanishes for $|z| < 1$ (Hida and Hitsuda, 1993, III §3 i).

   We assume that

$$\sum_{j=0}^{\infty} |\psi_j| < \infty, \tag{3.2.3}$$

   ensuring that the equality (3.2.2) holds with probability 1 (Brockwell and Davis, 1991, Prop. 3.1.1).

Statistical and theoretical aspects of second order stationary processes are treated extensively in literature, among many others in Brockwell and Davis (1991), Caines (1988) and Hannan and Deistler (1988). Many results on Gaussian processes can be found in Neveu (1968), Ibragimov and Rozanov (1978) and Hida and Hitsuda (1993). Indeed, (3.2.1) and (3.2.3) are standard assumptions in time series analysis, and a considerable variety of sufficient conditions for the assumptions to hold are known. We just note that if $f(\lambda), \lambda \in [-\pi, \pi]$, is the spectral density of the process $\{Y_n\}_{n=-\infty}^{\infty}$, then

$$2\pi f(\lambda) = \sigma_\epsilon^2 |\psi(e^{-i\lambda})|^2 \tag{3.2.4}$$

(Brockwell and Davis, 1991, eq. 4.4.3; Shiryayev, 1984, VI §6 eq. (16f.)), and

$$\int_{-\pi}^{\pi} \log f(\lambda) d\lambda > -\infty \tag{3.2.5}$$

is sufficient (and in fact also necessary) for the process to be purely nondeterministic (Shiryayev, 1984, VI §5 Theorem 4). The simplest setting in which (3.2.5) holds is the case when f happens to be bounded away from 0. Then the process is strongly mixing (Ibragimov and Linnik, 1971, Theorem 17.3.3). However, for the purpose of our analysis, we do not require this strong property.

We divide the problem of estimating $\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]$ into two steps:

1. *Approximation* of $\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]$ by some conditional expectation based on a fraction of the past only, say $\mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}]$ with $d \in \mathbb{N}$.

2. *Estimation* of the latter quantity from the observed data.

Here and in all the following, d should be taken as $d = d_n \in \mathbb{N}, d_n \nearrow \infty, d_n \leq n$, rather than as a mere constant. For the sake of simplicity of notation however, this will be suppressed most of the time.

3.2.1 An approximation result

As to the first step, the approximation step, note that by stationarity and Doob's conditional expectation continuity theorem (Doob, 1984, 2.I.5)

$$\left| \mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...] - \mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d_n+1}] \right| \to 0$$

in L_1 whenever $d_n \to \infty$. As will be seen, more stringent conditions are needed to obtain convergence with probability 1. Similar problems arise in the context of on-line order selection for AR(∞) models. Here, founded on Rissanen's (1989) stochastic complexity for model comparison, the influence of increasing dimensionality d_n on the prediction error of the estimated "best" AR(d_n) model is discussed in depth by Gerencsér (1992). The accuracy of order selection schemes based on least squares principles is also investigated in Davisson (1965) and Wax (1988). In contrast to these, the approximation of $\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]$ by $\mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d_n+1}]$ used in this section is not data-driven but chooses deterministic d_n according to the conditions given in the following lemma.

Lemma 3.2.1. *If the Taylor coefficients of*

$$\frac{1}{\psi(z)} = \sum_{k=0}^{\infty} \phi_k z^k \quad (|z| < 1)$$

satisfy

$$\sum_{k=d_n+1}^{\infty} |\phi_k|^2 \leq \left(\frac{const.}{\log n}\right)^r \tag{3.2.6}$$

for some $r > 1$ and sufficiently large n, then

$$\lim_{n\to\infty} \Big|\mathbf{E}[Y_{n+1}|Y_n, ..., Y_{d_n}] - \mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]\Big| = 0$$

with probability 1.

The proof of this result and of the next theorems is deferred to section 3.3.

3.2.2 An estimation algorithm

If we collect the autocovariances in the matrix

$$\Gamma_d := (\gamma(i-j))_{i,j=1,\ldots,d} = \begin{pmatrix} \gamma(0) & \gamma(1) & \ldots & \gamma(d-1) \\ \gamma(1) & \gamma(0) & \ldots & \gamma(d-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(d-1) & \gamma(d-2) & \ldots & \gamma(0) \end{pmatrix}$$

and

$$\gamma_d := (\gamma(d), ..., \gamma(1)),$$

we can obtain an explicit formula for $\mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}]$. In fact, assumption (3.2.3) implies

$$\gamma(k) \to 0 \quad (k \to \infty) \tag{3.2.7}$$

(Brockwell and Davis, 1991, Probl. 3.9), and from (3.2.1) and (3.2.7) it follows that Γ_d is non-singular (Brockwell and Davis, 1991, Prop. 5.1.1). For Gaussian processes one has (Brockwell and Davis, 1991, §5.4; Shiryayev, 1984, II §13 Theorem 2)

$$\mathbf{E}[Y_{d+1}|Y_d, ..., Y_1] = \gamma_d \, \Gamma_d^{-1} \, (Y_1, ..., Y_d)^T,$$

and the autoregression function

$$m_d(y_d, .., y_1) := \mathbf{E}[Y_{d+1}|Y_d = y_d, ..., Y_1 = y_1] = \gamma_d \, \Gamma_d^{-1} \, (y_1, ..., y_d)^T \tag{3.2.8}$$

is linear. Stationarity yields

$$m_d(y_d, ..., y_1) = \mathbf{E}[Y_{n+1}|Y_n = y_d, ..., Y_{n-d+1} = y_1]$$

and thus

$$\mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}] = m_d(Y_n, ...Y_{n-d+1}). \tag{3.2.9}$$

From (3.2.8) and (3.2.9) it is plausible to construct a simple estimator $\hat{E}_{d,n}$ for the conditional expectation $\mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}]$ by the following steps:

1. Estimate the autocovariances $\gamma(0), ..., \gamma(d)$ by the sample autocovariances

$$\hat{\gamma}_n(k) := \frac{1}{n} \sum_{i=1}^{n-|k|} Y_i Y_{i+|k|} \quad (k = -d, ..., d). \tag{3.2.10}$$

2. Set $\tilde{\Gamma}_{d,n} := (\hat{\gamma}_n(i-j))_{i,j=1,...,d}$. Posing $\tilde{\Gamma}_{d,n} = \frac{1}{n} A A^T$ with the matrix $A \in \mathbb{R}_{d \times 2n}$ formed by the first d rows of the matrix

$$\begin{pmatrix} 0 & 0 & — & 0 & Y_1 & Y_2 & \dots & Y_n \\ 0 & — & 0 & Y_1 & Y_2 & \dots & Y_n & 0 \\ | & / & / & & & / & / & | \\ 0 & Y_1 & Y_2 & \dots & Y_n & 0 & — & 0 \end{pmatrix} \in \mathbb{R}_{n \times 2n},$$

it is obvious that $\tilde{\Gamma}_{d,n}$ is non-negative definite. Thus

$$\hat{\Gamma}_{d,n} := \tilde{\Gamma}_{d,n} + \frac{1}{n} I_d = \left(\hat{\gamma}_n(i-j) + \frac{\delta_{i,j}}{n} \right)_{i,j=1,...,d}$$

is non-singular. Hence, with (3.2.10) and $\hat{\gamma}_{n,d} := (\hat{\gamma}_n(d), ..., \hat{\gamma}_n(1))$, define

$$\hat{m}_{d,n}(y_d, ..., y_1) := \hat{\gamma}_{n,d}\, \hat{\Gamma}_{d,n}^{-1}\, (T_{L_n}y_1, ..., T_{L_n}y_d)^T$$

on the analogy of (3.2.8). Here, $0 < L_n \nearrow \infty$ is a sequence of truncation heights, $T_L y := \operatorname{sgn}(y) \min\{L, |y|\}$ being the truncation operator.

3. Plug in the last d observations,

$$\hat{E}_{d,n} := \hat{m}_{d,n}(Y_n, ..., Y_{n-d+1}).$$

Remark. Even if $\hat{\gamma}_{n,d}\, \hat{\Gamma}_{d,n}^{-1}$ constitutes a strongly consistent estimate for the coefficients $\gamma_d\, \Gamma_d^{-1}$ of the autoregression function m_d, it may happen that estimation errors $(\hat{\gamma}_{n,d}\, \hat{\Gamma}_{d,n}^{-1} - \gamma_d\, \Gamma_d^{-1})$ which are per se "acceptable", occur together with large values of the $Y_n, ..., Y_{n-d+1}$ "plugged in". The resulting prediction error $|\mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}] - \hat{E}_{d,n}|$ becomes considerably large. Suitable truncation limits the size of the Y_i's without obscuring the information they contain.

Now, denoting the maximal absolute row sum of a real matrix $A = (a_{ij})_{i,j}$ by $\|A\|_\infty := \max_i \sum_j |a_{ij}|$, we establish the following convergence result for the proposed estimator:

Theorem 3.2.2. *Assume (3.2.1), (3.2.3), and choose d_n and L_n such that for some $r \geq 4$, some $\delta > 0$ and sufficiently large n*

$$\begin{aligned}
& d_n \leq n^{r/(2(r-2))}, \\
& L_n \|\Gamma_{d_n}^{-1}\|_\infty^2 d_n^{2(r+1)/r} \frac{(\log n)^{2/r} (\log\log n)^{2(1+\delta)/r}}{n^{1/2}} = O(1), \\
& \frac{L_n \|\Gamma_{d_n}^{-1}\|_\infty^2 d_n}{n} \to 0 \quad (n \to \infty), \\
& \sum_{n=1}^{\infty} \frac{d_n}{L_n} \exp\left(-\frac{1}{2\sigma^2} L_n^2\right) < \infty. \qquad (3.2.11)
\end{aligned}$$

Then

$$\lim_{n\to\infty} \left|\mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d_n+1}] - \hat{E}_{d_n,n}\right| = 0 \qquad (3.2.12)$$

with probability 1.

From (3.2.11), for the choice of d_n and L_n, one needs some bound on the possible growth of $\|\Gamma_{d_n}^{-1}\|_\infty$. Based on the spectral density f and its (essential) minimum $m_f \geq 0$ we distinguish the following cases:

Case 1: f is bounded away from 0, $m_f > 0$.

Case 2: f has finitely many zeros $\lambda_1, ..., \lambda_m \in (-\pi, \pi]$ of orders $p_1, ..., p_m$, that is, there exist constants $p_j^-, p_j^+ > 0, K \geq 1, \delta > 0$ such that

$$\frac{1}{K} < \frac{f(\lambda)}{|\lambda - \lambda_j|^{p_j^+}} < K$$

for all $\lambda \in (\lambda_j, \lambda_j + \delta)$ and

$$\frac{1}{K} < \frac{f(\lambda)}{|\lambda - \lambda_j|^{p_j^-}} < K$$

for all $\lambda \in (\lambda_j - \delta, \lambda_j)$. In this case we define the order of the jth zero as $p_j := \max\{p_j^-, p_j^+\}$ and set $p^* := \max\{p_j | j = 1, ..., m\}$.

Case 3: No restrictions are imposed on m_f, apart from those already implied by (3.2.1) and (3.2.3).

For each of the cases, upper bounds for $\|\Gamma_{d_n}^{-1}\|_\infty$ can easily be derived from classical results and results recently obtained by Serra (1998, 1999, 2000) and Böttcher and Grudsky (1998). These yield

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Corollary 3.2.3. *In cases 1-3 and under the assumptions (3.2.1) and (3.2.3) the strong consistency relation (3.2.12) holds if, for n sufficiently large,*

Case 1: $d_n \leq n^s$ *and* $L_n := (\log n)^t \quad (\frac{1}{6} > s > 0, t \geq 1)$.

Case 2: $d_n \leq n^s$ *and* $L_n := (\log n)^t \quad (\frac{1}{6+4p^*} > s > 0, t \geq 1)$.

Case 3: $d_n \leq \left(\frac{1}{q} \log n\right)^s$ *and* $L_n := (\log n)^t \quad (q > 4, 0 < s < 1, t \geq 1)$.

Before proving the results, we give some examples to illustrate the application of Lemma 3.2.1 and Corollary 3.2.3, such that for a suitable choice of d_n and L_n the consistency relation

$$\lim_{n \to \infty} \left| \hat{E}_{d_n,n} - \mathbf{E}[Y_{n+1} | Y_n, Y_{n-1}, ...] \right| \longrightarrow 0 \tag{3.2.13}$$

holds with probability 1 for all processes in large classes $\mathcal{G}_i$ of Gaussian processes.

Example 3.1: First, let the class $\mathcal{G}_1$ consist of all Gaussian processes satisfying (3.2.1) and (3.2.3) with spectral density bounded away from zero. We choose

$$d_n := \lfloor n^s \rfloor \quad (\frac{1}{4} > s > 0), \quad L_n := \log n$$

and obtain (3.2.13) for any element of $\mathcal{G}_1$.

Indeed, for every element of $\mathcal{G}_1$, $\psi(z)$ has no zeros for $|z| = 1$ by (3.2.4). Then $\psi(z)$ never vanishes in the closed unit disk, and $1/\psi(z)$ is analytic on a disk around 0 with radius $1 + \epsilon$ for some $\epsilon > 0$. Thus $\phi_k \left(1 + \frac{\epsilon}{2}\right)^k \to 0$ as $k \to \infty$, hence $|\phi_k| \leq c \left(1 + \frac{\epsilon}{2}\right)^{-k}$ with some constant $c > 0$. Set $\rho := (1 + \epsilon/2)^{-2} < 1$, then

$$\sum_{k=d_n+1}^{\infty} |\phi_k|^2 \leq c^2 \sum_{k=d_n+1}^{\infty} \rho^k = c^2 \frac{\rho}{1-\rho} \rho^{d_n} \leq (\log n)^{-3}$$

for n sufficiently large if only $d_n / \log\log n \to \infty$. Lemma 3.2.1 applies and Corollary 3.2.3 yields (3.2.13).

For $\mathcal{G}_1$ and the choice $d_n = O(\log n)$ it should be noted that (3.2.13) is also a consequence of An, Chen and Hannan (1982, Theorem 6). From there it follows that the Yule-Walker estimates $(\hat{\phi}_1, ..., \hat{\phi}_{d_n})$ of the first d_n coefficients in the AR(∞) representation satisfy the uniform convergence property

$$\sup_{1 \leq j \leq d_n} |\hat{\phi}_j - \phi_j| = O\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right)$$

with probability 1. Using the estimates $(\hat{\phi}_1, ..., \hat{\phi}_{d_n})$ and the fact that the true coefficients of the AR(∞) representation converge to zero at an exponential rate, one obtains (3.2.13). However, the next example illustrates that Lemma 3.2.1 and Corollary 3.2.3 are applicable in more general situations as well.

Example 3.2: Consider the class $\mathcal{G}_2$ of all Gaussian processes satisfying (3.2.1) and (3.2.3) such that for all elements of $\mathcal{G}_2$ the following two conditions hold:

a) The corresponding spectral density f has only finitely many zeros, each of which is of finite order in the sense of the above case 2 and

b) $|\phi_k| \leq \Phi_k$, where $\{\Phi_k\}_k$ is an eventually decreasing sequence satisfying $\sum\limits_{k=1}^{\infty} \Phi_k^{2-\epsilon} < \infty$ for some $\epsilon > 0$.

Note that the orders of the zeros as well as ϵ may be different for each element of $\mathcal{G}_2$. This class comprises $\mathcal{G}_1$ as well as Gaussian processes with transfer functions such as $\psi(z) = (1-z)^{1/3} := 1 + \sum_{k=1}^{\infty} \psi_k z^k$, $\psi_k = -\frac{2\cdot 5\cdot\ldots\cdot(3k-4)}{3\cdot 6\cdot\ldots\cdot(3k)}$, for which $\phi_k = \frac{1\cdot 4\cdot\ldots\cdot(3k-2)}{3\cdot 6\cdot\ldots\cdot(3k)}$ and $f(\lambda) = \frac{\sigma_\epsilon^2}{2\pi}\left(4\sin^2\frac{\lambda}{2}\right)^{1/3}$, the process being purely nondeterministic by (3.2.5).

With the choice

$$d_n := \lfloor (\log n)^{\log\log n} \rfloor \quad (\text{for } n \geq 2) \quad \text{and} \quad L_n := \log n,$$

(3.2.13) holds for any process in $\mathcal{G}_2$.

To see this, observe that

$$\sum_{k=d_n+1}^{\infty} \phi_k^2 \leq \Phi_{d_n+1}^{\epsilon} \sum_{k=d_n+1}^{\infty} \Phi_k^{2-\epsilon} \leq \Phi_{d_n+1}^{\epsilon} \sum_{k=1}^{\infty} \Phi_k^{2-\epsilon} = const. \cdot \Phi_{d_n+1}^{\epsilon}.$$

From $\sum\limits_{k=1}^{\infty} \Phi_k^2 < \infty$, Olivier's theorem (Knopp, 1956, 3.3 Theorem 1) allows us to infer $k\Phi_k^2 \longrightarrow 0 \quad (k \to \infty)$, hence

$$\sum_{k=d_n+1}^{\infty} \phi_k^2 \leq \frac{const.}{(d_n+1)^{\epsilon/2}}$$

for n sufficiently large. Thus (3.2.6) is fulfilled and Lemma 3.2.1 applies. On the other hand, $\lim_{n\to\infty} (\log n)^{\log\log n} n^{-s} = 0$ for any $s > 0$, and we can use Corollary 3.2.3 to obtain (3.2.13).

3.3 Proof of the approximation and estimation results

We first turn to the

Proof of Lemma 3.2.1. Note that the transfer function $\psi(z)$ never vanishes for $|z| < 1$ (Hida and Hitsuda, 1993, III §3 i). Hence its reciprocal is analytic for $|z| < 1$ and can be posed as

$$\phi(z) = \frac{1}{\psi(z)} = \sum_{k=0}^{\infty} \phi_k z^k \quad (|z| < 1)$$

with $\phi_0 = 1$. The innovations can be obtained by

$$\epsilon_{n+1} = \sum_{k=0}^{\infty} \phi_k Y_{n+1-k} = Y_{n+1} + \sum_{k=1}^{\infty} \phi_k Y_{n+1-k}$$

(Hida and Hitsuda, 1993, III §3 eq. 3.31). We observe that $\sigma(\epsilon_{n+1})$ is independent of $\sigma(Y_n, Y_{n-1}, ...)$, where the latter σ-algebra makes $\sum_{k=1}^{\infty} \phi_k Y_{n+1-k}$ measurable. Thus,

$$0 = \mathbf{E}[\epsilon_{n+1}|Y_n, Y_{n-1}, ...] = \mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...] + \sum_{k=1}^{\infty} \phi_k Y_{n+1-k}$$

and

$$\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...] = -\sum_{k=1}^{\infty} \phi_k Y_{n+1-k}. \tag{3.3.1}$$

Moreover,

$$\begin{aligned}
\mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}] &= \mathbf{E}\left[\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]\middle|Y_n, ..., Y_{n-d+1}\right] \\
&= -\sum_{k=1}^{d} \phi_k Y_{n+1-k} - \mathbf{E}\left[\sum_{k=d+1}^{\infty} \phi_k Y_{n+1-k}\middle|Y_n, ..., Y_{n-d+1}\right]. \qquad (3.3.2)
\end{aligned}$$

(3.3.1) and (3.3.2) imply

$$\begin{aligned}
&\mathbf{E}\left|\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...] - \mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}]\right|^2 \\
&\quad = \mathbf{E}\left|\mathbf{E}\left[\sum_{k=d+1}^{\infty} \phi_k Y_{n+1-k}\middle|Y_n, ..., Y_{n-d+1}\right] - \sum_{k=d+1}^{\infty} \phi_k Y_{n+1-k}\right|^2 \\
&\quad = \mathbf{E}\left|\sum_{k=d+1}^{\infty} \phi_k Y_{n+1-k}\right|^2 - \mathbf{E}\left(\mathbf{E}\left[\sum_{k=d+1}^{\infty} \phi_k Y_{n+1-k}\middle|Y_n, ..., Y_{n-d+1}\right]^2\right) \\
&\quad \leq \mathbf{E}\left|\sum_{k=d+1}^{\infty} \phi_k Y_{n+1-k}\right|^2.
\end{aligned}$$

Now, set

$$H_d(\lambda) := \sum_{k=d+1}^{\infty} \phi_k \exp(-ik\lambda).$$

Then $|H_d(\lambda)|^2 f(\lambda)$ is the spectral density of the linear filter $\sum_{k=d+1}^{\infty} \phi_k Y_{n+1-k}$ (Brockwell and Davis, 1991, Theorem 4.10.1), and we obtain

$$\begin{aligned}\mathbf{E}\Big|\sum_{k=d+1}^{\infty} \phi_k Y_{n+1-k}\Big|^2 &= \int_{-\pi}^{\pi} |H_d(\lambda)|^2 f(\lambda) d\lambda \\ &\leq \sup_{\lambda\in[-\pi,\pi]} f(\lambda) \int_{-\pi}^{\pi} |H_d(\lambda)|^2 d\lambda = 2\pi \sup_{\lambda\in[-\pi,\pi]} f(\lambda) \sum_{k=d+1}^{\infty} |\phi_k|^2.\end{aligned}$$

Since the difference $\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...] - \mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}]$ is the probability limit of the Gaussian variables $\mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-k+1}] - \mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}]$ as $k \to \infty$, it is itself Gaussian (Shiryayev, 1984, II §13.5).

We will apply the following lemma on the convergence of Gaussian random variables that can be found in Buldygin and Dočenko (1977, Lemma 3).

Lemma 3.3.1. *Let $\{W_n\}_{n=0}^{\infty}$ be a sequence of centered Gaussian random variables $W_n \sim N(0, \sigma_n^2)$ with $\sigma_n^2 \to 0$ $(n \to \infty)$. If for every $\epsilon > 0$*

$$\sum_{n=1}^{\infty} \exp\left(-\frac{\epsilon}{\sigma_n^2}\right) < \infty \tag{3.3.3}$$

then $W_n \to 0$ with probability 1 as $n \to \infty$.

In particular, (3.3.3) is fulfilled if

$$\sigma_n^2 \leq \left(\frac{const.}{\log n}\right)^r$$

for some $r > 1$ and sufficiently large n.

This follows immediately from Lemma 3.3.1 if we choose $1 < r' < r$ and observe that

$$\begin{aligned}\sum_{n=N}^{\infty} \exp\left(-\frac{\epsilon}{\sigma_n^2}\right) &\leq \sum_{n=N}^{\infty} \exp\left(-\epsilon\left(\frac{\log n}{const.}\right)^r\right) \\ &\leq \sum_{n=N}^{\infty} \exp\left(-(\log n)^{r'}\right) \leq \sum_{n=N}^{\infty} \left(\exp\left(-\log n\right)\right)^{r'} = \sum_{n=N}^{\infty} \frac{1}{n^{r'}} < \infty\end{aligned}$$

for N sufficiently large.

Now, by Lemma 3.3.1 and with $M_f := 2\pi \sup_{\lambda \in [-\pi,\pi]} f(\lambda)$, the inequality

$$\mathbf{Var}\Big(\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...] - \mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}]\Big) \\ \leq M_f \sum_{k=d_n+1}^{\infty} |\phi_k|^2 \leq \left(\frac{const.}{\log n}\right)^r$$

for some $r > 1$ and n sufficiently large implies

$$\Big|\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...] - \mathbf{E}[Y_{n+1}|Y_n, ..., Y_{n-d+1}]\Big| \to 0$$

with probability 1, and the proof of the lemma is finished. □

Remark. In order to obtain convergence of the W_n to zero with probability 1, one has to impose some condition on the rate of decay of σ_n^2. The conditions of Lemma 3.3.1 cannot be substantially weakened. Indeed, if we consider any sequence of independent random variables $W_n \sim N(0, \sigma_n^2)$ with $\sigma_n^2 = (\text{lb } n)^{-1}$, such a sequence cannot converge to zero with probability 1 (lb denoting the logarithm for base 2): In fact, the lower bound on $1 - \Phi$ in Feller (1968, VII.1. Lemma 2) gives

$$\mathbf{P}(|W_n| \geq \frac{1}{2}) > \left(\frac{4}{\sqrt{\text{lb } n}} - \frac{16}{\text{lb } n\sqrt{\text{lb } n}}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{8}\text{lb } n\right) =: q_n.$$

The q_n constitute an eventually positive, decreasing sequence. Thus, $\sum q_n$ has the same convergence properties as $\sum 2^n q_{2^n}$. The latter series diverges because of

$$\left(2^n q_{2^n}\right)^{1/n} = 2\left(\frac{4}{\sqrt{n}} - \frac{16}{n\sqrt{n}}\right)^{1/n} \frac{1}{(2\pi)^{1/2n}} \exp\left(-\frac{1}{8}\right) \to 2\exp\left(-\frac{1}{8}\right) > 1.$$

Keeping this in mind, assume we had $W_n \to 0$ with probability 1. With the characteristic function $f(y) := \mathbf{1}_{[-1/2,1/2]^C}(y)$, this yields independent random variables $f(W_n) \to f(0) = 0$ with probability 1. Since $f(W_n)$ is $\{0, 1\}$-valued, convergence to zero on a set of probability 1 implies (Shiryayev, 1984, II §10 Example 3)

$$\sum_{n=1}^{\infty} \mathbf{P}\Big(f(Y_n) = 1\Big) = \sum_{n=1}^{\infty} \mathbf{P}\Big(|W_n| \geq \frac{1}{2}\Big) < \infty,$$

a contradiction to the divergence of $\sum q_n$.

For the proof of Theorem 3.2.2 we need some preliminary observations. First we recall some simple facts from the theory of matrix norms. Let $\|\cdot\|$ be some vector norm on $\mathbb{R}^d$. By $\|\cdot\|$ also denote the corresponding matrix norm

$$\|A\| := \sup_{y\neq 0} \frac{\|Ay\|}{\|y\|} = \sup_{\|y\|=1} \|Ay\|$$

for $A \in \mathbb{R}_{d\times d}$. The spectrum of A is the collection of the moduli of the eigenvalues of A,

$$\mathrm{spr}(A) := \left\{ |\lambda| \,\middle|\, \lambda \text{ eigenvalue of } A \right\},$$

and the spectral radius of A is

$$\rho(A) := \max\ \mathrm{spr}(A).$$

Recall the following inequalities (Isaacson and Keller, 1994, Corollaries in Sec. 1.1 and 1.3):

Lemma 3.3.2. *Let $A, B, C \in \mathbb{R}_{d\times d}$.*
1. If $\|A\| < 1$, then $(I - A)^{-1}$ exists and we have

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

2. If B and C are non-singular with $\|I - B^{-1}C\| < 1$, the following inequalities hold:

$$\begin{aligned} \|C^{-1}\| &\leq \frac{\|B^{-1}\|}{1 - \|I - B^{-1}C\|}, \\ \|C^{-1} - B^{-1}\| &\leq \frac{\|B^{-1}\|\ \|I - B^{-1}C\|}{1 - \|I - B^{-1}C\|}. \end{aligned}$$

Proof. 1. Existence of $(I - A)^{-1}$ follows from the well known Neumann series. $(I - A)^{-1} = I + A(I - A)^{-1}$ implies $\|(I - A)^{-1}\| \leq 1 + \|A\|\ \|(I - A)^{-1}\|$, or using $\|A\| < 1$

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

2. Inversion of $B^{-1}C = I + B^{-1}C - I$ yields $C^{-1}B = (I + B^{-1}C - I)^{-1}$. Multiplying by B^{-1} from the right, we obtain $C^{-1} = (I - (I - B^{-1}C))^{-1}B^{-1}$

and using the first part of the lemma

$$\|C^{-1}\| \le \|B^{-1}\| \, \|(I-(I-B^{-1}C))^{-1}\| \le \frac{\|B^{-1}\|}{1-\|I-B^{-1}C\|}.$$

Finally, pose $C^{-1}-B^{-1}=(I-B^{-1}C)C^{-1}$ and obtain

$$\|C^{-1}-B^{-1}\| \le \|I-B^{-1}C\| \, \|C^{-1}\| \le \frac{\|B^{-1}\| \, \|I-B^{-1}C\|}{1-\|I-B^{-1}C\|}. \qquad \square$$

We will use the vector norms $\|y\|_2 := (\sum_{i=1}^d y_i^2)^{1/2}$ or $\|y\|_\infty := \max_{i=1,\dots,d} |x_i|$, and the corresponding matrix norms $\|A\|_2 := \rho(A^T A)^{1/2}$ and $\|A\|_\infty := \max_{i=1,\dots,d} \sum_{j=1}^d |a_{ij}|$, respectively. For the latter norm we have

Lemma 3.3.3. *For any symmetric and non-singular matrix $A \in \mathbb{R}_{d\times d}$*

$$\|A\|_\infty \le \frac{d^{1/2}}{\min\ \mathrm{spr}(A^{-1})}.$$

Proof. Let $y=(y_1,\dots,y_d)\in\mathbb{R}^d$. Then $\|y\|_\infty \le \|y\|_2 \le (d \max_i y_i^2)^{1/2} = d^{1/2}\|y\|_\infty$ and

$$\begin{aligned}\|A\|_\infty &= \sup_{y\neq 0}\frac{\|Ay\|_\infty}{\|y\|_\infty} \le d^{1/2}\sup_{y\neq 0}\frac{\|Ay\|_2}{\|y\|_2} = d^{1/2}\|A\|_2 = (d\rho(A^TA))^{1/2}\\ &= (d\rho(A^2))^{1/2} = (d\rho(A)^2)^{1/2} = d^{1/2}\rho(A) = d^{1/2}\frac{1}{\min \mathrm{spr}(A^{-1})}. \qquad \square\end{aligned}$$

The proposed estimation procedure requires convergence of $\hat\gamma_n(k)$ to $\gamma(k)$. Doob (1953, X §7 Theorem 7.1) gave the key result for this when proving that for a real-valued, centered, stationary and ergodic Gaussian process $\hat\gamma_n(k)\to\gamma(k)$ with probability 1 as $n\to\infty$ is equivalent to either one of the conditions (a) $\frac{1}{n+1}\sum_{k=0}^n |\gamma(k)|^2 \to 0$ as $n\to\infty$ or (b) the spectral distribution function F is continuous. In the present analysis, a more precise result featuring the rate of convergence will be needed. This important result was given by An, Chen and Hannan (1982, Theorem 1):

Lemma 3.3.4. *Let $\{Y_n\}_{n=1}^\infty$ be a stationary ergodic process with zero mean and finite variance, allowing for a representation*

$$Y_n = \sum_{j=0}^\infty \psi_j \epsilon_{n-j} \quad \textit{with} \quad \psi_0 = 1, \quad \sum_{j=0}^\infty |\psi_j| < \infty$$

and innovations ϵ_n satisfying

$$\begin{aligned}
&\mathbf{E}[\epsilon_n|\epsilon_m, m<n] = 0,\\
&\mathbf{E}[\epsilon_n^2|\epsilon_m, m<n] = \sigma_\epsilon^2,\\
&\mathbf{E}(|\epsilon_n|^r) < \infty \quad \textit{for some} \quad r \geq 4.
\end{aligned}$$

Then for any $\delta > 0$ and $d_n \leq n^{r/(2(r-2))}$

$$\lim_{n\to\infty} r_n \max_{0\leq k\leq d_n} |\gamma_n(k) - \hat{\gamma}_n(k)| = 0$$

with probability 1, where

$$r_n := \frac{n^{1/2}}{(d_n \log n)^{2/r} (\log\log n)^{2(1+\delta)/r}}.$$

In the "Gaussian case" considered here, the conditions on the innovations are fulfilled, since the ϵ_n's are independent, identically $N(0,\sigma_\epsilon^2)$-distributed, hence $\mathbf{E}[\epsilon_n|\epsilon_m, m<n] = \mathbf{E}\epsilon_n = 0$ and $\mathbf{E}[\epsilon_n^2|\epsilon_m, m<n] = \mathbf{E}(\epsilon_n^2) = \sigma_\epsilon^2$.

After these introductory remarks, we turn to the core of the

Proof of Theorem 3.2.2. $\bar{Y}_{n-d+1}^n :=$ is a shorthand notation for the d-past of the process $(Y_{n-d+1},...,Y_n)^T$, and we set $T_L\bar{Y}_{n-d+1}^n := (T_L Y_{n-d+1},...,T_L Y_n)^T$. The prediction error can be decomposed into

$$\begin{aligned}
&\left|\hat{E}_{d,n} - \mathbf{E}[Y_{n+1}|Y_n,...,Y_{n-d+1}]\right| = \left|\hat{\gamma}_{d,n}\hat{\Gamma}_{d,n}^{-1} T_{L_n}\bar{Y}_{n-d+1}^n - \gamma_d\Gamma_d^{-1}\bar{Y}_{n-d+1}^n\right|\\
&\quad\leq \left|(\hat{\gamma}_{d,n}\hat{\Gamma}_{d,n}^{-1} - \gamma_d\Gamma_d^{-1})T_{L_n}\bar{Y}_{n-d+1}^n\right| + \left|\gamma_d\Gamma_d^{-1}(T_{L_n}\bar{Y}_{n-d+1}^n - \bar{Y}_{n-d+1}^n)\right|\\
&\quad\leq dL_n\|\gamma_d\Gamma_d^{-1} - \hat{\gamma}_{d,n}\hat{\Gamma}_{d,n}^{-1}\|_\infty + d\|\gamma_d\|_\infty\|\Gamma_d^{-1}\|_\infty\|T_{L_n}\bar{Y}_{n-d+1}^n - \bar{Y}_{n-d+1}^n\|_\infty. \quad (3.3.4)
\end{aligned}$$

Convergence of the first term in (3.3.4): Observe that

$$\begin{aligned}
\|I - \Gamma_d^{-1}\hat{\Gamma}_{d,n}\|_\infty &= \|\Gamma_d^{-1}(\Gamma_d - \hat{\Gamma}_{d,n})\|_\infty \leq \|\Gamma_d^{-1}\|_\infty\ \|\Gamma_d - \hat{\Gamma}_{d,n}\|_\infty\\
&= \|\Gamma_d^{-1}\|_\infty \max_{i=1,...,d}\sum_{k=1}^d \left|\gamma(i-k) - \hat{\gamma}_{d,n}(i-k) - \frac{\delta_{ik}}{n}\right|\\
&\leq \|\Gamma_d^{-1}\|_\infty\left\{\max_{i=1,...,d}\sum_{k=1}^d |\gamma(i-k) - \hat{\gamma}_{d,n}(i-k)| + \frac{1}{n}\right\}\\
&\leq \|\Gamma_d^{-1}\|_\infty\left\{d\max_{i=0,...,d-1}|\gamma(i) - \hat{\gamma}_{d,n}(i)| + \frac{1}{n}\right\}. \quad (3.3.5)
\end{aligned}$$

From the An, Chen and Hannan (1982) result (Lemma 3.3.4), this tends to 0 with probability 1, if only (recall $d = d_n \le n^{r/(2(r-2))}$)

$$\frac{\|\Gamma_{d_n}^{-1}\|_\infty \, d_n}{r_n} = O(1) \tag{3.3.6}$$

and

$$\frac{\|\Gamma_{d_n}^{-1}\|_\infty}{n} \to 0 \quad (n \to \infty). \tag{3.3.7}$$

Thus, for all $\omega \in \Omega$ from a set of probability 1, for n sufficiently large

$$\|I - \Gamma_d^{-1}\hat{\Gamma}_{d,n}\|_\infty(\omega) < 1$$

and by the second part of Lemma 3.3.2

$$\begin{aligned}\|\hat{\Gamma}_{d,n}^{-1}\|_\infty &\le \frac{\|\Gamma_d^{-1}\|_\infty}{1 - \|I - \Gamma_d^{-1}\hat{\Gamma}_{d,n}\|_\infty},\\ \|\hat{\Gamma}_{d,n}^{-1} - \Gamma_d^{-1}\|_\infty &\le \frac{\|\Gamma_d^{-1}\|_\infty \, \|I - \Gamma_d^{-1}\hat{\Gamma}_{d,n}\|_\infty}{1 - \|I - \Gamma_d^{-1}\hat{\Gamma}_{d,n}\|_\infty}.\end{aligned}$$

Using those inequalities, it follows that

$$\begin{aligned}\|\gamma_d\Gamma_d^{-1} - \hat{\gamma}_{d,n}\hat{\Gamma}_{d,n}^{-1}\|_\infty &\le \|\gamma_d\Gamma_d^{-1} - \gamma_d\hat{\Gamma}_{d,n}^{-1}\|_\infty + \|\gamma_d\hat{\Gamma}_{d,n}^{-1} - \hat{\gamma}_{d,n}\hat{\Gamma}_{d,n}^{-1}\|_\infty\\ &\le \|\gamma_d\|_\infty \, \|\hat{\Gamma}_{d,n}^{-1} - \Gamma_d^{-1}\|_\infty + \|\gamma_d - \hat{\gamma}_{d,n}\|_\infty \, \|\hat{\Gamma}_{d,n}^{-1}\|_\infty\\ &\le \|\gamma_d\|_\infty \, \|\Gamma_d^{-1}\|_\infty \frac{\|I - \Gamma_d^{-1}\hat{\Gamma}_{d,n}\|_\infty}{1 - \|I - \Gamma_d^{-1}\hat{\Gamma}_{d,n}\|_\infty} + \|\Gamma_d^{-1}\|_\infty \frac{\|\gamma_d - \hat{\gamma}_{d,n}\|_\infty}{1 - \|I - \Gamma_d^{-1}\hat{\Gamma}_{d,n}\|_\infty}.\end{aligned}$$

For any second order stationary process, there exists a constant c_γ with

$$0 \le |\gamma(k)| \le c_\gamma < \infty \quad (k \in \mathbb{N}_0),$$

hence $\|\gamma_d\|_\infty \le c_\gamma$ (this can also be seen from (3.2.7)). We set $M_{d,n} := \max_{i=0,\ldots,d} |\gamma(i) - \hat{\gamma}_n(i)|$. Now, appealing to (3.3.5) again and tidying things up, we obtain

$$\|\gamma_d\Gamma_d^{-1} - \hat{\gamma}_{d,n}\hat{\Gamma}_{d,n}^{-1}\|_\infty \le \frac{\|\Gamma_d^{-1}\|_\infty(c_\gamma\|\Gamma_d^{-1}\|_\infty d + 1)M_{d,n} + \|\Gamma_d^{-1}\|_\infty^2 c_\gamma/n}{1 - (\|\Gamma_d^{-1}\|_\infty d M_{d,n} + \|\Gamma_d^{-1}\|_\infty/n)}$$

(for all $\omega \in \Omega$ from a set of probability 1 and for $n \ge N(\omega)$). Finally, according to Lemma 3.3.4, $dL_n\|\gamma_d\Gamma_d^{-1} - \hat{\gamma}_{d,n}\hat{\Gamma}_{d,n}^{-1}\|_\infty \to 0$ with probability 1 if

$$\frac{L_n\|\Gamma_{d_n}^{-1}\|_\infty^2 d_n^2}{r_n} = O(1), \tag{3.3.8}$$

$$\frac{L_n\|\Gamma_{d_n}^{-1}\|_\infty^2 d_n}{n} \to 0. \tag{3.3.9}$$

Convergence of the second term in (3.3.4): $d\|\gamma_d\|_\infty\|\Gamma_d^{-1}\|_\infty\|T_{L_n}\bar{Y}^n_{n-d+1}-\bar{Y}^n_{n-d+1}\|_\infty$ is readily bounded from above by $dc_\gamma\|\Gamma_d^{-1}\|_\infty\|T_{L_n}\bar{Y}^n_{n-d+1}-\bar{Y}^n_{n-d+1}\|_\infty$, so it suffices to ensure that (again $d=d_n$)

$$d\|\Gamma_d^{-1}\|_\infty\|T_{L_n}\bar{Y}^n_{n-d+1}-\bar{Y}^n_{n-d+1}\|_\infty \to 0$$

with probability 1 as $n\to\infty$. To this end, for $\epsilon>0$,

$$\begin{aligned}
&\mathbf{P}\left(d\|\Gamma_d^{-1}\|_\infty\|T_{L_n}\bar{Y}^n_{n-d+1}-\bar{Y}^n_{n-d+1}\|_\infty \ge \epsilon\right)\\
&\quad= \mathbf{P}\left(d\|\Gamma_d^{-1}\|_\infty \max_{i=0,\dots,d-1} |T_{L_n}Y_{n-i}-Y_{n-i}| \ge \epsilon\right)\\
&\quad\le \mathbf{P}\left(\exists i\in\{0,\dots,d-1\} : |Y_{n-i}|\ge L_n\right) = \mathbf{P}\left(\max_{i=0,\dots,d-1}|Y_{n-i}|\ge L_n\right)
\end{aligned}$$

since $|T_{L_n}Y_{n-i}-Y_{n-i}|>0$ implies $|Y_{n-i}|\ge L_n$. But

$$\mathbf{P}\left(\max_{i=0,\dots,d-1}|Y_{n-i}|\ge L_n\right) \le d\mathbf{P}\left(|Y_1|\ge L_n\right) = 2d\left(1-\Phi\left(\frac{L_n}{\sigma}\right)\right).$$

Here Φ denotes the distribution function of the standard normal distribution. Application of the standard bound on Φ,

$$1-\Phi(y) < \frac{1}{\sqrt{2\pi}y}\exp\left(-\frac{1}{2}y^2\right) \quad (y>0),$$

(Feller, 1968, VII.1. Lemma 2) yields

$$\mathbf{P}\left(d\|\Gamma_d^{-1}\|_\infty\|T_{L_n}\bar{Y}^n_{n-d+1}-\bar{Y}^n_{n-d+1}\|_\infty \ge \epsilon\right) \le 2d\frac{\sigma}{\sqrt{2\pi}L_n}\exp\left(-\frac{1}{2\sigma^2}L_n^2\right).$$

From this

$$d\|\gamma_d\|_\infty\|\Gamma_d^{-1}\|_\infty\|T_{L_n}\bar{Y}^n_{n-d+1}-\bar{Y}^n_{n-d+1}\|_\infty \to 0$$

with probability 1, if only

$$\sum_{n=1}^\infty \frac{d_n}{L_n}\exp\left(-\frac{1}{2\sigma^2}L_n^2\right) < \infty. \tag{3.3.10}$$

Putting things together: Apart from $d_n \le n^{r/(2(r-2))}$, the conditions to be fulfilled are (3.3.6) – (3.3.10). From (3.2.3) and (3.2.4), the spectral density f of the process is continuous, which yields $0<\sup_{\lambda\in[-\pi,\pi]} f(\lambda)<\infty$. With the standard bound on the spectrum of Γ_d (Brockwell and Davis, 1991, Prop. 4.5.3), one has

$$\|\Gamma_d^{-1}\|_\infty \ge \rho(\Gamma_d^{-1}) = \frac{1}{\min \operatorname{spr}(\Gamma_d)} \ge \frac{1}{2\pi\sup_{\lambda\in[-\pi,\pi]} f(\lambda)}.$$

Hence (3.3.6) and (3.3.7) are implied by (3.3.8) and (3.3.9), respectively. Rewriting (3.3.8) as

$$L_n \|\Gamma_{d_n}^{-1}\|_\infty^2 d_n^{2(r+1)/r} \frac{(\log n)^{2/r} (\log\log n)^{2(1+\delta)/r}}{n^{1/2}} = O(1),$$

we end up with the following four conditions

$$\begin{aligned}
& d_n \leq n^{r/(2(r-2))}, \\
& L_n \|\Gamma_{d_n}^{-1}\|_\infty^2 d_n^{2(r+1)/r} \\
& \qquad \cdot \frac{(\log n)^{2/r} (\log\log n)^{2(1+\delta)/r}}{n^{1/2}} = O(1), \\
& \frac{L_n \|\Gamma_{d_n}^{-1}\|_\infty^2 d_n}{n} \to 0 \quad (n \to \infty), \\
& \sum_{n=1}^{\infty} \frac{d_n}{L_n} \exp\left(-\frac{1}{2\sigma^2} L_n^2\right) < \infty
\end{aligned}$$

in order to obtain

$$\lim_{n\to\infty} \left| \mathbf{E}[Y_{n+1} | Y_n, ..., Y_{n-d_n+1}] - \hat{E}_{d_n,n} \right| = 0$$

with probability 1. □

Finally, we prove Corollary 3.2.3.

Proof of Corollary 3.2.3. Recently, Serra Capizzano obtained the following result (1999, Theorem 3.2; 2000, Theorem 1.2) on the "worst" rate of decay for the minimal eigenvalue $\mu_d^{\min}$ of Toeplitz matrices

$$T_d(f) := (\gamma(i-j))_{i,j=1,\ldots,d}$$

formed by the coefficients of the Fourier expansion

$$f(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma(j) \exp(-ij\lambda) \quad (\lambda \in [-\pi, \pi])$$

of some real-valued Lebesgue integrable function f. As to the situation of case 3, the following holds:

Lemma 3.3.5. *Let f be a real-valued Lebesgue integrable function with essential infimum m_f. Suppose there exists an interval $(a,b) \subseteq (-\pi,\pi), a < b$, and a number $\delta > 0$ such that $f(\lambda) > \delta$ for almost all $\lambda \in (a,b)$. Then*

$$\mu_d^{\min} \geq K \exp(-cd) + m_f \tag{3.3.11}$$

with some $c > 0$ and some $K > 0$ independent of d.

The constants c and K are related to the measure of the set where f essentially vanishes, not disclosed to the statistician. However, choosing some $0 < s < 1$, one has

$$K \exp(-cd) + m_f \geq \exp(-d^{1/s}) \tag{3.3.12}$$

for sufficiently large d. As already noted, from (3.2.3) and (3.2.4), the spectral density f of the process under consideration is continuous. Thus the requirements of Lemma 3.3.5 are met for $T_d(f) = \Gamma_d$, and Lemma 3.3.3 together with (3.3.11) and (3.3.12) yields

$$\|\Gamma_d^{-1}\|_\infty \leq \frac{d^{1/2}}{\mu_d^{\min}} \leq d^{1/2} \exp(d^{1/s}).$$

It remains to check the conditions (3.2.11) of Theorem 3.2.2 in the case of $q > 4, 0 < s < 1, t \geq 1$,

$$d_n \leq \left(\frac{1}{q} \log n\right)^s \quad \text{and} \quad L_n = (\log n)^t.$$

The first inequality in (3.2.11) is obvious, the second and third follow from

$$\frac{\|\Gamma_{d_n}^{-1}\|_\infty^2}{n^{1/2}} \leq \frac{d_n \exp(2(d_n)^{1/s})}{n^{1/2}} = d_n n^{2/q-1/2}$$

with $2/q - 1/2 < 0$. As to the fourth inequality, observe that $d_n/L_n \to 0$ as $n \to \infty$ and for n sufficiently large $-L_n^2/(2\sigma^2) \leq -2\log n$, hence

$$\exp\left(-\frac{1}{2\sigma^2} L_n^2\right) \leq \frac{1}{n^2}.$$

The lemma for the more restrictive cases 1 and 2 follows similarly, using

$$\mu_d^{\min} \geq 2\pi \inf_{\lambda \in [-\pi,\pi]} f(\lambda) > 0$$

(Brockwell and Davis, 1991, Proposition 4.5.3) for case 1 and

$$\mu_d^{\min} \geq \frac{const.}{d^{p^*}}$$

(Böttcher and Grudsky, 1998, Example 3.1 and Theorem 3.4; Serra Capizzano, 2000, Remark 1.2; less general in Serra, 1998, Theorem 2.3) for case 2. □

3.4 Simulations and examples

We first continue Examples 3.1 and 3.2 from Section 3.2.

Example 3.1 (continued): Consider the AR(1)-processes

$$Y_t - \Phi Y_{t-1} = Z_t \quad Z_t \text{ white noise with variance } \sigma^2$$

with $|\Phi| < 1$ (Brockwell and Davis, 1991, Example 4.4.2). The processes have the spectral densities

$$f(\lambda) = \frac{\sigma^2}{2\pi}(1 - 2\Phi\cos\lambda + \Phi^2)^{-1},$$

bounded away from zero. The autocovariance function is given by

$$\gamma(0) = \frac{\Phi^2 + 1}{(\Phi^2 - 1)^2}\sigma^2 \quad \text{and} \quad \gamma(k) = \left(\frac{2\Phi}{1 + \Phi^2}\right)^k \gamma(0),$$

from which we can calculate the conditional expectation $\mathbf{E}[Y_{n+1}|Y_n, ..., Y_1]$ of the next output given the past using (3.2.9). The latter conditional expectation acts as an approximation of the unknown true autoregression $\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]$.

Figure 3.1 a-b) shows two paths of the process (circles) for different values of σ^2 and Φ together with the corresponding autoregression (grey) and the estimated autoregression (black). The convergence of the estimates towards the true autoregression is clearly visible.

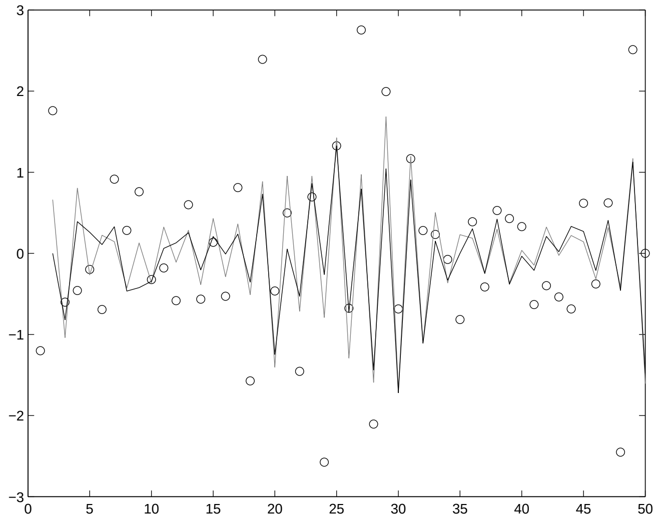
Example 3.2 (continued): The process from Example 3.2 in Section 3.2 has a spectral density

$$f(\lambda) = \frac{\sigma_\epsilon^2}{2\pi}\left(4\sin^2\frac{\lambda}{2}\right)^{1/3}.$$

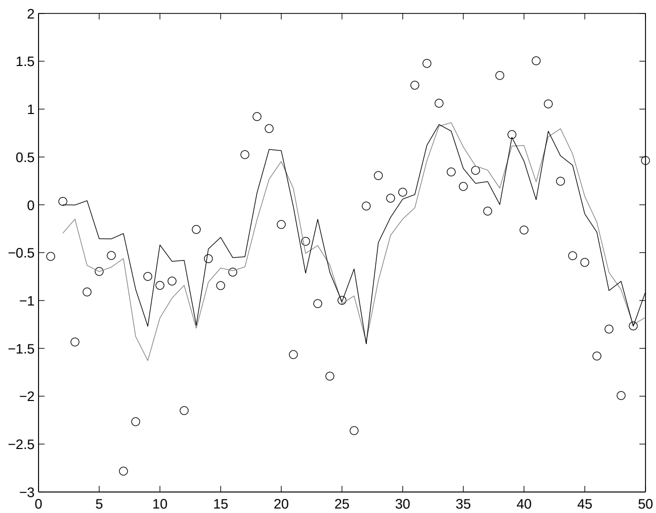
From this we calculate the autocovariances

$$\gamma(k) = \int_0^\pi f(\lambda)\cos(k\lambda)d\lambda$$

(Brockwell and Davis, 1991, eq. 4.3.10) using a compound trapezoidal integration rule with an error of at most 10^{-7}. Figure 3.2 a) shows the autocovariance function of the process for a variance $\sigma_\epsilon^2 = 0.01$ of the innovations. Again, $\mathbf{E}[Y_{n+1}|Y_n, Y_{n-1}, ...]$ is approximated by $\mathbf{E}[Y_{n+1}|Y_n, ..., Y_1]$, which is calculated by (3.2.9). The Gaussian process $\{Y_n\}_n$ itself is simulated by the method described

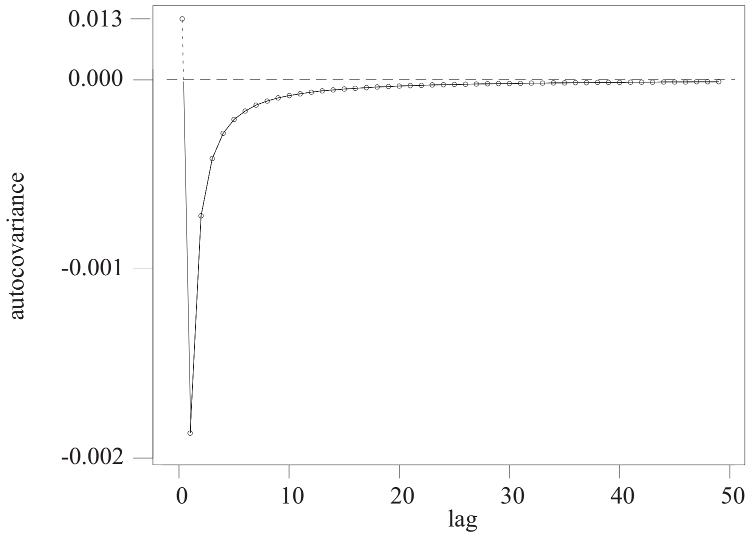


3.1 a) The AR(1) process from Example 3.1 with $\sigma^2 = 1, \Phi = -0.3$.

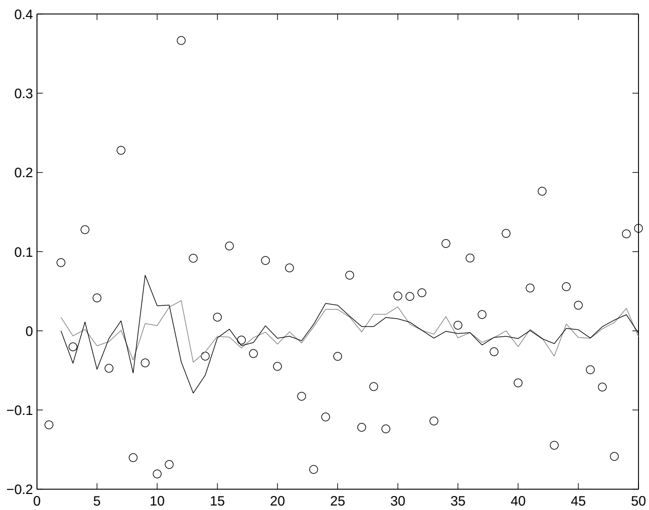


3.1 b) The AR(1) process from Example 3.1 with $\sigma^2 = 1, \Phi = 0.3$.

Figure 3.1: “True” (grey) and predicted (black) autoregression for two Gaussian AR(1) models.



3.2 a) The autocovariance function of the process.



3.2 b) “True” (grey) and predicted (black) autoregression.

Figure 3.2: The autocovariance function and a sample path of the process in Example 3.2.

in Brockwell and Davis (1991, Ex. 8.16, p. 271). Figure 3.2 b) shows the realisations of Y_n (circles), the "true" (grey) and the predicted (black) autoregression for 50 consecutive days. Again, the convergence result is convincing.

The following two examples illustrate the performance of the greedy strategy from Section 3.1.

Example 3.3: We run the greedy strategy from Section 3.1 in a market with a stock whose price follows a geometrical Brownian motion (Luenberger, 1998, Sec. 11.7; Korn and Korn, 1999, Ch. 2) with a mean return of 2% p.a. and a volatility of $\sigma = 10\%$ p.a.. The bond offers a riskless return of 2% p.a. (i.e. we set $r = 2\%/365$). The algorithm of Section 3.2 is used to predict the log-returns of the stocks. Figure 3.3 shows the value of an investment of \$1 either solely in the stock (grey, solid) or in the bond (grey, dashed). The value of the greedy strategy is shown by the solid black line. In times when the share price is likely to plummet the investor takes refuge in the bond. Thus he participates

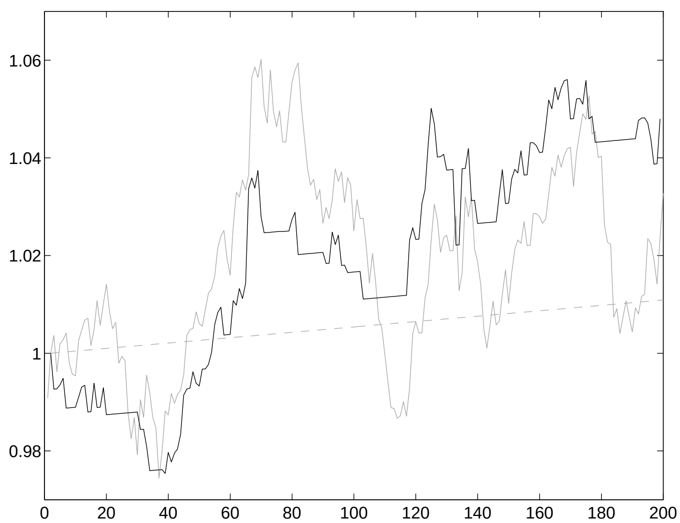
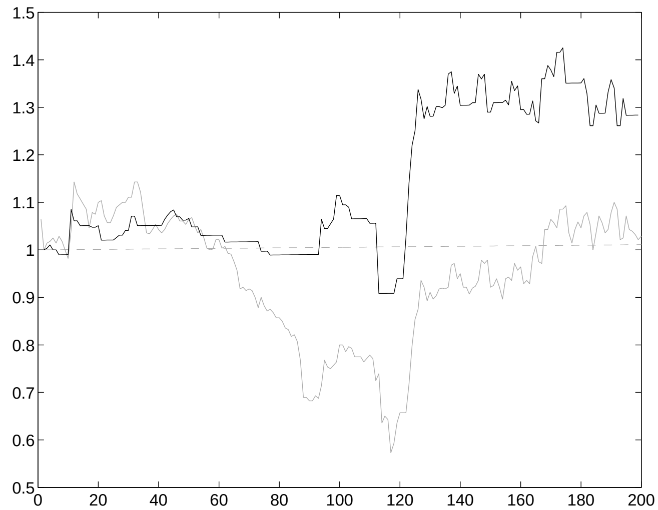
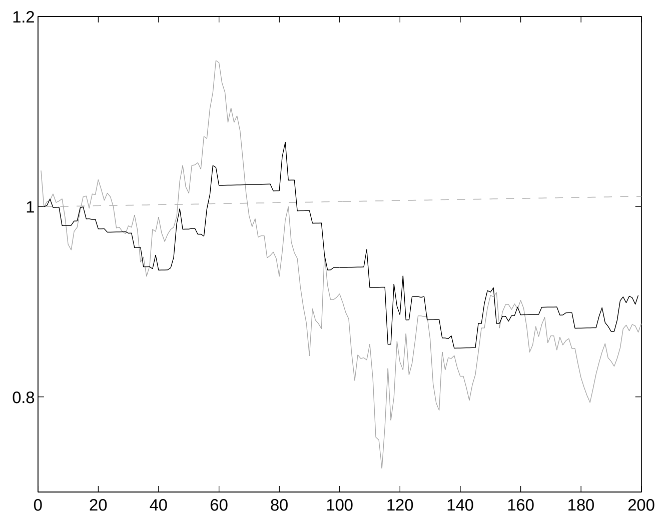


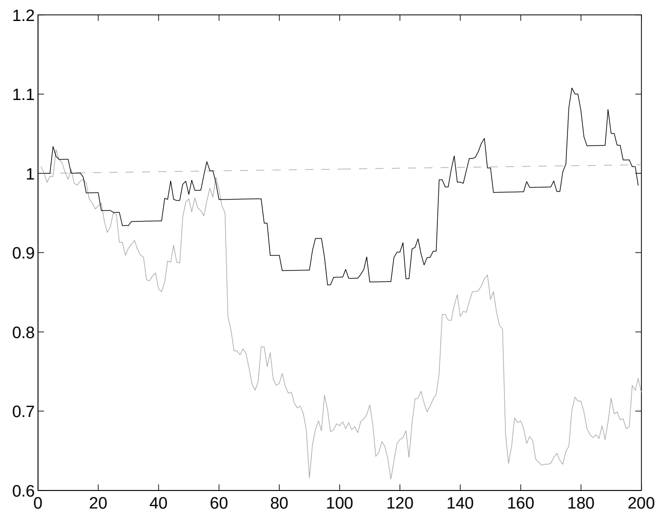
Figure 3.3: Daily value of a \$1 investment in a stock following a geometrical Brownian motion with $\mu = 2\%/365, \sigma = 10\%/\sqrt{365}$ (grey, solid), in a bond with short rate 2% p.a. (grey, dashed) and in a greedy strategy (black).



3.4 a) Yellow Corp. (YELL)



3.4 b) SONY (SNE).



3.4 c) Boeing Co. (BA).

Figure 3.4: Daily value of a \$1 investment in some shares from Dow Jones Indices at NYSE 24/4/1998-8/2/1999 (grey, solid), in a bond with short rate 2% p.a. (grey, dashed) or in a greedy strategy (black), respectively.

in the rise of the share price more than in its fall, increasing his annual yield beyond 2%. As in Section 2.2 this is the phenomenon of volatility pumping (Luenberger, 1998, Examples 15.2 and 15.3). The share's volatility is used to draw an above average return from the stock.

Example 3.4: We replace the geometrical Brownian motion by various real stock price processes on 200 days of trading, using the NYSE closing prices 24/4/1998-8/2/1999 (data from www.wallstreetcity.com). Figure 3.4 a-c) shows the corresponding charts. Although the greedy strategy does not manage to yield a return at least as large as the bond's return in all cases (Fig. 3.4 b), it typically outperformed the stock, considerably reducing the investor's risk of financial loss from pure share investment.

CHAPTER 4

A Markov model with transaction costs: probabilistic view

In Chapter 1 (Theorem 1.3.4) we have seen that investment according to the log-optimal portfolio is optimal in an asymptotic and a non-asymptotic sense. In an m stock market with i.i.d. returns for example, the log-optimal portfolio is a constant $b^* \in S$. This, however, does *not* mean that once the investor allocated his wealth according to b^*, he need not rebalance his portfolio in the following market periods. On the contrary, since the price of each stock evolves in a different way, the proportions of wealth held in the stocks will differ from b^* already after the next market period. Thus, selling and purchasing stocks becomes necessary after basically each market period. In a market where transactions such as selling and buying generate transaction costs, one should therefore adopt a more carefully chosen strategy, combining the task of maximizing the portfolio return with the task of having to pay as little transaction fees as possible. In the setting of a Markov market model, this chapter gives an optimal solution to this problem.

In Section 4.1 we will set up a market model with transaction costs and formulate our investment goals. The chapter is devoted to probabilistic aspects of the model, i.e. the analysis assumes the distribution of the return process to be known. Statistical aspects, in particular what to do if the distribution is not revealed to the investor, will be investigated in Chapter 5. In Section 4.2 we propose an optimal strategy (Theorem 4.2.1) whose optimality is proven in the remainder of the section. Section 4.3 concludes the chapter with results that will be needed in Chapter 5 when dealing with the statistical aspects of the model.

Historically, the necessity of modelling investment problems including transaction fees arose at a time when most research dealt with continuous-time models, and consequently, most of the work on transaction costs concerns the continuous-time case. Not much emphasis was put on statistical discrete-time strategy building. A detailed overview of literature can be found in Davis and Norman (1990), Fleming (1999), Cadenillas (2000) or Bielecki and Pliska (2000). However, many models and strategies involved considerable computational effort, which made it necessary to use approximation techniques (see e.g. Fitzpatrick and Fleming, 1991, and Atkinson et al., 1997). Nowadays, continuous-time models are frequently approximated by time-discrete models (for example in Bielecki, Hernandez and Pliska, 1999, to discretize the continuous-time model in Bielecki and Pliska, 1999). This brought about a renaissance of discrete-time modelling. Up to date surveys of strategy planning under transaction costs in discrete-time markets can be found in Carassus and Jouini (2000) for a Cox-Rubinstein type model, in Blum and Kalai (1999) for optimal constant-rebalanced portfolios and in Bobryk and Stettner (1999) for a market with a bond and one stock having i.i.d. returns.

4.1 Strategies in markets with transaction fees

In this section we set up a market with a bond and several stocks whose returns form a d-stage Markov process. Markets of this kind arise when discretizing markets driven by stochastic differential equations even beyond the famous Black-Scholes model. Consider a market of m stocks and a risk-free bond (which we will think of as a bank account). Again, $X_{i,j}$ denotes the return of the jth stock from time $i-1$ to time i and $r \geq 0$ is the interest rate of the bond. The return process $\{X_i := (X_{i,1}, ..., X_{i,m})^T\}_{i=-d+1}^{\infty}$ is assumed to be a ***d*-stage Markov process with continuous autoregression**, i.e., if we denote the last d observed returns by $\bar{X}_{i+1} = (X_{i-d+1}, ..., X_i)$, the process satisfies the following conditions:

V1: $\{X_i\}_{i=-d+1}^{\infty}$ is a stationary $[A,B]^m$-valued stochastic process on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ ($0 < A < B < \infty$ need not be known),

V2: $\mathbf{E}[h(b, \bar{X}_{i+1})|\mathcal{F}_i] = \mathbf{E}[h(b, \bar{X}_{i+1})|\bar{X}_i] \quad \mathbf{P}-a.s.$,

V3: $\mathbf{E}[h(b, \bar{X}_{i+1})|\bar{X}_i = \bar{x}]$ is a continuous function of $(b, \bar{x}) \in S \times [A,B]^{dm}$, for all continuous functions $h : S \times [A, B]^{dm} \longrightarrow \mathbb{R}$ and all i. Thus, at time $i-1$, the further evolution of the market depends upon a sub-σ-algebra $\sigma(\bar{X}_i)$ of the total information field $\mathcal{F}_i := \sigma(X_{-d+1}, ..., X_{i-1})$.

Following Blum and Kalai (1999) we assume that only *purchasing* shares in the market generates **transaction costs** (brokerage fees, commission) proportional to the total value of the transaction, i.e.

$$\text{transaction costs} = c \cdot \text{value of purchased shares}$$

with a **commission factor** $c \in [0, 1)$. Paying into and drawing money from the risk-free account does not generate any fees. In case two commission factors apply for selling and purchasing shares, say c_{sell} and c_{purchase}, one may use $c := (c_{\text{sell}} + c_{\text{purchase}})/(1 + c_{\text{purchase}})$ as a compound commission factor applying to purchases only. This approach never underestimates the capital reducing effect of transaction costs. Indeed, with 1 unit of money in a stock one can purchase $(1 - c_{\text{sell}})/(1 + c_{\text{purchase}}) = 1 - c$ value in another stock or pay $1 - c_{\text{sell}} \geq 1 - c$ units of money into the account. Conversely, 1 unit of money in the bond can purchase $1/(1 + c_{\text{purchase}}) \geq 1 - c$ value in a stock.

To see how investment actions are limited by transaction fees, consider a fixed time instant i. Then the investor's wealth W_i is used to acquire a new portfolio, given by an enhanced portfolio vector $b_{i+1} := (b_{i+1,-1}, ..., b_{i+1,m})^T$, which now is $(m+2)$-dimensional. Here

$b_{i+1,-1}$ is the proportion of W_i needed to settle the transaction costs that arise when the portfolio is restructured,

$b_{i+1,0}$ is the proportion of W_i to be held in the bond and, as usual,

$b_{i+1,j}$ is the proportion of W_i to be held in stock j $(j = 1, ..., m)$.

No short selling or consumption is considered, i.e. $\sum_{j=-1}^{m} b_{i+1,j} = 1$ and $b_{i+1,j} \geq 0$. Thus, the portfolio vector b_{i+1} chosen at time i becomes a member of the simplex $S := \{b = (b_{-1}, ..., b_m)^T \in \mathbb{R}^{m+2} | b_j \geq 0 \text{ for all } j, \sum_{j=-1}^{m} b_j = 1\}$.

Now, in the market period from time $i-1$ to time i the investor's wealth W_{i-1} generated a value of $(1+r)b_{i,0}W_{i-1}$ in the bond and of $X_{i,j}b_{i,j}W_{i-1}$ in the jth stock. An amount of $b_{i,-1}W_{i-1}$ was used to settle transaction fees and is no longer available. The resulting wealth at time i becomes $W_i = (1+r)b_{i,0}W_{i-1} + \sum_{j=1}^m X_{i,j}b_{i,j}W_{i-1}$, or equivalently

$$\frac{W_i}{W_{i-1}} = (1+r)b_{i,0} + \sum_{j=1}^m X_{i,j}b_{i,j}. \tag{4.1.1}$$

Rebalancing the portfolio b_i to b_{i+1} generates transaction costs of total amount $c\sum_{j=1}^m (b_{i+1,j}W_i - X_{i,j}b_{i,j}W_{i-1})^+$ which are settled using the amount $b_{i+1,-1}W_i$. Hence the investor has to observe the **self-financing condition**

$$b_{i+1,-1}W_i = c\sum_{j=1}^m (b_{i+1,j}W_i - X_{i,j}b_{i,j}W_{i-1})^+. \tag{4.1.2}$$

Using (4.1.1), (4.1.2) is equivalent to

$$\begin{aligned} g_c(b_i, \bar{X}_{i+1}, b_{i+1}) := {} & b_{i+1,-1}\left((1+r)b_{i,0} + \sum_{k=1}^m X_{i,k}b_{i,k}\right) \\ & -c\sum_{j=1}^m \left(b_{i+1,j}\left((1+r)b_{i,0} + \sum_{k=1}^m X_{i,k}b_{i,k}\right) - X_{i,j}b_{i,j}\right)^+ = 0. \end{aligned} \tag{4.1.3}$$

If $\bar{x}$ is the matrix formed from the last d observed return vectors and s denotes the last portfolio vector, we call the collection of all portfolios satisfying the self-financing condition,

$$S(s,\bar{x}) = \{b \in S \mid g_c(s,\bar{x},b) = 0\}, \tag{4.1.4}$$

the **admissible set** corresponding to $(s,\bar{x}) \in S \times [A,B]^{dm}$. Note that for all $(s,\bar{x}) \in S \times [A,B]^{dm}$

$$a^* := (0,1,0,...,0)^T \in S(s,\bar{x}), \tag{4.1.4a}$$

i.e. there is always one option open to the investor: He can pay all his wealth into the risk-free account at any time.

The investor can only follow non-anticipating portfolio strategies which comply with the self-financing condition:

Definition 4.1.1. *A sequence $\{b_i\}_{i=0}^\infty$ of random variables $\Omega \to S$ is called* **admissible portfolio strategy** *if for all $i \in \mathbb{N}$ the following holds:*

1. $b_i \in S(b_{i-1}, \bar{X}_i)$ $\mathbf{P}$-a.s.,

2. b_i *is* $\mathcal{F}_i$*-measurable and*

3. $b_0 = a^*$.

Condition 1 enforces that an admissible strategy never generates more transaction fees than affordable. Because of condition 2, investment decisions require no more information than currently available. Finally, condition 3 provides for a standardized setting in so far as the investor's wealth is accumulated in the bond at the beginning of the investment process.

As we have seen, the pair $(b_{i-1}, \bar{X}_i)$ carries complete information about both, the stochastic regime of the next market period (from the d-stage Markov property) and about the admissible set (from the self-financing condition (4.1.3)). Hence, at time $i-1$, the investment decision may be taken by applying a so-called portfolio selection function $\phi : S \times [A, B]^{dm} \longrightarrow S$ to the pair $(b_{i-1}, \bar{X}_i)$. This approach wastes no information.

Definition 4.1.2. *An admissible portfolio strategy* $\{b_i\}_{i=0}^{\infty}$ *is based on the* **portfolio selection function** $\phi : S \times [A, B]^{dm} \longrightarrow S$ *if* ϕ *is measurable and, for all* i*,*

$$b_i = \phi(b_{i-1}, \bar{X}_i) \quad \mathbf{P} - a.s..$$

We now move on to defining the **investment goal**. From the previous chapters we know that the logarithmic utility function $f : S \times ([A, B]^m)^d \longrightarrow \mathbb{R}$,

$$f(b, \bar{X}_{i+1}) = \log\big((0, 1 + r, X_i^T)b\big)$$

is the optimal choice for long-run and short-term investment targets alike (note that the entry 0 in the first vector of the scalar product corresponds to the amount of transaction costs which is lost). In this chapter we therefore assume that the investor aims to choose an admissible strategy $\{b_i\}_i$ such that, in the long run, the expected mean utility $\mathbf{E}(\frac{1}{n}\sum_{i=0}^{n-1} f(b_i, \bar{X}_{i+1}))$ is larger than for any other strategy based on some portfolio selection function. This is formalized by inequality (4.2.4) below.

It should be pointed out that the process $\{X_i\}_i$ need not contain mere return information, but may contain additional **factors and side information** in some of its coordinates as well. – Provided of course that these occur in the form of a d-stage Markov process with continuous autoregression function and that the joint vector of returns and factors satisfies V1, V2 and V3. The utility function f then simply ignores the coordinates containing the factors and side information.

4.2 An optimal strategy

For the rest of this chapter, assume that V1, V2 and V3 hold and that a $(0,1)$-valued sequence of *discount factors* $\{\delta_i\}_{i=1}^{\infty}$ is fixed for which

$$\begin{aligned}
&\delta_i \searrow 0 \quad \text{monotonic decreasing as} \quad i \to \infty, \\
&i\delta_i \longrightarrow \infty \quad \text{as} \quad i \to \infty, \\
&\delta_i \quad \text{piecewise constant:} \quad \delta_1 = \delta_2, \delta_3 = \delta_4 = \delta_5, \delta_6 = \ldots = \delta_9, \ldots, \\
&(\delta_i - \delta_{i+1})/\delta_{i+1}^2 \leq 1 \quad \text{for all} \quad i \geq 1.
\end{aligned} \tag{4.2.1}$$

Then the following theorem gives an algorithm for optimal investment based on the so-called **Bellman equation**. Using stationarity, let

$$m(b, \bar{x}) := \mathbf{E}[f(b, \bar{X}_{i+1})|\bar{X}_i = \bar{x}] = \mathbf{E}[f(b, \bar{X}_{d+1})|\bar{X}_d = \bar{x}].$$

Theorem 4.2.1. *Let $h_i \in C(S \times [A,B]^{dm})$ be a solution of the Bellman equation*

$$h_i(s, \bar{x}) = \max_{b \in S(s,\bar{x})} \left\{ m(b, \bar{x}) + (1 - \delta_i)\mathbf{E}[h_i(b, \bar{X}_{d+1})|\bar{X}_d = \bar{x}] \right\}. \tag{4.2.2}$$

With $V_i(b, \bar{x}) := m(b, \bar{x}) + (1 - \delta_i)\mathbf{E}[h_i(b, \bar{X}_{d+1})|\bar{X}_d = \bar{x}]$ we obtain an admissible portfolio strategy by

$$\begin{aligned}
b_0^* &:= a^* \\
b_i^* &:= \arg \max_{b \in S(b_{i-1}^*, \bar{X}_i)} V_i(b, \bar{X}_i).
\end{aligned} \tag{4.2.3}$$

This strategy is optimal in the sense that for any portfolio strategy $\{b_i\}_i$ based on a portfolio selection function

$$\liminf_{n \to \infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \mathbf{E} f(b_i^*, \bar{X}_{i+1}) - \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{E} f(b_i, \bar{X}_{i+1}) \right) \geq 0. \tag{4.2.4}$$

Before proving the theorem we make some remarks:

1. (4.2.4) compares the mean utility of the investment schemes $\{b_i^*\}_i$ and $\{b_i\}_i$ in the worst case (lim inf) that may occur for remote time horizon. Hence, the result of Theorem 4.2.1 is a worst case analysis. The Bellman equation (4.2.2) penalizes the "target" function m to obtain a target that is adjusted to loss through transaction costs. It thus balances the need to make as many transactions as needed, but, at the same time, as few as possible.

2. The strategy b_i^* is a generalization of the log-optimal strategy in Chapter 1. If no transaction costs occur, then $c = 0$ and $S(s, \bar{x}) = \{(0, b_0, ..., b_m) | b_j \geq 0, \sum_j b_j = 1\}$ independent of s. Hence, $h_i(s, \bar{x}) = h_i(\bar{x})$, and $b_i^* = \arg\max_{b \in S(b_{i-1}^*, \bar{X}_i)} m(b, \bar{X}_i)$ coincides with the classical log-optimal strategy.

3. In dynamic programming, a solution h_i of the Bellman equation (4.2.2) is frequently referred to as **value function**. The existence of a value function for $\delta_i = 0$ can be obtained under more restrictive extra conditions such as a finite state space of the Markov chain and certain recurrence properties of the transition matrix (Ross, 1970, Sec. 6.7 and Bertsekas, 1976, Sec. 8.1). To avoid these extra coditions we use a variant of the so-called *vanishing discount approach* (Hernández-Lerma and Lasserre, 1996, Sec. 5.3 - 5.5) where solutions of the Bellman equation (4.2.2) are produced for a sequence of discount factors $\delta_i \to 0$.

4. A sequence $\{\delta_i\}_i$ satisfying the above conditions can be obtained recursively by

$$d_1 \in (0, 1) \quad \text{arbitrary}, \quad d_{k+1} := \frac{1}{2}\left(\sqrt{1 + 4d_k} - 1\right)$$

and

$$\begin{aligned} &\delta_1 = \delta_2 := d_1 \\ &\delta_3 = \delta_4 = \delta_5 := d_2 \\ &\delta_6 = \delta_7 = \delta_8 = \delta_9 := d_3 \\ &\delta_{10} = ... \end{aligned}$$

Note that d_k coverges monotonically decreasing to 0, that $k(k+1)d_k \to \infty$ $(k \to \infty)$ and that $(d_k - d_{k+1})/d_{k+1}^2 = 1$.

The Bellman equation is closely linked with the theory of Markov control processes and stochastic dynamic programming (SDP). These have been applied to financial mathematics since the 1960s, e.g. by Samuelson (1969), Merton (1969) and Bertsekas (1976, Sec. 3.3). A good introduction to SDP and Markov control can be found in Bertsekas (1976), Bertsekas and Shreve (1978), in Hernández-Lerma and Lasserre (1996 and 1999) or in Bather (2000). Among recent applications to discrete-time finance we only mention Hernández-Lerma and Lasserre (1996, Example 1.3.2) and Duffie (1988, Sec. III.19) – both contain more references. It should be pointed out, however, that none of the classical models can properly deal with the transaction cost problem we are dealing with. Before giving details of the proof of Theorem 4.2.1, we should briefly comment on that.

4.2.1 Some comments on Markov control

In the terminology of Hernández-Lerma and Lasserre (1996 and 1999), a discrete-time **Markov-control process** is a five-tuple $(X, A, \{A(x)|x \in X\}, Q, c)$, with state space X and a set A of control actions. $\{A(x)|x \in X\}$ is a class of nonempty sets $A(x) \subseteq A$, where $A(x)$ contains the admissible control actions in state x. x_t denotes the state of the system at time t. Q is the transition probability distribution $Q(dy|x,a) = \mathbf{P}_{x_{t+1}|x_t=x,a_t=a}(dy)$, i.e. the distribution of x_{t+1} given the system is in state $x_t = x$ at time t and control action $a_t = a$ is taken. Finally, c is the one step cost function, $c(x,a)$ being the cost incurred when choosing control action a in state x.

One then seeks for a sequential choice of control actions $a_i \in A(x_i)$ such as to minimize

$$\limsup_{n\to\infty} \mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1} c(x_i, a_i)$$

or maximize

$$\liminf_{n\to\infty} \mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1}(-c)(x_i, a_i)$$

($-c$ thus becomes the utility function). Optimal strategies can be generated from solutions (ρ^*, h) of the Bellman equation

$$\rho^* + h(x) = \min_{a\in A(x)} \left\{ c(x,a) + \int_X h(y)Q(dy|x,a) \right\}$$

with $\rho^* \in \mathbb{R}$ and continuous function h. In order to solve this equation, either appropriate boundedness and continuity properties of the solutions of the *discounted* Bellman equation

$$\rho^* + h(x) = \min_{a \in A(x)} \left\{ c(x,a) + \delta \int_X h(y) Q(dy|x,a) \right\}$$

$(0 < \delta < 1)$ are needed (Hernández-Lerma and Lasserre, 1996, Theorems 5.4.3 and 5.5.4) or recurrence and irreducibility conditions for the Markov chain with the transition probability distribution Q (Ross, 1970, Cor. 6.20 or Bertsekas, 1976, Prop. 3). Typically, most research assumes the state space X to be countable and the additional existence of a state x^* with $Q(x^*|x,a) \geq const. > 0$ for all $x \in X, a \in A(x)$. Such conditions are hard to verify from market data and –what is even worse– they are *not* satisfied for the control problem of portfolio selection under transaction costs. Indeed, we have seen that the collection of admissible actions (portfolio choices b_i) at time $i-1$ depends on the last d observed return vectors as well as on the last chosen control b_{i-1}. Therefore, we have no other choice than to describe the state of the system by the joint vector $(b_{i-1}, \bar{X}_i)$. Then the transition dynamics under control b_i is given by $(b_{i-1}, \bar{X}_i) \longmapsto (b_i, \bar{X}_{i+1})$. We thus end up with a transition probability distribution Q that clearly does not satisfy the mentioned recurrence and irreducibility conditions. This drawback was also noted by Bielecki, Hernández and Pliska (2000). They observe that the typical conditions imposed to ensure the existence of a solution of the Bellman equation are to rigid to be applicable in transaction cost problems. On the other hand they found it still possible to characterize optimal strategies in terms of optimality equations which correspond to the classical Bellman equation. This is supported by the earlier findings of Stettner (1999) and our results.

4.2.2 Proof of Theorem 4.2.1

In this section we prove the main result, Theorem 4.2.1. The proof requires several steps. We first have to show the existence of a solution of the Bellman equation and investigate certain properties of the solution. We then have to show that the strategy b_i^* calculated by (4.2.3) consists of portfolio choices that are admissible. In a third step we will derive a technical tool to approximate admissible strategies based on a portfolio selection function by simpler periodic strategies (Lemma 4.2.6). It is only in the fourth step that, using the technical tools derived before, we will completely prove Theorem 4.2.1.

1st step in the proof of Theorem 4.2.1: Solving the Bellman equation. In dynamic programming, the Bellman equation usually takes the form

$$\lambda + h(s,\bar{x}) = \max_{b \in S(s,\bar{x})} \left\{ m(b,\bar{x}) + (1-\delta)\mathbf{E}[h(b,\bar{X}_{d+1})|\bar{X}_d = \bar{x}] \right\}, \tag{4.2.5}$$

which is to be solved for $h \in C(S \times [A,B]^{dm})$ and $\lambda \in \mathbb{R}$ (see e.g. Bertsekas, 1976, Sec. 8.1 or Hernández-Lerma and Lasserre, 1996, Sec. 5.2). We first derive some basic facts about the existence of solutions and their properties. The following proposition is well known from the theory of dynamic programming, we nonetheless outline a proof.

Proposition 4.2.2. *For all $\delta \in (0,1)$ there exists a solution $(h,\lambda) \in C(S \times [A,B]^{dm}) \times \mathbb{R}$ and a solution $(h,0) \in C(S \times [A,B]^{dm}) \times \{0\}$ of the Bellman equation (4.2.5).*

Proof. In the following let $g \in C(S \times [A,B]^{dm})$, $\lambda \in \mathbb{R}$. The seminorm

$$\|g\| := \max_{(s,\bar{x}) \in S \times [A,B]^{dm}} g(s,\bar{x}) - \min_{(s,\bar{x}) \in S \times [A,B]^{dm}} g(s,\bar{x})$$

on $C(S \times [A,B]^{dm})$ makes the factor space

$$C^* := C(S \times [A,B]^{dm}) \big/ \{\text{constant functions}\}$$

a Banach space with norm $\|[g]\| := \|g\|$, where $[g] := \{g(\cdot,\cdot) + r | r \in \mathbb{R}\}$ denotes the equivalence class of $g \in C(S \times [A,B]^{dm})$.

Indeed, $K := \{\text{constant functions}\}$ is a closed subspace of the space $(C(S \times [A,B]^{dm}, \|\cdot\|_\infty)$. Then $(C(S \times [A,B]^{dm}) \big/ K, \|\cdot\|^*)$ with

$$\begin{aligned} \|\cdot\|^* : \quad & C(S \times [A,B]^{dm}) \big/ K \longrightarrow \mathbb{R}_0^+ : \\ & [f] \longmapsto \|[f]\|^* := \inf_{k \in K} \|f + k\|_\infty = \inf_{c \in \mathbb{R}} \|f + c\|_\infty \end{aligned}$$

is a Banach space (Hirzebruch and Scharlau, 1996, Lemma 5.10). The norms $\|\cdot\|^*$ and $\|\cdot\|$ are equivalent on $C(S \times [A,B]^{dm}) \big/ K$ because of $\inf_{c \in \mathbb{R}} \|f + c\|_\infty = \frac{1}{2}\|f\|$.

Note that, V3 implies that for any $g \in C(S \times [A,B]^{dm})$, the conditional expectation $\mathbf{E}[g(b,\bar{X}_{d+1})|\bar{X}_d = \bar{x}]$ is continuous in $(b,\bar{x}) \in S \times [A,B]^{dm}$, and $S(\cdot,\cdot)$ is continuous in the sense of Aliprantis and Border (1999, Definition 16.2 and Theorems 16.20, 16.21). Now, by Berge's Maximum Theorem (Aliprantis and Border, 1999, Theorem 16.31), $\max_{b\in S(s,\bar{x})}\{m(b,\bar{x}) + (1-\delta)\mathbf{E}[g(b,\bar{X}_{d+1})|\bar{X}_d = \bar{x}]\}$ is continuous on $S \times [A,B]^{dm}$. Hence we can define the operator

$$\begin{aligned} M: \quad & C(S\times[A,B]^{dm}) \longrightarrow C(S\times[A,B]^{dm}): \\ & g(s,\bar{x}) \longmapsto \max_{b\in S(s,\bar{x})} \left\{m(b,\bar{x}) + (1-\delta)\mathbf{E}[g(b,\bar{X}_{d+1})|\bar{X}_d = \bar{x}]\right\}. \end{aligned}$$

On C^*, M operates according to $M[g] := [Mg]$, observe that the right hand side is independent of the chosen representative of $[g]$. Solving (4.2.5) thus becomes equivalent to solving the functional equation

$$M[h] = [h] \tag{4.2.6}$$

in C^*. This can be accomplished by an application of Banach's Fixed Point Theorem. According to the Banach Fixed Point Theorem (Aliprantis und Border, 1999, Theorem 3.36), (4.2.6) can be solved using the iteration $[h]_{n+1} := M[h]_n$ (*value iteration*), if only M is a contraction mapping. This will be shown in the following, using standard techniques.

For functions $g, h \in C(S\times[A,B]^{dm})$ we have

$$\begin{aligned} (Mh)(s,\bar{x}) &= \max_{b\in S(s,\bar{x})} \left\{m(b,\bar{x}) + (1-\delta)\mathbf{E}[h(b,\bar{X}_{d+1})|\bar{X}_d = \bar{x}]\right\} \\ &= m(b^*,\bar{x}) + (1-\delta)\mathbf{E}[h(b^*,\bar{X}_{d+1})|\bar{X}_d = \bar{x}], \\ (Mg)(s,\bar{x}) &= \max_{b\in S(s,\bar{x})} \left\{m(b,\bar{x}) + (1-\delta)\mathbf{E}[g(b,\bar{X}_{d+1})|\bar{X}_d = \bar{x}]\right\} \\ &\geq m(b^*,\bar{x}) + (1-\delta)\mathbf{E}[g(b^*,\bar{X}_{d+1})|\bar{X}_d = \bar{x}] \end{aligned}$$

for some $b^* \in S(s,\bar{x})$. Hence, writing $\max_{(s,\bar{x})}$ instead of $\max_{(s,\bar{x})\in S\times[A,B]^{dm}}$,

$$\begin{aligned} (Mh)(s,\bar{x}) - (Mg)(s,\bar{x}) &\leq (1-\delta)\mathbf{E}[h(b^*,\bar{X}_{d+1}) - g(b^*,\bar{X}_{d+1})|\bar{X}_d = \bar{x}] \\ &\leq (1-\delta)\max_{(s,\bar{x})}(h(s,\bar{x}) - g(s,\bar{x})) \end{aligned}$$

for all $(s,\bar{x}) \in S\times[A,B]^{dm}$, which yields

$$\max_{(s,\bar{x})}((Mh)(s,\bar{x}) - (Mg)(s,\bar{x})) \leq (1-\delta)\max_{(s,\bar{x})}(h(s,\bar{x}) - g(s,\bar{x})).$$

From this we find that

$$\|Mh - Mg\| = \max_{(s,\bar{x})}((Mh)(s,\bar{x}) - (Mg)(s,\bar{x}))$$

$$\begin{aligned}
&\quad - \min_{(s,\bar{x})} ((Mh)(s,\bar{x}) - (Mg)(s,\bar{x})) \\
&= \max_{(s,\bar{x})} ((Mh)(s,\bar{x}) - (Mg)(s,\bar{x})) \\
&\quad + \max_{(s,\bar{x})} ((Mg)(s,\bar{x}) - (Mh)(s,\bar{x})) \\
&\leq (1-\delta) \max_{(s,\bar{x})} (h(s,\bar{x}) - g(s,\bar{x})) \\
&\quad + (1-\delta) \max_{(s,\bar{x})} (g(s,\bar{x}) - h(s,\bar{x})) \\
&= (1-\delta)\|g-h\|
\end{aligned}$$

for $0 < 1-\delta < 1$, a contraction property that implies the existence of a solution of (4.2.5).

Finally, let $(h,\lambda) \in C(S \times [A,B]^{dm}) \times \mathbb{R}$ be an arbitrary solution of (4.2.5),

$$\lambda + h(s,\bar{x}) = \max_{b \in S(s,\bar{x})} \left\{ m(b,\bar{x}) + (1-\delta)\mathbf{E}[h(b,\bar{X}_{d+1})|\bar{X}_d = \bar{x}] \right\}.$$

This is equivalent to ($c \in \mathbb{R}$ arbitrary)

$$(\lambda - \delta c) + (h+c)(s,\bar{x}) = \max_{b \in S(s,\bar{x})} \left\{ m(b,\bar{x}) + (1-\delta)\mathbf{E}[(h+c)(b,\bar{X}_{d+1})|\bar{X}_d = \bar{x}] \right\}.$$

In particular, choosing $c := \lambda/\delta$, for $\tilde{h} := h + c$, we obtain the relation

$$\tilde{h}(s,\bar{x}) = (\lambda - \delta c) + \tilde{h}(s,\bar{x}) = \max_{b \in S(s,\bar{x})} \left\{ m(b,\bar{x}) + (1-\delta)\mathbf{E}[\tilde{h}(b,\bar{X}_{d+1})|\bar{X}_d = \bar{x}] \right\},$$

and $(\tilde{h}, 0)$ also solves the Bellman equation. □

The next lemma gives some technical properties of arbitrary solutions $(h, 0)$ of the Bellman equation.

Lemma 4.2.3. *Let $\{\delta_i\}_i$ be a monotonic decreasing sequence with $\delta_i \in (0,1)$ and $(\delta_i - \delta_{i+1})/\delta_{i+1}^2 \leq 1$. If $(h_i, 0)$ is a solution of the Bellman equation (4.2.5, $\delta = \delta_i$), then*

1. $\delta_i \|h_i\|_\infty \leq \|f\|_\infty$.
2. $\|h_{i+1} - h_i\|_\infty \leq \|f\|_\infty$.
3. *Along any admissible portfolio sequence $\{b_i\}_i$ we have*

$$\mathbf{E}h_i(b_{i-1}, \bar{X}_i) - \mathbf{E}h_i(b_i, \bar{X}_{i+1}) \geq -2\|f\|_\infty. \tag{4.2.7}$$

In particular, for all j,

$$\begin{aligned}&\mathbf{E}h_i(b_{i-1},\bar{X}_i)-\mathbf{E}h_i(a^*,\bar{X}_j)\\&\qquad=\mathbf{E}h_i(b_{i-1},\bar{X}_i)-\mathbf{E}h_i(a^*,\bar{X}_{i+1})\geq -2\|f\|_\infty. \quad (4.2.8)\end{aligned}$$

Proof. 1. The Bellman equation (4.2.5) with $\lambda=0$ implies that $\|f\|_\infty+(1-\delta_i)\|h_i\|_\infty\geq\|h_i\|_\infty$, which can be rewritten as $\delta_i\|h_i\|_\infty\leq\|f\|_\infty$.

2. Similarly as in the proof of Proposition 4.2.2,

$$\begin{aligned}h_i(s,\bar{x}) &= \max_{b\in S(s,\bar{x})}\left\{m(b,\bar{x})+(1-\delta_i)\mathbf{E}[h_i(b,\bar{X}_{d+1})|\bar{X}_d=\bar{x}]\right\}\\&= m(b^*,\bar{x})+(1-\delta_i)\mathbf{E}[h_i(b^*,\bar{X}_{d+1})|\bar{X}_d=\bar{x}],\\h_j(s,\bar{x}) &= \max_{b\in S(s,\bar{x})}\left\{m(b,\bar{x})+(1-\delta_j)\mathbf{E}[h_j(b,\bar{X}_{d+1})|\bar{X}_d=\bar{x}]\right\}\\&\geq m(b^*,\bar{x})+(1-\delta_j)\mathbf{E}[h_j(b^*,\bar{X}_{d+1})|\bar{X}_d=\bar{x}]\end{aligned}$$

for some $b^*\in S(s,\bar{x})$. Taking differences yields

$$\begin{aligned}&h_i(s,\bar{x})-h_j(s,\bar{x})\\&\quad\leq(1-\delta_i)\mathbf{E}[h_i(b^*,\bar{X}_{d+1})|\bar{X}_d=\bar{x}]-(1-\delta_j)\mathbf{E}[h_j(b^*,\bar{X}_{d+1})|\bar{X}_d=\bar{x}]\\&\quad\leq\max_{(s,\bar{x})\in S\times[A,B]^{dm}}\{(1-\delta_i)h_i(s,\bar{x})-(1-\delta_j)h_j(s,\bar{x})\}\\&\quad\leq(1-\delta_i)\|h_i-h_j\|_\infty+|\delta_j-\delta_i|\|h_j\|_\infty\\&\quad\leq\max\{1-\delta_i,1-\delta_j\}\|h_i-h_j\|_\infty+|\delta_j-\delta_i|\max\{\|h_i\|_\infty,\|h_j\|_\infty\}.\end{aligned}$$

The right hand side of this chain of inequalities remains the same when swapping i and j, therefore

$$\|h_i-h_j\|_\infty\leq\max\{1-\delta_i,1-\delta_j\}\|h_i-h_j\|_\infty+|\delta_j-\delta_i|\max\{\|h_i\|_\infty,\|h_j\|_\infty\}.$$

Using part 1 of the lemma, we conclude that

$$\|h_i-h_j\|_\infty \;\leq\; \frac{|\delta_j-\delta_i|\cdot\max\{\|h_i\|_\infty,\|h_j\|_\infty\}}{1-\max\{1-\delta_i,1-\delta_j\}}\leq\frac{|\delta_j-\delta_i|}{\min\{\delta_i,\delta_j\}^2}\|f\|_\infty.$$

Monotonicity of δ_i and the assumption $(\delta_i-\delta_{i+1})/\delta_{i+1}^2\leq 1$ allows us to infer

$$\|h_i-h_{i+1}\|_\infty\leq\frac{\delta_i-\delta_{i+1}}{\delta_{i+1}^2}\|f\|_\infty\leq\|f\|_\infty.$$

3. The relation (4.2.8) is a direct consequence of (4.2.7) because of the stationarity of $\{X_i\}_i$ and $a^* \in S(b_{i-1}, \bar{X}_i)$ being deterministic. To prove (4.2.7) observe that the Bellman equation implies

$$h_i(b_{i-1}, \bar{X}_i) \geq m(b_i, \bar{X}_i) + (1 - \delta_i)\mathbf{E}[h_i(b_i, \bar{X}_{i+1})|\bar{X}_i],$$

and

$$\mathbf{E}h_i(b_{i-1}, \bar{X}_i) - \mathbf{E}h_i(b_i, \bar{X}_{i+1}) \geq \mathbf{E}m(b_i, \bar{X}_i) - \delta_i \mathbf{E}h_i(b_i, \bar{X}_{i+1}) \geq -\|f\|_\infty - \delta_i \|h_i\|_\infty.$$

Plugging in the result from part 1 of the lemma yields

$$\mathbf{E}h_i(b_{i-1}, \bar{X}_i) - \mathbf{E}h_i(b_i, \bar{X}_{i+1}) \geq -2\|f\|_\infty,$$

and the proof is finished. □

2nd step in the proof of Theorem 4.2.1: Admissibility of $\{b_i^*\}_i$.
We will show that the maximization problem (4.2.3) is solved by a measurable solution procedure $b_i^* = \phi_i(b_{i-1}^*, \bar{X}_i)$ with suitable portfolio selection functions ϕ_i. Thus $\{b_i^*\}_i$ becomes an admissible strategy. The argument involves some notions from set-valued analysis.

The admissible set $S(s, \bar{x})$ for $(s, \bar{x}) \in S \times [A, B]^{dm}$ is a member of the family $\mathcal{C} = \mathcal{C}(S)$ of closed subsets of S. $\mathcal{C}$ is equipped with the σ-algebra generated by families of the form $\mathcal{F}_K := \{A \in \mathcal{C}(S) | A \cap K \neq \emptyset\}$ (K ranging over all compact $K \subseteq S$) (Matheron, 1975, p. 27, or Molchanov, 1993, Chapter 1). At time $i-1$, an admissible strategy $\{b_i\}_i$ picks some element of the random set $S_i := S(b_{i-1}, \bar{X}_i) = \{b \in S | g_c(b_{i-1}, \bar{X}_i, b) = 0\} \in \mathcal{C}$. S_i is a so-called *random closed set (RACS)*, a measurable mapping $C : \Omega \to \mathcal{C}$. b_i itself is a *selector*, a random variable $\Omega \to S$ such that $b_i \in S_i$ with probability 1. A short introduction to RACS and selectors can be found in Hernández-Lerma and Lasserre (1996, Appendix D) and Bertsekas and Shreve (1978, Sec. 7.5).

S_i is compact. Thus the solution of the maximization problem (4.2.3) is the random non-empty set

$$\arg \max_{b \in S(b_{i-1}, \bar{X}_i)} V_i(b, \bar{X}_i) := \left\{ b \in S(b_{i-1}, \bar{X}_i) \;\middle|\; V_i(b, \bar{X}_i) = \sup_{c \in S(b_{i-1}, \bar{X}_i)} V_i(c, \bar{X}_i) \right\}.$$

We need only show that this is a RACS for which a suitable selector exists:

Lemma 4.2.4. *The mapping*

$$\Omega \to \mathcal{C} : \omega \longmapsto \arg \max_{b \in S(b_{i-1}(\omega), \bar{X}_i(\omega))} V_i(b, \bar{X}_i(\omega))$$

is a RACS for which a selector of the form $\phi_i(b_{i-1}, \bar{X}_i)$ exists, with a measurable function

$$\phi_i : S \times [A, B]^{dm} \longrightarrow S.$$

From Lemma 4.2.4 it follows that the recurrence relation (4.2.3) is solved by

$$b_0^* \;:=\; a^*, \quad b_i^* := \phi_i(b_{i-1}^*, \bar{X}_i).$$

In particular, b_i^* is a selector for $\arg\max_{b \in S(b_{i-1}, \bar{X}_i)} V_i(b, \bar{X}_i)$, hence $\mathcal{F}_i$-measurable, so that $\{b_i^*\}_i$ constitutes an admissible strategy.

Proof of Lemma 4.2.4. The proof will be given in three steps. Fix some $i \in \mathbb{N}$.

First note that $S(\cdot, \cdot) : S \times [A, B]^{dm} \to \mathcal{C} : (s, \bar{x}) \mapsto S(s, \bar{x})$ is a measurable mapping, so that $S(b_i, \bar{X}_{i+1})$ is a RACS: The continuity of g_c implies that $\{b \in S | g_c(s, \bar{x}, b) = 0\}$ is a closed subset of S. For the measurability of $S(\cdot, \cdot)$ we need only show that for any compact $K \subseteq S$

$$S^{-1}(\mathcal{F}_K) \in \mathcal{B}(S \times [A, B]^{dm}),$$

with the σ-algebra $\mathcal{B}(S \times [A, B]^{dm})$ of Borelian sets on $S \times [A, B]^{dm}$. To this end, choose a countable dense subset K' of K. Using the continuity of g_c again, it is easily verified that

$$S^{-1}(\mathcal{F}_K) = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in K'} \left\{ (s, \bar{x}) \,\middle|\, |g_c(s, \bar{x}, k)| < \frac{1}{n} \right\},$$

which implies the measurability of $S^{-1}(\mathcal{F}_K)$.

Secondly, we will see that $\alpha : \mathcal{C} \times [A, B]^{dm} \to \mathcal{C} : (C, \bar{x}) \mapsto \arg\max_{a \in C} V_i(a, \bar{x})$ is measurable. This combined with $S(b_i, \bar{X}_{i+1})$ being a RACS yields that

$$\arg \max_{b \in S(b_{i-1}, \bar{X}_i)} V_i(b, \bar{X}_i)$$

is itself a RACS. As to the measurability of α we consider $C \in \mathcal{C}$ and a compact subset K of S. With

$$\alpha^{-1}(\mathcal{F}_K) = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in K'} \left\{ (C, \bar{x}) \,\middle|\, \sup_{y \in C} V_i(y, \bar{x}) \le V_i(k, \bar{x}) + \frac{1}{n} \right\},$$

it suffices to verify that each of the sets

$$\left\{ (C, \bar{x}) \,\middle|\, \sup_{y \in C} V_i(y, \bar{x}) \le c \right\} \quad (c \in \mathbb{R})$$

is measurable. Indeed, if S' is a countable dense subset of S, then

$$\begin{aligned} &\left\{ (C, \bar{x}) \,\middle|\, \sup_{y \in C} V_i(y, \bar{x}) > c \right\} = \{(C, \bar{x}) | \exists y \in C \cap S' \; V_i(y, \bar{x}) > c\} \\ &\quad = \bigcup_{y \in S'} \{(C, \bar{x}) | y \in C, V_i(y, \bar{x}) > c\} = \bigcup_{y \in S'} (\{C | y \in C\} \times \{\bar{x} | V_i(y, \bar{x}) > c\}) \\ &\quad = \bigcup_{y \in S'} \left(\mathcal{F}_{\{y\}} \times V_i(y, \cdot)^{-1}(c, \infty) \right), \end{aligned}$$

and the desired measurability follows.

Thirdly, we apply Theorem 7.33 in Bertsekas and Shreve (1978) (or Aliprantis and Border, 1999, Theorem 17.18). From there, for the closed set $D := \{(s, \bar{x}, b) | b \in S(s, \bar{x})\}$, we can find a measurable function $\phi_i : \{(s, \bar{x}) | \exists b : (s, \bar{x}, b) \in D\} \to S$ with

$$V_i(\phi_i(s, \bar{x}), \bar{x}) = \max_{b \in \{b | (s, \bar{x}, b) \in D\}} V_i(b, \bar{x}).$$

Now, the proof is finished observing that $\{b | (s, \bar{x}, b) \in D\} = S(s, \bar{x})$, $\{(s, \bar{x}) \,|\, \exists b : (s, \bar{x}, b) \in D\} = S \times [A, B]^{dm}$ (because of $a^* \in S(s, \bar{x})$). Thus

$$\phi_i(b_{i-1}, \bar{X}_i) \in \arg \max_{b \in S(b_{i-1}, \bar{X}_i)} V_i(b, \bar{X}_i)$$

with probability 1. $\square$

3rd step in the proof of Theorem 4.2.1: Approximation of strategies that are based on portfolio selection functions.
For the analysis of the strategy b_i^* it will be convenient to approximate strategies based on portfolio selection functions by members of a smaller class of strategies which we call "periodic" strategies:

Definition 4.2.5. *An admissible strategy $\{b_i\}_i$ is called N-**periodic** $(N \in \mathbb{N})$ if for all $k \in \mathbb{N}_0$*
$$b_{kN} = a^* \ \mathbf{P} - a.s..$$

For any admissible strategy $\{b_i\}_i$ and any $\epsilon > 0$ we define
$$N(\{b_i\}_i, \epsilon) := \min \left\{ n \,\middle|\, \left| \frac{1}{n} \sum_{i=1}^{n} \mathbf{E} f(b_i, \bar{X}_{i+1}) - \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{E} f(b_i, \bar{X}_{i+1}) \right| \leq \epsilon \right\}.$$
$N(\{b_i\}_i, \epsilon)$ measures how long it takes the strategy $\{b_i\}_i$ to approach the long-term optimum for the first time up to an error of ϵ. Note that there is some $N \in \mathbb{N}$ such that
$$\left| \frac{1}{N} \sum_{i=1}^{N} \mathbf{E} f(b_i, \bar{X}_{i+1}) - \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{E} f(b_i, \bar{X}_{i+1}) \right| \leq \epsilon,$$
so that $N(\{b_i\}_i, \epsilon) < \infty$. Using $N(\{b_i\}_i, \epsilon)$ one can aproximate any strategy based on a portfolio selection function arbitrarily closely by a periodic strategy in the following sense:

Lemma 4.2.6. *Let $\{b_i\}_i$ be an admissible strategy based on a portfolio selection function c. Then for any $\epsilon > 0$ there exists an admissible N-periodic strategy $\{\tilde{b}_i\}_i$ $(N \in \mathbb{N})$ with*
$$\liminf_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \mathbf{E} f(\tilde{b}_i, \bar{X}_{i+1}) - \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{E} f(b_i, \bar{X}_{i+1}) \right) \geq -\epsilon.$$

Proof. Let $N := N((\{b_i\}_i, \epsilon)$, $\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{E} f(b_i, \bar{X}_{i+1})$ and $s := \limsup_{n\to\infty} \mu_n$, i.e.
$$|\mu_N - s| \leq \epsilon. \tag{4.2.9}$$
Considering the fact that, at any stage, the investor may choose the portfolio a^*, we define an N-periodic admissible strategy $\tilde{b}_i$ by
$$\begin{aligned}
&\tilde{b}_0 := a^*, \tilde{b}_1 := c(a^*, \bar{X}_1), ..., \tilde{b}_{N-1} := c(\tilde{b}_{N-2}, \bar{X}_{N-1}),\\
&\tilde{b}_N := a^*, \tilde{b}_{N+1} := c(a^*, \bar{X}_{N+1}), ..., \tilde{b}_{2N-1} := c(\tilde{b}_{2N-2}, \bar{X}_{2N-1}),\\
&\tilde{b}_{2N} := a^*, \quad \text{etc. ... .}
\end{aligned}$$

In particular, $\tilde{b}_0 = b_0, ..., \tilde{b}_{N-1} = b_{N-1}$. Hence, for all $k \in \mathbb{N}_0$ (convention $\frac{1}{0}\sum_{i=0}^{-1} ... = 0$):

$$\tilde{\mu}_{kN} := \frac{1}{kN}\sum_{i=0}^{kN-1} \mathbf{E} f(\tilde{b}_i, \bar{X}_{i+1}) = \frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{N}\sum_{i=jN}^{(j+1)N-1} \mathbf{E} f(\tilde{b}_i, \bar{X}_{i+1}) = \frac{1}{k}\sum_{j=0}^{k-1}\mu_N = \mu_N. \tag{4.2.10}$$

This follows from the construction of $\tilde{b}_i$ with the selection function c. Indeed, the matrix $(\tilde{b}_{jN}, ..., \tilde{b}_{(j+1)N-1}, \bar{X}_{jN+1}, ..., \bar{X}_{(j+1)N})$ is a function of $(\bar{X}_{jN+1}, ..., \bar{X}_{(j+1)N})$ and as such is distributed as $(b_0, ..., b_{N-1}, \bar{X}_1, ..., \bar{X}_N)$ which is used to calculate μ_N (by stationarity).

(4.2.9) and (4.2.10) imply that for all $k \in \mathbb{N}_0$: $|\tilde{\mu}_{kN} - s| \le \epsilon$. Now, let $\lfloor n \rfloor_N$ be the largest $kN (k \in \mathbb{N})$ with $kN \le n$. Then

$$\begin{aligned}
|\tilde{\mu}_n - s| &= \left| \frac{\lfloor n \rfloor_N}{n}\left(\tilde{\mu}_{\lfloor n \rfloor_N} - s\right) + \frac{1}{n}\sum_{i=\lfloor n \rfloor_N}^{n-1} \mathbf{E} f(\tilde{b}_i, \bar{X}_{i+1}) + \left(\frac{\lfloor n \rfloor_N}{n} - 1\right) s \right| \\
&\le \frac{\lfloor n \rfloor_N}{n}\left|\tilde{\mu}_{\lfloor n \rfloor_N} - s\right| + \frac{N-1}{n}\|f\|_\infty + \frac{n - \lfloor n \rfloor_N}{n}|s| \\
&\le 1 \cdot \epsilon + \frac{N-1}{n}\|f\|_\infty + \frac{N-1}{n}|s|,
\end{aligned}$$

where we used $n - \lfloor n \rfloor_N \le N - 1$. It follows that $\limsup_{n\to\infty} |\tilde{\mu}_n - s| \le \epsilon$ and finally that

$$\begin{aligned}
&\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1} \mathbf{E} f(\tilde{b}_i, \bar{X}_{i+1}) - \frac{1}{n}\sum_{i=0}^{n-1} \mathbf{E} f(b_i, \bar{X}_{i+1})\right) \ge \liminf_{n\to\infty} \tilde{\mu}_n - \limsup_{n\to\infty} \mu_n \\
&\quad = \liminf_{n\to\infty} \tilde{\mu}_n - s \ge \liminf_{n\to\infty} - |\tilde{\mu}_n - s| = -\limsup_{n\to\infty} |\tilde{\mu}_n - s| \ge -\epsilon,
\end{aligned}$$

proving Lemma 4.2.6. □

In the proof of Theorem 4.2.1, Lemma 4.2.6 will enable us to restrict ourselves to the class of periodic strategies competing with $\{b_i^*\}_i$. These will turn out to be much more tractable. We are now in the position to turn to the

4th step in the proof of Theorem 4.2.1: Finishing the proof of Theorem 4.2.1.

Consider a given admissible strategy $\{b_i\}_i$ based on a portfolio selection function. Let $\epsilon > 0$ be arbitrary but fixed. According to Lemma 4.2.6 there exists an $N := N(\{b_i\}_i, \epsilon)$-periodic admissible strategy $\{\tilde{b_i}\}_i$ with

$$\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(\tilde{b}_i,\bar{X}_{i+1})-\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(b_i,\bar{X}_{i+1})\right)\geq -\epsilon.$$

We will show that

$$\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(b_i^*,\bar{X}_{i+1})-\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(\tilde{b}_i,\bar{X}_{i+1})\right)\geq 0. \tag{4.2.11}$$

This yields

$$\begin{aligned}
&\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(b_i^*,\bar{X}_{i+1})-\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(b_i,\bar{X}_{i+1})\right)\\
&\quad\geq \liminf_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(b_i^*,\bar{X}_{i+1})-\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(\tilde{b}_i,\bar{X}_{i+1})\right)\\
&\qquad\quad+\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(\tilde{b}_i,\bar{X}_{i+1})-\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(b_i,\bar{X}_{i+1})\right)\\
&\quad\geq -\epsilon,
\end{aligned}$$

and the assertion follows from ϵ being arbitrary.

It remains to show (4.2.11). To this end, observe that by V2

$$\begin{aligned}
\mathbf{E}f(b_i,\bar{X}_{i+1}) &= \mathbf{E}\left(\mathbf{E}[f(b_i,\bar{X}_{i+1})|\mathcal{F}_i]\right)=\mathbf{E}\left(\mathbf{E}[f(b,\bar{X}_{i+1})|\mathcal{F}_i]_{|b=b_i}\right)\\
&= \mathbf{E}\left(\mathbf{E}[f(b,\bar{X}_{i+1})|\bar{X}_i]_{|b=b_i}\right)=\mathbf{E}m(b_i,\bar{X}_i).
\end{aligned}$$

Replacing f by m does not alter the value of $\mathbf{E}(\frac{1}{n}\sum_{i=0}^{n-1}f(b_i,\bar{X}_{i+1}))$, the target function, hence we may prove (4.2.11) in the form

$$\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}m(b_i^*,\bar{X}_i)-\frac{1}{n}\sum_{i=1}^{n}\mathbf{E}m(\tilde{b}_i,\bar{X}_i)\right)\geq 0.$$

First note that the Bellman equation implies that for all admissible strategies $\{b_i\}_i$

$$\begin{aligned}
&(1-\delta_{i+1})\mathbf{E}[h_{i+1}(b_{i+1},\bar{X}_{i+2})|\mathcal{F}_{i+1}]\\
&\quad= m(b_{i+1},\bar{X}_{i+1})+(1-\delta_{i+1})\mathbf{E}[h_{i+1}(b_{i+1},\bar{X}_{i+2})|\mathcal{F}_{i+1}]-m(b_{i+1},\bar{X}_{i+1})\\
&\quad\leq \max_{b\in S(b_i,\bar{X}_{i+1})}\left\{m(b,\bar{X}_{i+1})+(1-\delta_{i+1})\mathbf{E}[h_{i+1}(b,\bar{X}_{i+2})|\mathcal{F}_{i+1}]\right\}-m(b_{i+1},\bar{X}_{i+1})\\
&\quad= h_{i+1}(b_i,\bar{X}_{i+1})-m(b_{i+1},\bar{X}_{i+1}).
\end{aligned}$$

Taking expectations,

$$\begin{aligned}(1-\delta_{i+1})\mathbf{E}h_{i+1}(b_{i+1},\bar{X}_{i+2}) &= (1-\delta_{i+1})\mathbf{E}\left(\mathbf{E}[h_{i+1}(b_{i+1},\bar{X}_{i+2})|\mathcal{F}_{i+1}]\right)\\ &\leq \mathbf{E}h_{i+1}(b_i,\bar{X}_{i+1}) - \mathbf{E}m(b_{i+1},\bar{X}_{i+1}),\end{aligned}$$

and summing up we are left with

$$\begin{aligned}&\mathbf{E}\left(\frac{1}{n}\sum_{i=-1}^{n-2} m(b_{i+1},\bar{X}_{i+1})\right)\\ &\qquad \leq -\mathbf{E}\left(\frac{1}{n}\sum_{i=-1}^{n-2}\left((1-\delta_{i+1})h_{i+1}(b_{i+1},\bar{X}_{i+2}) - h_{i+1}(b_i,\bar{X}_{i+1})\right)\right).\end{aligned}$$

Equality holds for the strategy $\{b_i^*\}_i$, i.e.

$$\begin{aligned}&\mathbf{E}\left(\frac{1}{n}\sum_{i=0}^{n-1} m(b_i^*,\bar{X}_i)\right) - \mathbf{E}\left(\frac{1}{n}\sum_{i=0}^{n-1} m(\tilde{b}_i,\bar{X}_i)\right)\\ &\qquad \geq \mathbf{E}\frac{1}{n}\sum_{i=-1}^{n-2}\left(h_{i+1}(\tilde{b}_{i+1},\bar{X}_{i+2}) - h_{i+1}(\tilde{b}_i,\bar{X}_{i+1})\right)\\ &\qquad\quad -\mathbf{E}\frac{1}{n}\sum_{i=-1}^{n-2}\left(h_{i+1}(b_{i+1}^*,\bar{X}_{i+2}) - h_{i+1}(b_i^*,\bar{X}_{i+1})\right)\\ &\qquad\quad +\mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1}\delta_i\left(h_i(b_i^*,\bar{X}_{i+1}) - h_i(\tilde{b}_i,\bar{X}_{i+1})\right)\\ &\qquad =: A - B + C.\end{aligned} \tag{4.2.12}$$

We now investigate the asymptotic behaviour of the terms on the right hand side.

The first and the second term A and B of (4.2.12) are of the same form and tend to 0 as $n \to \infty$ since

$$\lim_{n\to\infty}\mathbf{E}\frac{1}{n}\sum_{i=-1}^{n-2}\left(h_{i+1}(b_{i+1},\bar{X}_{i+2}) - h_{i+1}(b_i,\bar{X}_{i+1})\right) = 0$$

for any admissible strategy $\{b_i\}_i$. To prove this, we set $h_{-1} := 0$ and consider the decomposition

$$\frac{1}{n}\sum_{i=-1}^{n-2}\left(h_{i+1}(b_{i+1},\bar{X}_{i+2}) - h_{i+1}(b_i,\bar{X}_{i+1})\right) = D + E$$

into

$$D := \frac{1}{n} \sum_{i=-1}^{n-2} \left(h_{i+1}(b_{i+1}, \bar{X}_{i+2}) - h_i(b_i, \bar{X}_{i+1}) \right)$$

and

$$E := \frac{1}{n} \sum_{i=-1}^{n-2} \left(h_i(b_i, \bar{X}_{i+1}) - h_{i+1}(b_i, \bar{X}_{i+1}) \right).$$

D is a telescopic sum, and using part 1 of Lemma 4.2.3 and the assumptions about $\{\delta_i\}_i$, we find that

$$|D| \;=\; \left| \frac{h_{n-1}(b_{n-1}, \bar{X}_n) - h_{-1}(b_{-1}, \bar{X}_0)}{n} \right| \le \frac{\|h_{n-1}\|_\infty}{n} \le \frac{\|f\|_\infty}{\delta_{n-1} n} \to 0.$$

As to E we note that $\delta_1 = \delta_2, \delta_3 = \delta_4 = \delta_5, \delta_6 = ... = \delta_9$, etc. implies that $h_1 = h_2, h_3 = h_4 = h_5, h_6 = ... = h_9, ...$. Hence there are at most $\sqrt{2n} + 1$ non-zero differences in E. By virtue of Lemma 4.2.3 these are bounded in absolute value by $\max_i \|h_{i+1} - h_i\|_\infty \le \|f\|_\infty$, which yields

$$|E| \le \frac{\sqrt{2n}+1}{n} \|f\|_\infty \longrightarrow 0.$$

The third term C of (4.2.12) is decomposed into

$$\begin{aligned}
&\mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1} \delta_i \left(h_i(b_i^*, \bar{X}_{i+1}) - h_i(\tilde{b}_i, \bar{X}_{i+1}) \right) \\
&\quad= \mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1} \delta_i \left(h_i(b_i^*, \bar{X}_{i+1}) - h_{i+1}(b_i^*, \bar{X}_{i+1}) \right) \\
&\qquad+ \mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1} \delta_i \left(h_{i+1}(b_i^*, \bar{X}_{i+1}) - h_{i+1}(a^*, \bar{X}_{\lfloor i \rfloor_N + 1}) \right) \\
&\qquad+ \mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1} \delta_i \left(h_{i+1}(a^*, \bar{X}_{\lfloor i \rfloor_N + 1}) - h_{\lfloor i \rfloor_N + 1}(a^*, \bar{X}_{\lfloor i \rfloor_N + 1}) \right) \\
&\qquad+ \mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1} \delta_i \left(h_{\lfloor i \rfloor_N + 1}(a^*, \bar{X}_{\lfloor i \rfloor_N + 1}) - h_i(\tilde{b}_i, \bar{X}_{i+1}) \right)
\end{aligned}$$

The absolute value of the first expectation in the decomposition is bounded from above by

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_i \|f\|_\infty \longrightarrow 0 \quad (n \to \infty)$$

(Lemma 4.2.3, part 2). Using Lemma 4.2.3, (4.2.8), we can bound the second expectation from below by

$$-2\|f\|_\infty \frac{1}{n}\sum_{i=0}^{n-1}\delta_i \longrightarrow 0 \quad (n\to\infty).$$

The third expectation has a lower bound

$$-\|f\|_\infty \frac{1}{n}\sum_{i=0}^{n-1}\delta_i(i-\lfloor i\rfloor_N) \geq -\|f\|_\infty (N-1)\frac{1}{n}\sum_{i=0}^{n-1}\delta_i \longrightarrow 0 \quad (n\to\infty)$$

(Lemma 4.2.3, part 2). Therefore, we need only show that the fourth expectation satisfies

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\delta_i\left(\mathbf{E}h_{\lfloor i\rfloor_N+1}(a^*,\bar{X}_{\lfloor i\rfloor_N+1})-\mathbf{E}h_i(\tilde{b}_i,\bar{X}_{i+1})\right)\geq 0. \tag{4.2.13}$$

This will be done exploiting the periodicity of $\{\tilde{b}_i\}_i$. In order to prove (4.2.13) we first assume that $n=kN$ with $k\in\mathbb{N}_0$:

$$\begin{aligned}
&\frac{1}{kN}\sum_{i=0}^{kN-1}\delta_i\left(\mathbf{E}h_{\lfloor i\rfloor_N+1}(a^*,\bar{X}_{\lfloor i\rfloor_N+1})-\mathbf{E}h_i(\tilde{b}_i,\bar{X}_{i+1})\right)\\
&\qquad=\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{N}\sum_{i=jN}^{(j+1)N-1}\delta_i\left(\mathbf{E}h_{\lfloor i\rfloor_N+1}(a^*,\bar{X}_{\lfloor i\rfloor_N+1})-\mathbf{E}h_i(\tilde{b}_i,\bar{X}_{i+1})\right).
\end{aligned}$$

Here, for $i\in\{jN,...,(j+1)N-1\}$

$$\begin{aligned}
&\mathbf{E}h_{\lfloor i\rfloor_N+1}(a^*,\bar{X}_{\lfloor i\rfloor_N+1})-\mathbf{E}h_i(\tilde{b}_i,\bar{X}_{i+1})\\
&\quad= \mathbf{E}h_{jN+1}(\tilde{b}_{jN},\bar{X}_{jN+1})-\mathbf{E}h_{jN+1}(\tilde{b}_{jN+1},\bar{X}_{jN+2})\\
&\qquad\qquad +\mathbf{E}h_{jN+1}(\tilde{b}_{jN+1},\bar{X}_{jN+2})-\mathbf{E}h_{jN+2}(\tilde{b}_{jN+1},\bar{X}_{jN+2})\\
&\qquad +\mathbf{E}h_{jN+2}(\tilde{b}_{jN+1},\bar{X}_{jN+2})-\mathbf{E}h_{jN+2}(\tilde{b}_{jN+2},\bar{X}_{jN+3})\\
&\qquad\qquad +\mathbf{E}h_{jN+2}(\tilde{b}_{jN+2},\bar{X}_{jN+3})-\mathbf{E}h_{jN+3}(\tilde{b}_{jN+2},\bar{X}_{jN+3})\\
&\qquad +\mathbf{E}h_{jN+3}(\tilde{b}_{jN+2},\bar{X}_{jN+3})-\mathbf{E}h_{jN+3}(\tilde{b}_{jN+3},\bar{X}_{jN+4})\\
&\qquad\qquad + \quad ... \quad - \quad ...\\
&\qquad\qquad +\mathbf{E}h_{i-1}(\tilde{b}_{i-1},\bar{X}_{i})-\mathbf{E}h_{i}(\tilde{b}_{i-1},\bar{X}_{i})\\
&\qquad +\mathbf{E}h_{i}(\tilde{b}_{i-1},\bar{X}_{i})-\mathbf{E}h_{i}(\tilde{b}_{i},\bar{X}_{i+1})\\
&\quad\geq -2\|f\|_\infty(i-jN)-\|f\|_\infty(i-jN-1),
\end{aligned} \tag{4.2.14}$$

where the 1st, 3rd, etc. line after the equality (the non-indented terms) can be bounded by (4.2.7), the 2nd, 4th, etc. term (the indented terms) by Lemma 4.2.3, part 2. Consequently,

$$\mathbf{E}h_{\lfloor i\rfloor_N+1}(a^*, \bar{X}_{\lfloor i\rfloor_N+1}) - \mathbf{E}h_i(\tilde{b}_i, \bar{X}_{i+1}) \geq -3\|f\|_\infty(i - jN)$$

and

$$\begin{aligned}
&\frac{1}{kN}\sum_{i=0}^{kN-1}\delta_i\left(\mathbf{E}h_{\lfloor i\rfloor_N+1}(a^*, \bar{X}_{\lfloor i\rfloor_N+1}) - \mathbf{E}h_i(\tilde{b}_i, \bar{X}_{i+1})\right)\\
&\geq -\frac{3\|f\|_\infty}{k}\sum_{j=0}^{k-1}\frac{1}{N}\sum_{i=jN}^{(j+1)N-1}\delta_i(i-jN) \geq -\frac{3\|f\|_\infty}{k}\sum_{j=0}^{k-1}\frac{\delta_{jN}}{N}\sum_{i=jN}^{(j+1)N-1}(i-jN)\\
&= -3\|f\|_\infty\left(\frac{1}{k}\sum_{j=0}^{k-1}\delta_{jN}\right)\left(\frac{1}{N}\sum_{i=0}^{N-1}i\right) = -3\|f\|_\infty\left(\frac{1}{k}\sum_{j=0}^{k-1}\delta_{jN}\right)\frac{N-1}{2}. \qquad (4.2.15)
\end{aligned}$$

For arbitrary $n \in \mathbb{N}$ (not necessarily $n = kN$), we find that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=0}^{n-1}\delta_i\left(\mathbf{E}h_{\lfloor i\rfloor_N+1}(a^*, \bar{X}_{\lfloor i\rfloor_N+1}) - \mathbf{E}h_i(\tilde{b}_i, \bar{X}_{i+1})\right)\\
&\quad = \left\{\frac{\lfloor n\rfloor_N}{n}\cdot\frac{1}{\lfloor n\rfloor_N}\sum_{i=0}^{\lfloor n\rfloor_N-1}\delta_i\left(\mathbf{E}h_{\lfloor i\rfloor_N+1}(a^*, \bar{X}_{\lfloor i\rfloor_N+1}) - \mathbf{E}h_i(\tilde{b}_i, \bar{X}_{i+1})\right)\right\}\\
&\qquad + \left\{\frac{1}{n}\sum_{i=\lfloor n\rfloor_N}^{n-1}\delta_i\left(\mathbf{E}h_{\lfloor i\rfloor_N+1}(a^*, \bar{X}_{\lfloor i\rfloor_N+1}) - \mathbf{E}h_i(\tilde{b}_i, \bar{X}_{i+1})\right)\right\}. \qquad (4.2.16)
\end{aligned}$$

Set $k := \lfloor n/N\rfloor$ and bound the first bracket in (4.2.16) from below by

$$-\frac{\lfloor n\rfloor_N}{n}\cdot\frac{3\|f\|_\infty(N-1)}{2}\left(\frac{1}{k}\sum_{j=0}^{k-1}\delta_{jN}\right) \longrightarrow 0 \quad (n\to\infty)$$

(use (4.2.15) and note that $n \to \infty$ implies $k \to \infty$). The absolute value of the second bracket is bounded from above by

$$\begin{aligned}
&\frac{1}{n}\sum_{i=\lfloor n\rfloor_N}^{n-1}\delta_i\left(\|h_{\lfloor i\rfloor_N+1}\|_\infty + \|h_i\|_\infty\right)\\
&\quad\leq \frac{1}{n}\sum_{i=\lfloor n\rfloor_N}^{n-1}\left(\delta_{\lfloor i\rfloor_N}\|h_{\lfloor i\rfloor_N}\|_\infty + \delta_{\lfloor i\rfloor_N}\|h_{\lfloor i\rfloor_N+1} - h_{\lfloor i\rfloor_N}\|_\infty + \delta_i\|h_i\|_\infty\right)
\end{aligned}$$

$$\leq \frac{1}{n} \sum_{i=\lfloor n \rfloor_N}^{n-1} 3\|f\|_\infty = 3\|f\|_\infty \frac{n - \lfloor n \rfloor_N}{n} \leq 3\|f\|_\infty \frac{N-1}{n} \longrightarrow 0 \quad (n \to \infty),$$

which concludes the proof. □

4.3 Further properties of the value function

As a preparation for the next chapter, where we will be dealing with the case when the distribution of the return process is unknown, we prove the following result concerning the Lipschitz continuity of the value function h_i:

Proposition 4.3.1. *Let $(h_i, 0)$ be a solution of the Bellman equation (4.2.5) for $\delta = \delta_i$ $(i = 1, 2, ...)$, then for sufficiently small commission factor $c \geq 0$ there exists a constant $K > 0$ such that*

$$|\delta_i \cdot h_i(b_1, \bar{x}) - \delta_i \cdot h_i(b_2, \bar{x})| \leq K \cdot \|b_1 - b_2\|_\infty$$

for all $b_1, b_2 \in S$, all $\bar{x} \in [A, B]^{dm}$ and all $i \in \mathbb{N}$.

Proof. The argument requires some less known notions from analysis, especially Clarke's generalized derivative for Lipschitz continuous functions (Clarke, 1981). Let $W \subseteq \mathbb{R}^{d'}$ and $Z \subseteq \mathbb{R}^{d''}$ be Banach spaces (whose supremum-norms are both denoted by $\|\cdot\|_\infty$). Given a Lipschitz continuous mapping $\Phi : W \times Z \to \mathbb{R}$, *Clarke's generalized derivative* is defined as the convex hull of a limit set,

$$\partial_w \Phi(w, z) := conv \left\{ \lim_{i \to \infty} \nabla_{w_i} \Phi(w_i, z) \middle| w_i \to w \right\},$$

where only those sequences w_i are considered for which all gradients $\nabla_{w_i} \Phi(w_i, z)$ and the limit $\lim_{i \to \infty} \nabla_{w_i} \Phi(w_i, z)$ exist (note that the gradients exist almost everywhere due to Rademacher's Theorem). $H(M_1, M_2)$ denotes the *Hausdorff distance* between two subsets M_1 and M_2 of $\mathbb{R}^{d'}$, defined by

$$H(M_1, M_2) := \max \left\{ \sup_{w_2 \in M_2} \rho(w_2, M_1), \sup_{w_1 \in M_1} \rho(w_1, M_2) \right\}$$

with $\rho(w_2, M_1) := \inf_{w_1 \in M_1} \|w_1 - w_2\|_\infty$.

We will also need the following proposition by Ledyaev (1984, Theorem 1) concerning Lipschitz continuity of implicitly defined set-valued functions:

Proposition 4.3.2. *(Ledyaev, 1984, Theorem 1) Let $W \subseteq \mathbb{R}^{d'}$ and $Z \subseteq \mathbb{R}^{d''}$ be Banach spaces and $\Phi : W \times Z \longrightarrow \mathbb{R}$ a mapping such that:*

1. *For all $z \in Z$ the function $\Phi(\cdot, z)$ is Lipschitz continuous.*
2. *There exists a constant $L > 0$ with*
$$|\Phi(w, z_1) - \Phi(w, z_2)| \leq L\|z_1 - z_2\|_\infty$$
for all $w \in W, z_1, z_2 \in Z$.
3. *There exists a constant Δ such that for all $(w, z) \in W \times Z$ with $\Phi(w, z) > 0$*
$$\inf_{v \in \partial_w \Phi(w,z)} \|v\|_\infty > \Delta.$$

Then the set-valued mapping $M(z) := \{w \in W | \Phi(w, z) \leq 0\}$ satisfies the Lipschitz property
$$H(M(z_1), M(z_2)) \leq \frac{L}{\Delta}\|z_1 - z_2\|_\infty.$$

Finally, we need the *modulus of continuity*, defined for any continuous function $g : S \times [A, B]^{dm} \to \mathbb{R}$ as ($\bar{x} \in [A, B]^{dm}$ fixed)
$$\omega\Big(g(\cdot, \bar{x}), \epsilon\Big) := \sup_{s,t \in S, \|s-t\|_\infty \leq \epsilon} \Big|g(s, \bar{x}) - g(t, \bar{x})\Big|.$$

Combining the Hausdorff distance and the modulus of continuity it is easily seen that
$$\left|\max_{b \in S(s,\bar{x})} g(b, \bar{x}) - \max_{b \in S(t,\bar{x})} g(b, \bar{x})\right| \leq \omega\Big(g(\cdot, \bar{x}), H(S(s, \bar{x}), S(t, \bar{x}))\Big). \tag{4.3.1}$$

Thus, having all the tools we need at hand, we can embark on the proof of Proposition 4.3.1.

Let $b = (b_{-1}, ..., b_m) \in S, \bar{x} = (x_1, ..., x_m) \in [A, B]^{dm}$. For fixed $\bar{x}$ we set $\Phi : S \times S \to \mathbb{R} : (b, s) \mapsto \Phi(b, s) := |g_c(s, \bar{x}, b)|$, recall g_c from (4.1.3). Clearly, Φ is Lipschitz continuous in the first argument. Moreover,
$$|\Phi(b, s) - \Phi(b, t)| \leq |g_c(s, \bar{x}, b) - g_c(t, \bar{x}, b)| \leq c \cdot const(r, B, m) \cdot \|s - t\|_\infty$$

(to see this, note that the self-financing condition forces $b_{-1} \leq c$). Taking the gradient for g_c (under $\Phi(b,s) > 0$) yields

$$\partial_b \Phi(b,s) \subseteq \Big(\{1\} \times \{0\} \times [0,c]^m\Big) \cdot \Big((1+r)b_0 + \sum_{k=1}^m x_k b_k\Big),$$

so that

$$\begin{aligned}\inf_{v \in \partial_b \Phi(b,s)} \|v\|_\infty &\geq (1+r)b_0 + \sum_{k=1}^m x_k b_k \\ &\geq \min\{1+r, A\}(1-b_{-1}) \geq const(r,A)(1-c).\end{aligned}$$

As a consequence, the conditions of Propositon 4.3.2 are fulfilled. We can choose $\Delta := const(r,A) \cdot (1-c)$ and $L := c \cdot const(r,B,m)$ to obtain

$$H(S(s,\bar{x}), S(t,\bar{x})) \leq \frac{L}{\Delta} \|s-t\|_\infty. \tag{4.3.2}$$

For c sufficiently small, $L/\Delta \leq 1$. Recall the function

$$V_i(b,\bar{x}) = m(b,\bar{x}) + (1-\delta_i)\mathbf{E}[h_i(b,\bar{X}_{d+1})|\bar{X}_d = \bar{x}]$$

that defined h_i in (4.2.2). The Bellman equation yields

$$\Big|h_i(s,\bar{x}) - h_i(t,\bar{x})\Big| = \Big|\max_{b \in S(s,\bar{x})} V_i(b,\bar{x}) - \max_{b \in S(t,\bar{x})} V_i(b,\bar{x})\Big| \leq \omega\Big(V_i(\cdot,\bar{x}), \|s-t\|_\infty\Big)$$

by (4.3.1) and (4.3.2). Hence,

$$\omega\Big(h_i(\cdot,\bar{x}),\epsilon\Big) \leq \omega\Big(V_i(\cdot,\bar{x}),\epsilon\Big) \leq \omega\Big(f(\cdot,\bar{x}),\epsilon\Big) + (1-\delta_i)\omega\Big(h_i(\cdot,\bar{x}),\epsilon\Big).$$

The Lipschitz property of f yields $\omega\Big(f(\cdot,\bar{x}),\epsilon\Big) \leq const. \cdot \epsilon$ and from the latter chain of inequalities we obtain

$$\omega\Big(h_i(\cdot,\bar{x}),\epsilon\Big) \leq \frac{const.}{\delta_i} \cdot \epsilon.$$

Thus $\delta_i \cdot h_i(\cdot,\bar{x})$ is Lipschitz continuous with the same Lipschitz constant as f (independent of $\bar{x}$), and the proof is finished. □

CHAPTER 5

A Markov model with transaction costs: statistical view

Chapter 4 completely remained within the probabilistic framework, i.e. the point of view of an investor with full knowledge of the underlying return distribution. This, of course, is highly unrealistic. In practice, the investor's view is one of a statistician rather than a probabilist. From observations of stock returns, factors and side information, he assembles a market "picture", an idea of the stochastic laws of the market. He then decides on an investment strategy. In the following it will be shown how he can balance the need to avoid transaction costs and the need to boost his wealth in his investment decisions – without knowing the underlying return distribution. The model is the same as in Chapter 4.

In particular, Section 5.1 sets up an empirical counterpart of the Bellman equation which we used in Theorem 4.2.1 to construct an optimal strategy. Based on this empirical version of the Bellman equation we construct a portfolio selection algorithm in Section 5.1.1 (cf. (5.1.5)). This algorithm has virtually the same optimality properties as the algorithm in Chapter 4 (Theorems 5.1.1 and 5.1.2). To verify this, we need results on uniformly consistent regression estimation which will be given in Section 5.2, Theorem 5.2.1 and Corollary 5.2.2 being the central results featuring the speed of uniform convergence of kernel regression estimates. Section 5.3 finally gives the proof of the optimality properties of the algorithm.

5.1 The empirical Bellman equation

The whole procedure described in the last chapter was founded on the Bellman equation. In particular, in Theorem 4.2.1, we used the Bellman equation

$$\lambda_i + h_i(s, \bar{x}) = \max_{b \in S(s,\bar{x})} \left\{ m(b, \bar{x}) + (1 - \delta_i)\mathbf{E}[h_i(b, \bar{X}_{d+1}) | \bar{X}_d = \bar{x}] \right\}, \qquad (5.1.1)$$

$\lambda_i \in \mathbb{R}, h_i \in C(S \times [A, B]^{dm})$, to construct an optimal investment strategy in a stationary d-stage Markov process $\{X_i\}_{i=-d+1}^{\infty}$ of return vectors in a financial market. Here,

$$m(b, \bar{x}) := \mathbf{E}[f(b, \bar{X}_{i+1}) | \bar{X}_i = \bar{x}] = \mathbf{E}[f(b, \bar{X}_{d+1}) | \bar{X}_d = \bar{x}],$$

f being the (logarithmic) utility function and $S(s, \bar{x})$ being the set of admissible portfolio vectors when $\bar{x}$ denotes the last d observed return vectors and s the last chosen portfolio. The Bellman equation was solved by a value iteration type of algorithm in Chapter 4, which crucially relies on the distribution of the stationary process $\{\bar{X}_i\}_i$. This distribution, in general, is unknown to the investor. Nonetheless he may try to estimate a solution of the Bellman equation.

A natural way to obtain an estimated solution is to replace the conditional expectations $\mathbf{E}[h_i(b, \bar{X}_{d+1}) | \bar{X}_d = \bar{x}]$ and $m(b, \bar{x})$ in (5.1.1) by kernel estimates

$$\sum_{j=-d+1}^{i-1} h_i(b, \bar{X}_{j+1}) \cdot \frac{K\Big((\bar{x} - \bar{X}_j)/w_i\Big)}{\sum_{k=-d+1}^{i-1} K\Big((\bar{x} - \bar{X}_k)/w_i\Big)}$$

and

$$\sum_{j=-d+1}^{i-1} f(b, \bar{X}_{j+1}) \cdot \frac{K\Big((\bar{x} - \bar{X}_j)/w_i\Big)}{\sum_{k=-d+1}^{i-1} K\Big((\bar{x} - \bar{X}_k)/w_i\Big)},$$

respectively. K is a bounded, Lipschitz continuous kernel function $[A, B]^{dm} \longrightarrow \mathbb{R}_0^+$ with $\int_{\mathbb{R}^{dm}} K(x)dx = 1$ and $\int_{\mathbb{R}^{dm}} K(x)\|x\|_\infty dx < \infty$. As we shall see, the bandwidths can be chosen as $w_i \sim i^{-1/(dm+2)}$. For simplicity, we us the shorthand notation

$$K_i(\bar{X}_j, \bar{x}) := \frac{K\Big((\bar{x} - \bar{X}_j)/w_i\Big)}{\sum_{k=-d+1}^{i-1} K\Big((\bar{x} - \bar{X}_k)/w_i\Big)} \quad \text{with} \quad w_i \sim i^{-1/(dm+2)}.$$

We thus obtain what we call the **empirical Bellman equation**,

$$\hat{\lambda}_i + \hat{h}_i(s, \bar{x}) = \max_{b \in S(s,\bar{x})} \left\{ \sum_{j=-d+1}^{i-1} \left(f(b, \bar{X}_{j+1}) + (1-\delta_i)\hat{h}_i(b, \bar{X}_{j+1}) \right) K_i(\bar{X}_j, \bar{x}) \right\}, \tag{5.1.2}$$

which is to be solved for $\hat{\lambda}_i \in \mathbb{R}, \hat{h}_i \in C(S \times [A,B]^{dm})$.

5.1.1 An optimal strategy

If we set up an investment strategy as in Theorem 4.2.1, however, using the *empirical* Bellman equation instead of the original Bellman equation (4.2.2), two questions arise: Is there a solution to the empirical Bellman equation and if so, is the corresponding strategy optimal?

We first tackle the existence of solutions of (5.1.2). Using the fact that $K_i(\bar{X}_j, \bar{x})$ is a continuous function of $\bar{x}$ and appealing to Berge's Maximum Theorem (Aliprantis and Border, 1999, Theorem 16.31) again, we can define an operator $\hat{M}_i$ by

$$\begin{aligned} \hat{M}_i : \quad & C(S \times [A,B]^{dm}) \longrightarrow C(S \times [A,B]^{dm}) : \\ & h \longmapsto \max_{b \in S(s,\bar{x})} \left\{ \sum_{j=-d+1}^{i-1} \left(f(b, \bar{X}_{j+1}) + (1-\delta_i) h(b, \bar{X}_{j+1}) \right) K_i(\bar{X}_j, \bar{x}) \right\}. \end{aligned}$$

$\hat{M}_i$ is the empirical counterpart of the operator M in the proof of Proposition 4.2.2. Because of

$$\sum_{j=-d+1}^{i-1} K_i(\bar{X}_j, \bar{x}) = 1, \tag{5.1.3}$$

$\hat{M}_i$ operates on $C(S \times [A,B]^{dm})/\{\text{constant functions}\}$ according to $\hat{M}_i[h] := [\hat{M}_i h]$. Now, arguing similarly as in the proof of Proposition 4.2.2, we find that $\hat{M}_i$ is a contraction mapping in the norm $\|g\| := \max g - \min g$. Indeed, for functions $g, h \in C(S \times [A,B]^{dm})$ there exists a $b^* \in S(s, \bar{x})$ with

$$\begin{aligned} (\hat{M}_i h)(s, \bar{x}) &= \sum_{j=-d+1}^{i-1} \left(f(b^*, \bar{X}_{j+1}) + (1-\delta_i) h(b^*, \bar{X}_{j+1}) \right) K_i(\bar{X}_j, \bar{x}), \\ (\hat{M}_i g)(s, \bar{x}) &\geq \sum_{j=-d+1}^{i-1} \left(f(b^*, \bar{X}_{j+1}) + (1-\delta_i) g(b^*, \bar{X}_{j+1}) \right) K_i(\bar{X}_j, \bar{x}). \end{aligned}$$

Using (5.1.3), this implies

$$\begin{aligned}&(\hat{M}_i h)(s,\bar{x}) - (\hat{M}_i g)(s,\bar{x})\\ &\quad\le (1-\delta_i)\sum_{j=-d+1}^{i-1}\big(h(b^*,\bar{X}_{j+1}) - g(b^*,\bar{X}_{j+1})\big)\,K_i(\bar{X}_j,\bar{x}) \le (1-\delta_i)\|h-g\|_\infty.\end{aligned}$$

Starting from here, one can argue exactly as in the proof of Proposition 4.2.2 to obtain $\|\hat{M}_i h - \hat{M}_i g\| \le (1-\delta_i)\|h-g\|$. Hence there exists a solution $\hat{h}_i, \hat{\lambda}_i$ of the empirical Bellman equation. Due to (5.1.3) this can be normalized to

$$\hat{h}_i(s,\bar{x}) = \max_{b\in S(s,\bar{x})}\left\{\sum_{j=-d+1}^{i-1}\Big(f(b,\bar{X}_{j+1}) + (1-\delta_i)\hat{h}_i(b,\bar{X}_{j+1})\Big)\,K_i(\bar{X}_j,\bar{x})\right\}. \tag{5.1.4}$$

In the sequel, $\hat{h}_i$ denotes a solution of (5.1.4).

We are now in a position to define the strategy which will turn out to have the same optimality properties as the strategy in Chapter 4. On the analogy of (4.2.3) the investor follows the strategy

$$\hat{b}_0 \;:=\; a^* \quad\text{and}\quad \hat{b}_i := \arg\max_{b\in S(\hat{b}_{i-1},\bar{X}_i)} \hat{V}_i(b,\bar{X}_i) \tag{5.1.5}$$

with

$$\hat{V}_i(b,\bar{x}) := \sum_{j=-d+1}^{i-1}\Big(f(b,\bar{X}_{j+1}) + (1-\delta_i)\hat{h}_i(b,\bar{X}_{j+1})\Big)\,K_i(\bar{X}_j,\bar{x}).$$

Observe that, in contrast to b_i^* from Chapter 4, this strategy can be constructed using observed data only, we need not know the underlying distribution of the return process.

The important feature is: This strategy is still optimal if the kernel estimates work sufficiently well. A sufficient condition is given in the following theorem.

Theorem 5.1.1. *Under the assumptions of Theorem 4.2.1, let h_i be the solutions of the Bellman equation (5.1.1, $\lambda_i = 0$) and define the class $\mathcal{G} := \{\delta_i \cdot h_i, \delta_i \cdot f \,|\, i = 1,2,...\}$.*

Then

$$\lim_{i\to\infty}\frac{1}{\delta_{i+1}^2}\sup_{g\in\mathcal{G}}\mathbf{E}\sup_{b\in S,\bar{x}\in[A,B]^{dm}}\Bigg|\sum_{j=-d+1}^{i} g(b,\bar{X}_{j+1})K_{i+1}(\bar{X}_j,\bar{x}) \\ -\mathbf{E}[g(b,\bar{X}_{i+2})|\bar{X}_{i+1}=\bar{x}]\Bigg| = 0 \tag{5.1.6}$$

implies that

$$\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(\hat{b}_i,\bar{X}_{i+1})-\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{E}f(b_i,\bar{X}_{i+1})\right)\geq 0 \tag{5.1.7}$$

for any admissible strategy $\{b_i\}_i$ based on a portfolio selection function.

This theorem translates the investment problem into a regression estimation problem. Of course, it is desirable to have practical sufficient conditions for the assumptions of Theorem 5.1.1 to hold. As we shall see, choosing $\delta_i \geq 1/\log i$ suffices, for example we may choose $\delta_1 = \delta_2 := d_1$, $\delta_3 = \delta_4 = \delta_5 := d_2$, etc. with $d_i \sim 1/\log i$ (cf. Theorem 5.1.2 below).

In the following, a stochastic process $\{X_i\}_{i=-\infty}^{\infty}$ is called a **stationary GSM-process (geometrically strongly mixing)**, if beyond stationarity the following holds: There exist constants $c > 0$ and $\rho \in [0,1)$ such that the **α-mixing coefficients**

$$\alpha(k) := \alpha(\sigma(X_i, i\leq 0),\sigma(X_i, i\geq k)) := \sup_{\substack{B\in\sigma(X_i,i\leq 0)\\ C\in\sigma(X_i,i\geq k)}}|\mathbf{P}(B\cap C)-\mathbf{P}(B)\mathbf{P}(C)|$$

satisfy

$$\alpha(k)\leq c\cdot\rho^k \quad (k\geq 1).$$

The behaviour of the α-mixing coefficients $\alpha(k)$ mirrors how fast dependency in the process variables decays for large time lags k. Under mild assumptions, the class of GSM-processes comprises linear processes (Bosq, 1996, Sec. 1.3, 2.3), polynomial AR-processes (Doukhan, 1994, Sec. 2.4.1, Th. 5) such as ARMA-processes (Doukhan, 1994, Sec. 2.4.1.2, Th. 6, Cor. 3), and Doeblin- or Harris-recurrent Markov chains (Doukhan, 1994, Sec. 2.4, Th. 1 and 3).

Theorem 5.1.2. *The assumptions of Theorem 5.1.1 are fulfilled if the following hold:*

1. *$\{X_i\}_{i=-d+1}^{\infty}$ is a stationary $[A,B]^m$-valued d-stage Markov process (cf. V1-V3 in Section 4.1) and geometrically strongly mixing.*

2. *There exist densities $f_{\bar{X}_0}$ and $f_{X_0|\bar{X}_0}$ of the distributions $\mathbf{P}_{\bar{X}_0}$ and $\mathbf{P}_{X_0|\bar{X}_0}$, respectively, such that*

   – *$f_{\bar{X}_0}$ is Lipschitz continuous, i.e., for some $C > 0$,*

   $$|f_{\bar{X}_0}(\bar{x}) - f_{\bar{X}_0}(\bar{y})| \le C\|\bar{x} - \bar{y}\|_\infty \quad \text{for all } \bar{x}, \bar{y} \in [A,B]^{dm},$$

   – *the level sets $\{\bar{x} : f_{\bar{X}_0}(\bar{x}) \ge 1/n\}$ of $f_{\bar{X}_0}$ satisfy*

   $$H(\operatorname{supp} f_{\bar{X}_0}, \{\bar{x} : f_{\bar{X}_0}(\bar{x}) \ge 1/n\}) \le n^{-k}$$

   for some $k > 0$, H denoting the Hausdorff distance (cf. Section 4.3),

   – *$f_{X_0|\bar{X}_0}$ is Lipschitz continuous such that for some $C > 0$,*

   $$|f_{X_0|\bar{X}_0}(x, \bar{x}) - f_{X_0|\bar{X}_0}(x, \bar{y})| \le C\|\bar{x} - \bar{y}\|_\infty$$

   for all $x \in [A,B]^m, \bar{x}, \bar{y} \in [A,B]^{dm}$.

3. *The commission factor c ist sufficiently small.*

4. *$\delta_i \ge 1/\log i$ satisfies (4.2.1).*

Hence, under the (not too restrictive) conditions of Theorem 5.1.2 we are able to construct an admissible strategy $\{\hat{b}_i\}_i$ that is superior to any other admissible strategy $\{b_i\}_i$ based on a portfolio selection function in the sense of

$$\liminf_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \mathbf{E} f(\hat{b}_i, \bar{X}_{i+1}) - \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{E} f(b_i, \bar{X}_{i+1}) \right) \ge 0.$$

It should be stressed again that this is a conservative, i.e. worst case analysis, the lim inf giving the worst possible performance of our strategy $\{\hat{b}_i\}_i$.

Remark. There are a number of sufficient conditions for $\{X_i\}_i$ to be a stationary GSM-process (see, e.g., Doukhan, 1994). For example, the GSM property holds if there exists a continuous function $r : [A,B]^m \to \mathbb{R}_0^+$ with

$$f_{X_0|\bar{X}_0}(x_0|\bar{x}_0) \ge r(x_0)$$

for all $\bar{x}_0 \in [A, B]^{dm}, x_0 \in [A, B]^m$ and

$$\int_{[A,B]^m} r(x_0) dx_0 > 0.$$

To verify this, one may use the Doeblin condition in Theorem 1 of Doukhan (1994, Sec. 2.4): The transition probabilities of the d-stage Markov process $\{X_i\}_i$ are

$$\mathbf{P}(\bar{x}_0, C) = \int_C f_{X_0|\bar{X}_0}(x_0|\bar{x}_0) dx_0$$

for $\bar{x}_0 \in [A, B]^{dm}, C \in \mathcal{B}([A, B]^m)$. In particular, with the measure $\mu := r \cdot \lambda$ (λ denoting the Lebesgue-Borel-measure on $[A, B]^m$),

$$\mathbf{P}(\bar{x}_0, C) \geq \int_C r(x_0) dx_0 = \mu(C)$$

and

$$\mu([A, B]^m) = \int_{[A,B]^m} r(x_0) dx_0 \in (0, 1].$$

Under these circumstances, the cited theorem of Doukhan (1994) allows us to conclude that $\{X_i\}_i$ is a GSM-process.

5.1.2 How to prove optimality

Before we actually prove the main results, Theorems 5.1.1 and 5.1.2, we sketch the way in which we are going to proceed. Before we can start with the proof, we need to establish some results on uniformly consistent regression estimation. This will be done in the next section. Once having got results on the speed of uniform almost sure convergence we can embark on the core of the proof of the theorems in Section 5.3. There, we argue along the lines of the corresponding results in Chapter 4, using uniform consistency results to pass from the solution of the empirical Bellman equation to the solution of the original Bellman equation.

5.2 Uniformly consistent regression estimation

We start with the following curve estimation problem. Given data $X_0, X_1, ..., X_n$

from a stationary $\mathbb{R}^d$-valued stochastic process $\{X_i\}_{i=-\infty}^{\infty}$ and a function $g : S \times \mathbb{R}^d \longrightarrow \mathbb{R}$ ($S \subseteq \mathbb{R}^{d'}$ compact), the objective is to estimate both, the function

$$\phi(g, b, x) := \mathbf{E}[g(b, X_1)|X_0 = x] \cdot f_{X_0}(x)$$

as well as the regression function

$$R(g, b, x) := \mathbf{E}[g(b, X_1)|X_0 = x].$$

The first will be estimated uniformly in $b \in S$ and $x \in \mathbb{R}^d$ by the kernel estimate

$$Z_n(g, b, x) := \frac{1}{nw_n^d} \sum_{i=0}^{n-1} g(b, X_{i+1}) K\left(\frac{x - X_i}{w_n}\right), \tag{5.2.1}$$

the latter by

$$R_n(g, b, x) := \frac{Z_n(g, b, x)}{Z_n(\mathbf{1}, b, x)}.$$

$K : \mathbb{R}^d \longrightarrow \mathbb{R}_0^+$ is assumed to be a fixed bounded, Lipschitz continuous kernel function with $\int_{\mathbb{R}^d} K(x)dx = 1$ and $\int_{\mathbb{R}^d} K(x)\|x\|_\infty dx < \infty$. $w_n \in \mathbb{R}^+$ is a sequence of bandwidths to be chosen later (such that $\lim_{n\to\infty} w_n = 0$). $\mathbf{1}$ denotes the constant function $(b, x) \mapsto 1$. Note that $Z_n(\mathbf{1}, b, x)$ is an estimate of the density $f_{X_0}(x)$.

Bosq (1996, Theorems 2.2, 3.3, 3.3) derives rates for the uniform almost sure convergence of $Z_n(\mathbf{1}, b, x)$ and $Z_n(g, b, x)$ for a fixed g in stationary GSM-processes. Now, the following theorem generalizes the results of Bosq such as to feature the rate of convergence of the expected $\|\cdot\|_\infty$-error of the estimate (5.2.1) in GSM-processes *uniformly* over a huge class of functions g. For this, we agree on the following notation: $\mathcal{F}$ is the class of sequences $\{c\ n^s \log^t n\}_{n\in\mathbb{N}}$ $(c > 0, s > 0, t \in \mathbb{R}$ or $c > 0, s = 0, t > 1)$. For any set $\mathcal{X} \subseteq \mathbb{R}^d$ we denote the $\|\cdot\|_\infty$-diameter of $\mathcal{X}$ by $\operatorname{diam}(\mathcal{X}) := \sup_{x,y\in\mathcal{X}} \|x - y\|_\infty$. f_{X_0} is a density of X_0, $f_{X_1|X_0}$ a density of the conditional distribution $\mathbf{P}_{X_1|X_0}$. For the estimation of ϕ, we will prove the

---

Theorem 5.2.1. *Let $\{X_i\}_{i=0}^{\infty}$ be a stationary GSM-process of $\mathbb{R}^d$-valued random variables with a Lipschitz continuous density f_{X_0}. Moreover, let $\mathcal{G}$ be a class of functions $g : S \times \mathbb{R}^d \longrightarrow \mathbb{R}$ ($S \subseteq \mathbb{R}^{d'}$ compact) with the following property: There exists a constant $C > 0$ such that for all $g \in \mathcal{G}$ and all*

$a, b \in S, x, y \in \mathbb{R}^d$

$$\|g\|_\infty \leq C, \tag{5.2.2}$$
$$\int_{\mathbb{R}^d} |g(a,x)|dx \leq C, \tag{5.2.3}$$
$$|g(a,x) - g(b,x)| \leq C \cdot \|a-b\|_\infty, \tag{5.2.4}$$
$$|\mathbf{E}[g(b,X_1)|X_0 = x] - \mathbf{E}[g(b,X_1)|X_0 = y]| \leq C \cdot \|x-y\|_\infty. \tag{5.2.5}$$

Furthermore, choose $w_n := n^{-1/(d+2)}$. Then

$$\sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x\in\mathcal{X}_n, b\in S} |Z_n(g,b,x) - \phi(g,b,x)| = o\left(\frac{\log^\beta n}{n^{1/(d+2)}}\right)$$

for any $\beta > 1$ and any sequence $\mathcal{X}_n \subseteq \mathbb{R}^d$ with $\mathrm{diam}(\mathcal{X}_n) \in \mathcal{F}$.

Now, if the support of f_{X_0}, supp $f_{X_0} = \overline{\{x : f_{X_0}(x) > 0\}}$, is a compact subset of $\mathbb{R}^d$, the theorem may be used to derive the following result concerning the estimation of the regression function R. Recall that H denotes the Hausdorff distance (cf. Section 4.3).

Corollary 5.2.2. *Under the assumptions of Theorem 5.2.1, if f_{X_0} has compact support and the level sets*

$$\mathcal{X}_n := \left\{x : f_{X_0}(x) \geq \frac{1}{n}\right\} \tag{5.2.6}$$

satisfy

$$H\left(\mathrm{supp} f_{X_0}, \mathcal{X}_n\right) \leq \frac{const.}{n^k} \quad (0 < k \leq \infty), \tag{5.2.7}$$

then

$$\sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x\in\mathrm{supp} f_{X_0}, b\in S} |R_n(g,b,x) - R(g,b,x)| = O\left(\frac{\log^\beta n}{n^{k/((d+2)(k+1))}}\right)$$

for any $\beta > 1$.

Remark. The additional assumption (5.2.7) is not too restrictive. In particular, if $\inf_{x\in\mathcal{X}} f(x) \geq const. > 0$, we can chose $k = \infty$ to obtain the "optimal" rate

$$\sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x\in \mathrm{supp} f_{X_0}, b\in S} |R_n(g,b,x) - R(g,b,x)| = O\left(\frac{\log^\beta n}{n^{1/(d+2)}}\right)$$

for any $\beta > 1$ (for the optimality of rates see, e.g., Györfi et al., 2002).

The proofs of Theorems 5.2.1 and Corollary 5.2.2 refine arguments used in the proofs of Theorems 2.2 and 3.2 in Bosq (1996). We first give the

Proof of Theorem 5.2.1. We write $\sup_{x,b}$ instead of $\sup_{x\in\mathcal{X}_n, b\in S}$.The estimation error can be decomposed into stochastic and deterministic error,

$$\begin{aligned}
&\sup_{x,b} |Z_n(g,b,x) - \phi(g,b,x)| \\
&\qquad \le \sup_{x,b} |Z_n(g,b,x) - \mathbf{E}Z_n(g,b,x)| + \sup_{x,b} |\mathbf{E}Z_n(g,b,x) - \phi(g,b,x)|.
\end{aligned}$$

1st step: Analysis of the deterministic error $\sup |\mathbf{E}Z_n - \phi|$.
Write $\mathbf{E}Z_n(g,b,x)$ as

$$\begin{aligned}
\mathbf{E}Z_n(g,b,x) &= \frac{1}{nw_n^d}\sum_{i=0}^{n-1}\int_{\mathbb{R}^d} \mathbf{E}\left[g(b,X_{i+1})K\left(\frac{x-X_i}{w_n}\right)\middle| X_i = y\right]\cdot f_{X_i}(y)dy \\
&= \int_{\mathbb{R}^d} \phi(g,b,x-zw_n)K(z)dz,
\end{aligned}$$

so that, with $\int K(z)dz = 1$,

$$\sup_{x,b} |\mathbf{E}Z_n(g,b,x) - \phi(g,b,x)| \le \sup_{x,b}\int_{\mathbb{R}^d} |\phi(g,b,x-zw_n) - \phi(g,b,x)|K(z)dz.$$

The integrand is bounded by

$$\begin{aligned}
&|\phi(g,b,x-zw_n) - \phi(g,b,x)| \\
&\qquad \le \Big|\mathbf{E}[g(b,X_1)|X_0 = x-zw_n] - \mathbf{E}[g(b,X_1)|X_0 = x]\Big| \cdot f_{X_0}(x-zw_n) \\
&\qquad\qquad + \Big|f_{X_0}(x-zw_n) - f_{X_0}(x)\Big| \cdot \mathbf{E}\left[|g(b,X_1)|\,\middle|\, X_0 = x\right] \qquad (5.2.8)\\
&\qquad \le const.\cdot \|z\|_\infty w_n,
\end{aligned}$$

where we have used (5.2.2) - (5.2.5). By assumption, the integral $\int K(z)\|z\|_\infty dz$ is finite, which yields

$$\sup_{x,b} |\mathbf{E}Z_n(g,b,x) - \phi(g,b,x)| \le const.\cdot w_n. \qquad (5.2.9)$$

Note that the constant only depends on K, f_{X_0} and on C in (5.2.2) - (5.2.5), not however on the specific $g \in \mathcal{G}$ under consideration.

2nd step: Analysis of the deterministic error $\sup |Z_n - \mathbf{E}Z_n|$.
We first cover $\mathcal{X}_n$ by $\lceil \nu_n \rceil^d$ ($\nu_n \geq 1$) cubes

$$C(j,n) := \left\{ x \,\middle|\, \|x - x_{j,n}\|_\infty \leq \frac{\mathrm{diam}(\mathcal{X}_\mathrm{n})}{2 \cdot \lceil \nu_n \rceil} \right\}$$

with side lengths $\mathrm{diam}(\mathcal{X}_n)/\lceil \nu_n \rceil$ and centres $x_{j,n}$ ($j = 1, ..., \lceil \nu_n \rceil^d$). Analogously, S is covered by $\lceil \nu_n \rceil^{d'}$ cubes

$$S(k,n) := \left\{ b \in S \,\middle|\, \|b - b_{k,n}\|_\infty \leq \frac{\mathrm{diam}(\mathrm{S})}{2 \cdot \lceil \nu_n \rceil} \right\}$$

($k = 1, ..., \lceil \nu_n \rceil^{d'}$). From this,

$$\begin{aligned}
\sup_{x,b} |Z_n(g,b,x) - \mathbf{E}Z_n(g,b,x)| &= \sup_{j,k} \sup_{x \in C(j,n), b \in S(k,n)} |Z_n(g,b,x) - \mathbf{E}Z_n(g,b,x)| \\
\leq \sup_{j,k} & \; |Z_n(g,b_{k,n},x_{j,n}) - \mathbf{E}Z_n(g,b_{k,n},x_{j,n})| \\
&+ \sup_{j,k} \sup_{x \in C(j,n), b \in S(k,n)} |Z_n(g,b,x) - Z_n(g,b_{k,n},x_{j,n})| \\
&+ \sup_{j,k} \sup_{x \in C(j,n), b \in S(k,n)} |\mathbf{E}Z_n(g,b,x) - \mathbf{E}Z_n(g,b_{k,n},x_{j,n})|.
\end{aligned}$$

On the other hand, using (5.2.2) - (5.2.4),

$$\begin{aligned}
&|Z_n(g,b,x) - Z_n(g,b_{k,n},x_{j,n})| \\
&\quad \leq \frac{1}{nw_n^d} \sum_{i=0}^{n-1} \left| g(b, X_{i+1}) - g(b_{k,n}, X_{i+1}) \right| \cdot K\left(\frac{x - X_i}{w_n}\right) \\
&\qquad + \frac{1}{nw_n^d} \sum_{i=0}^{n-1} |g(b_{k,n}, X_{i+1})| \cdot \left| K\left(\frac{x - X_i}{w_n}\right) - K\left(\frac{x_{j,n} - X_i}{w_n}\right) \right| \\
&\quad \leq const. \cdot \frac{1}{w_n^d} \cdot \|b - b_{k,n}\|_\infty + const. \cdot \frac{1}{w_n^d} \cdot \frac{\|x - x_{j,n}\|_\infty}{w_n} \\
&\quad \leq const. \cdot \frac{\max(\mathrm{diam}(\mathcal{X}_n), \mathrm{diam}(S))}{w_n^{d+1} \nu_n},
\end{aligned}$$

where without loss of generality we have assumed that $w_n \leq 1$. Again, the constant does not depend on $b \in S, x \in \mathbb{R}^d$ and $g \in \mathcal{G}$.

Using the same argument, we find that

$$|\mathbf{E}Z_n(g,b,x) - \mathbf{E}Z_n(g,b_{k,n},x_{j,n})| \leq const. \cdot \frac{\max(\mathrm{diam}(\mathcal{X}_n), \mathrm{diam}(S))}{w_n^{d+1} \nu_n},$$

and hence

$$\begin{aligned}&\sup_{x,b}|Z_n(g,b,x)-\mathbf{E}Z_n(g,b,x)|\\ &\quad\le \sup_{j,k}\ |Z_n(g,b_{k,n},x_{j,n})-\mathbf{E}Z_n(g,b_{k,n},x_{j,n})|\\ &\qquad +2\cdot const.\cdot\frac{\max(\operatorname{diam}(\mathcal{X}_n),\operatorname{diam}(S))}{w_n^{d+1}\nu_n}.\end{aligned}$$

This result yields ($0<r_n\to\infty$ will represent the desired rate of convergence at a later stage)

$$\begin{aligned}
r_n\cdot&\mathbf{E}\sup_{x,b}|Z_n(g,b,x)-\mathbf{E}Z_n(g,b,x)|\\
&\le \mathbf{E}\left(r_n\cdot\sup_{j,k}\ |Z_n(g,b_{k,n},x_{j,n})-\mathbf{E}Z_n(g,b_{k,n},x_{j,n})|\right)\\
&\qquad +2\cdot const.\cdot r_n\cdot\frac{\max(\operatorname{diam}(\mathcal{X}_n),\operatorname{diam}(S))}{w_n^{d+1}\nu_n}\\
&= \int_0^{2\|g\|_\infty\|K\|_\infty r_n/w_n^d}\mathbf{P}\left(\sup_{j,k}\ |Z_n(g,b_{k,n},x_{j,n})-\mathbf{E}Z_n(g,b_{k,n},x_{j,n})|>\frac{\epsilon}{r_n}\right)d\epsilon\\
&\qquad +2\cdot const.\cdot r_n\cdot\frac{\max(\operatorname{diam}(\mathcal{X}_n),\operatorname{diam}(S))}{w_n^{d+1}\nu_n}\\
&\le \mu+\sum_{j,k}\int_\mu^{2\|g\|_\infty\|K\|_\infty r_n/w_n^d}\mathbf{P}\left(|Z_n(g,b_{k,n},x_{j,n})-\mathbf{E}Z_n(g,b_{k,n},x_{j,n})|>\frac{\epsilon}{r_n}\right)d\epsilon\\
&\qquad +2\cdot const.\cdot r_n\cdot\frac{\max(\operatorname{diam}(\mathcal{X}_n),\operatorname{diam}(S))}{w_n^{d+1}\nu_n},
\end{aligned}\tag{5.2.10}$$

where $\mu>0$ is arbitrary.

3rd step: Combining the results of step 1 and step 2.
Assume for the moment that

$$\mathbf{P}\left(|Z_n(g,b_{k,n},x_{j,n})-\mathbf{E}Z_n(g,b_{k,n},x_{j,n})|>\frac{\epsilon}{r_n}\right)$$

has an upper bound $p_n^{(1)}(\epsilon)+p_n^{(2)}(\epsilon)$ independent of g,b and x with the following four properties:

1. We have
$$\int_\mu^\infty p_n^{(1)}(\epsilon)d\epsilon<\infty.\tag{5.2.11}$$

2. For all $\epsilon > 0$:
$$\lim_{n\to\infty} (\nu_n + 1)^{d+d'} p_n^{(1)}(\epsilon) = 0. \tag{5.2.12}$$

3. For all $\mu > 0$ there exists a $N(\mu) \in \mathbb{N}$ such that for all $\epsilon \geq \mu$ we have that
$$(\nu_n + 1)^{d+d'} p_n^{(1)}(\epsilon) \quad \text{is monotonic decreasing for } n \geq N(\mu). \tag{5.2.13}$$

4. We have
$$\lim_{n\to\infty} \quad (\nu_n + 1)^{d+d'} \int\limits_{\mu}^{2\|g\|_\infty \|K\|_\infty r_n / w_n^d} p_n^{(2)}(\epsilon) d\epsilon = 0. \tag{5.2.14}$$

We then infer from (5.2.9) and (5.2.10) that

$$\begin{aligned}
&\limsup_{n\to\infty} \quad \sup_{g\in\mathcal{G}} \quad r_n \cdot \mathbf{E} \sup_{\substack{x\in\mathcal{X}_n \\ b\in S}} |Z_n(g,b,x) - \phi(g,b,x)| \\
&\quad \leq \mu + \limsup_{n\to\infty} \quad (\nu_n + 1)^{d+d'} \int\limits_{\mu}^{\infty} p_n^{(1)}(\epsilon) d\epsilon \\
&\qquad + \limsup_{n\to\infty} \quad (\nu_n + 1)^{d+d'} \int\limits_{\mu}^{2\|g\|_\infty \|K\|_\infty r_n / w_n^d} p_n^{(2)}(\epsilon) d\epsilon \\
&\qquad + \limsup_{n\to\infty} \quad const. \cdot r_n \cdot \left(w_n + \frac{\max(\operatorname{diam}(\mathcal{X}_n), \operatorname{diam}(S))}{w_n^{d+1} \nu_n} \right).
\end{aligned}$$

The second term is zero using (5.2.11)-(5.2.13) and the monotone convergence theorem, the third term is zero because of (5.2.14). μ being arbitrary we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \quad \sup_{g\in\mathcal{G}} \quad r_n \cdot \mathbf{E} \sup_{x,b} |Z_n(g,b,x) - \phi(g,b,x)| \\
&\quad \leq \limsup_{n\to\infty} \; const. \cdot r_n \cdot \left(w_n + \frac{\max(\operatorname{diam}(\mathcal{X}_n), \operatorname{diam}(S))}{w_n^{d+1} \nu_n} \right),
\end{aligned} \tag{5.2.15}$$

from which we shall determine r_n.

4th step: Finding a bound $p_n^{(1)}(\epsilon) + p_n^{(2)}(\epsilon)$ for the 3rd step.
To this end let

$$W_{i,n} := W_{i,n}(b,x) := \frac{1}{w_n^d} \left\{ g(b, X_{i+1}) K\left(\frac{x - X_i}{w_n}\right) - \mathbf{E} g(b, X_{i+1}) K\left(\frac{x - X_i}{w_n}\right) \right\}.$$

Simple calculations yield

$$\begin{aligned}\mathbf{Var}W_{i,n} &\leq \frac{\|K\|_\infty \|f_{X_0}\|_\infty \|g\|_\infty^2}{w_n^d},\\ |\mathbf{Cov}(W_{i,n}, W_{j,n})| &\leq \|f_{X_0}\|_\infty \|g\|_\infty^2 \left(\frac{\|K\|_\infty}{w_n^d} + \|f_{X_0}\|_\infty\right),\end{aligned}$$

and we have

$$Z_n(g,b,x) - \mathbf{E}Z_n(g,b,x) = \frac{1}{n}\sum_{i=0}^{n-1} W_{i,n}. \tag{5.2.16}$$

The property of $\{X_i\}_i$ being a stationary GSM-process is inherited by $\{W_{i,n}\}_i$ whose kth α-mixing coefficient is less than $\alpha(k-1)$, the $(k-1)$st α-mixing coefficient of $\{X_i\}_i$ (independent of $b \in S, x \in \mathbb{R}^d$). To bound the tails of $Z_n - \mathbf{E}Z_n$, we exploit the expansion (5.2.16). We also use Theorem 1.3 in Bosq (1996) which states a tail inequality for empirical means of centered random variables in terms of their α-mixing coefficients:

Proposition 5.2.3. *(Bosq, 1996, Theorem 1.3) Let $\{Y_i\}_{i=-\infty}^{\infty}$ be a centered real-valued stochastic process with $\sup_{1\leq i\leq n}\|Y_i\|_\infty \leq D$. Then for any $q \in [1, n/2]$ and any $\epsilon > 0$*

$$\mathbf{P}\left(\left|\frac{1}{n}\sum_{i=0}^{n-1} Y_i\right| > \epsilon\right) \leq 4\exp\left(-\frac{\epsilon^2 q}{8v^2}\right) + 22\left(1+\frac{4D}{\epsilon}\right)^{1/2} \lceil q\rceil \alpha\left(\left\lfloor \frac{n}{2q}\right\rfloor\right),$$

where

$$\begin{aligned}p &:= \frac{n}{2q}\\ v^2 &:= \frac{2}{p^2}\sigma(q)^2 + \frac{D\epsilon}{2}\\ \sigma^2(q) &:= \max_{0\leq j\leq 2\lceil q\rceil -1} \mathbf{E}\Big(\Big(\lfloor jp\rfloor + 1 - jp\Big)Y_{\lfloor jp\rfloor+1} + Y_{\lfloor jp\rfloor+2} + \dots\\ &\qquad + Y_{\lfloor (j+1)p\rfloor} + \Big((j+1)p - \lfloor (j+1)p\rfloor\Big)Y_{\lfloor (j+1)p+1\rfloor}\Big)^2.\end{aligned}$$

The proposition will be applied to the centered GSM-process $\{W_{i,n}\}_i$. Multiplying out the sum defining $\sigma^2(q)$ we obtain at most $(p+2)^2$ terms. Above variance and covariance bounds for $W_{i,n}$ then yield that for any $q = q_n \in [1, n/2]$

$$\frac{2}{p^2}\sigma^2(q) \leq \frac{const.}{w_n^d},$$

where the constant depends on nothing but K, f_{X_0} and C from (5.2.2)-(5.2.3). Hence (set $D := C \cdot \|K\|_\infty w_n^{-d}$)

$$\mathbf{P}\left(|Z_n(g,b,x) - \mathbf{E}Z_n(g,b,x)| > \frac{\epsilon}{r_n} \right) \le p_n^{(1)}(\epsilon) + p_n^{(2)}(\epsilon)$$

with (*const.* being another suitable constant)

$$\begin{aligned} p_n^{(1)}(\epsilon) &:= 4\exp\left(-\frac{\epsilon^2 q_n w_n^d}{const. \cdot (1+\epsilon) r_n^2} \right), \\ p_n^{(2)}(\epsilon) &:= 22\left(1 + \frac{const. \cdot r_n}{\epsilon w_n^d}\right)^{1/2} \lceil q_n \rceil \alpha\left(\left\lfloor \frac{n}{2q_n} \right\rfloor - 1 \right). \end{aligned}$$

5th step: We can now move on to *finding appropriate w_n, r_n, ν_n in (5.2.15).* The crux is to satisfy (5.2.11)-(5.2.14) with the above $p_n^{(1)}$ and $p_n^{(2)}$. (5.2.11) is fulfilled because of $\int_\mu^\infty \exp(-\epsilon) d\epsilon < \infty$. Elementary calculations show that (5.2.12) and (5.2.13) are satisfied if only

$$\frac{q_n w_n^d}{r_n^2} \in \mathcal{F} \quad \text{and} \quad \nu_n^{d+d'} \in \mathcal{F} \tag{5.2.17}$$

(observe that $\{a_n\}, \{b_n\} \in \mathcal{F}$ and $0 \le \rho < 1$ implies $a_n \rho^{b_n} \searrow 0$). As to (5.2.14) we use

$$\begin{aligned} &(\nu_n + 1)^{d+d'} \int_\mu^{2\|g\|_\infty \|K\|_\infty r_n / w_n^d} p_n^{(2)}(\epsilon) d\epsilon \\ &\le 22 \cdot (\nu_n+1)^{d+d'} \cdot \lceil q_n \rceil \alpha\left(\left\lfloor \frac{n}{2q_n} \right\rfloor - 1 \right) \int_0^{2\|g\|_\infty \|K\|_\infty r_n / w_n^d} \left(1 + \frac{const. \cdot r_n}{\epsilon w_n^d}\right)^{1/2} d\epsilon \\ &\le const. \cdot (\nu_n+1)^{d+d'} \cdot \lceil q_n \rceil \alpha\left(\left\lfloor \frac{n}{2q_n} \right\rfloor - 1 \right) \cdot \frac{r_n}{w_n^d}. \end{aligned}$$

Thus (5.2.14) is satisfied if only

$$\lim_{n\to\infty} \frac{r_n \nu_n^{d+d'}}{w_n^d} \cdot \lceil q_n \rceil \alpha\left(\left\lfloor \frac{n}{2q_n} \right\rfloor - 1 \right) = 0. \tag{5.2.18}$$

So it suffices to satisfy (5.2.17) and (5.2.18). This is done by the choice

$$r_n := \frac{1}{w_n \log^\beta n} \qquad (\beta > 1)$$

$$\begin{aligned}
\nu_n &:= \frac{\mathrm{diam}(S) + \mathrm{diam}(\mathcal{X}_n)}{w_n^{d+2}} \\
w_n &:= \frac{1}{n^{1/(d+2)}} \\
q_n &:= \frac{n}{\log^a n} \qquad (2\beta - 1 > a > 1).
\end{aligned}$$

Indeed, from $q_n w_n^d / r_n^2 = \log^{2\beta - a} n \in \mathcal{F}$ and $\mathrm{diam}(\mathcal{X}_n) \in \mathcal{F}$ we have (5.2.17). Moreover,

$$\begin{aligned}
&\frac{r_n \nu_n^{d+d'}}{w_n^d} \cdot \lceil q_n \rceil \alpha \left(\left\lfloor \frac{n}{2q_n} \right\rfloor - 1 \right) \\
&\quad \le const. \cdot (\mathrm{diam}(S) + \mathrm{diam}(\mathcal{X}_n))^{d+d'} \cdot \frac{n^{1+d+d'+(1+d)/(2+d)}}{\log^{a+\beta} n} \cdot \alpha \left(\left\lfloor \frac{\log^a n}{2} \right\rfloor - 1 \right),
\end{aligned}$$

and the GSM-property yields (5.2.18) (observe again that $\{a_n\}, \{b_n\} \in \mathcal{F}$ and $0 \le \rho < 1$ implies $a_n \rho^{b_n} \searrow 0$).

Finally, (5.2.15) now reads

$$\limsup_{n\to\infty} \sup_{g\in\mathcal{G}} r_n \cdot \mathbf{E} \sup_{x,b} |Z_n(g,b,x) - \phi(g,b,x)| \le \limsup_{n\to\infty} \ const. \cdot \frac{1}{\log^\beta n} = 0. \quad \square$$

The proof of Corollary 5.2.2 is more straightforward.

Proof of Corollary 5.2.2. Set $\mathcal{X} = \mathrm{supp}\, f_{X_0}$ and $\mathcal{X}_n' := \mathcal{X}_{n^\gamma}$, where $\gamma > 0$ is adjusted later. We write $\sup_{x,b}$ instead of $\sup_{x\in\mathcal{X}, b\in S}$. Clearly,

$$\begin{aligned}
&\sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x,b} |R_n(g,b,x) - R(g,b,x)| \\
&\quad \le \sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x\in\mathcal{X}_n', b\in S} |R_n(g,b,x) - R(g,b,x)| \\
&\qquad + \sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x,b} \inf_{x^*\in\mathcal{X}_n'} |R_n(g,b,x) - R_n(g,b,x^*)| \\
&\qquad + \sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x,b} \inf_{x^*\in\mathcal{X}_n'} |R(g,b,x) - R(g,b,x^*)|.
\end{aligned}$$

Condition (5.2.7) implies

$$\sup_{x\in\mathcal{X}} \inf_{x^*\in\mathcal{X}_n'} \|x - x^*\|_\infty \le const. \cdot n^{-k\gamma}.$$

Using (5.2.2)-(5.2.5), we can bound the second and the third term from above by

$$const. \cdot \sup_{x\in\mathcal{X}} \inf_{x^*\in\mathcal{X}_n'} \|x - x^*\|_\infty \le const. \cdot n^{-k\gamma}.$$

Using Theorem 5.2.1, the first term satisfies

$$\begin{aligned}
&\sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x\in\mathcal{X}_n', b\in S} |R_n(g,b,x) - R(g,b,x)| \\
&\quad = \sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x\in\mathcal{X}_n', b\in S} \left| \left(R_n(g,b,x) - \frac{Z_n(g,b,x)}{f_{X_0}(x)} \right) + \left(\frac{Z_n(g,b,x)}{f_{X_0}(x)} - \frac{\phi(g,b,x)}{f_{X_0}(x)} \right) \right| \\
&\quad \le \sup_{g\in\mathcal{G}} \mathbf{E} \frac{\sup_{x\in\mathcal{X}_n', b\in S} |R_n(g,b,x)|}{\inf_{x\in\mathcal{X}_n'} f_{X_0}(x)} \cdot \sup_{x\in\mathcal{X}_n', b\in S} |f_{X_0}(x) - Z_n(\mathbf{1},b,x)| \\
&\qquad\quad + \frac{1}{\inf_{x\in\mathcal{X}_n'} f_{X_0}(x)} \quad \sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x\in\mathcal{X}_n', b\in S} |Z_n(g,b,x) - \phi(g,b,x)| \\
&\quad \le const. \cdot n^\gamma \frac{\log^\beta n}{n^{1/(d+2)}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sup_{g\in\mathcal{G}} \mathbf{E} \sup_{x,b} |R_n(g,b,x) - R(g,b,x)| \\
&\qquad = O\left(\frac{\log^\beta n}{n^{1/(d+2)-\gamma}} + \frac{1}{n^{k\gamma}} \right) = O\left(\frac{\log^\beta n}{n^{k/((d+2)(k+1))}} \right),
\end{aligned}$$

the latter equality for the balanced choice $\gamma = 1/((d+2)(k+1))$. □

5.3 Proving the optimality of the strategy

With the results of the previous section, we are in a position to give the

Proof of Theorem 5.1.1. Analyzing the strategy $\{\hat{b}_i\}_i$, our first task is to derive an inequality as (4.2.12) for $\hat{h}_i$ instead of h_i and $\hat{b}_i$ instead of b_i^*.

From the empirical Bellman equation, we find that

$$\begin{aligned}
&\hat{h}_{i+1}(b_i, \bar{X}_{i+1}) \\
&\quad = \max_{b\in S(b_i,\bar{X}_{i+1})} \sum_{j=-d+1}^{i} \Big(f(b, \bar{X}_{j+1}) + (1-\delta_{i+1})\hat{h}_{i+1}(b, \bar{X}_{j+1}) \Big) K_{i+1}(\bar{X}_j, \bar{X}_{i+1}) \\
&\quad \ge \sum_{j=-d+1}^{i} \Big(f(b_{i+1}, \bar{X}_{j+1}) + (1-\delta_{i+1})\hat{h}_{i+1}(b_{i+1}, \bar{X}_{j+1}) \Big) K_{i+1}(\bar{X}_j, \bar{X}_{i+1})
\end{aligned}$$

for any admissible portfolio strategy $\{b_i\}_i$. Equality holds for $\{\hat{b}_i\}_i$, which yields

$$m(\hat{b}_{i+1}, \bar{X}_{i+1}) - m(b_{i+1}, \bar{X}_{i+1}) \tag{5.3.1}$$

$$\begin{aligned}
&\geq H_i^{(1)}(b_{i+1}) + H_i^{(2)}(b_{i+1}) - \hat{h}_{i+1}(b_i, \bar{X}_{i+1}) + (1-\delta_{i+1})\mathbf{E}[\hat{h}_{i+1}(b_{i+1}, \bar{X}_{i+2})|\mathcal{F}_{i+1}] \\
&\quad - H_i^{(1)}(\hat{b}_{i+1}) - H_i^{(2)}(\hat{b}_{i+1}) + \hat{h}_{i+1}(\hat{b}_i, \bar{X}_{i+1}) - (1-\delta_{i+1})\mathbf{E}[\hat{h}_{i+1}(\hat{b}_{i+1}, \bar{X}_{i+2})|\mathcal{F}_{i+1}]
\end{aligned}$$

with

$$H_i^{(1)}(b) := \sum_{j=-d+1}^{i} f(b, \bar{X}_{j+1}) K_{i+1}(\bar{X}_j, \bar{X}_{i+1}) - m(b, \bar{X}_{i+1}),$$

and

$$H_i^{(2)}(b) := (1-\delta_{i+1}) \left(\sum_{j=-d+1}^{i} \hat{h}_{i+1}(b, \bar{X}_{j+1}) K_{i+1}(\bar{X}_j, \bar{X}_{i+1}) - \mathbf{E}[\hat{h}_{i+1}(b, \bar{X}_{i+2})|\mathcal{F}_{i+1}] \right).$$

Now, we investigate the asymptotics of the terms $H_i^{(1)}$ and $H_i^{(2)}$ in (5.3.1). Clearly,

$$\sup_{b \in S} |H_i^{(1)}(b)| \leq \sup_{b \in S, \bar{x} \in [A,B]^{dm}} \left| \sum_{j=-d+1}^{i} f(b, \bar{X}_{j+1}) K_{i+1}(\bar{X}_j, \bar{x}) - m(b, \bar{x}) \right|. \quad (5.3.2)$$

To analyse $H_i^{(2)}$ we define a $\mathcal{F}_i$-measurable random variable c_i,

$$c_i := \arg\min_{c \in \mathbb{R}} \|\hat{h}_i - h_i + c\|_\infty,$$

and obtain

$$\begin{aligned}
&\sup_{b \in S} |H_i^{(2)}(b)| \\
&\quad \leq \sup_{b \in S} \left| \sum_{j=-d+1}^{i} h_{i+1}(b, \bar{X}_{j+1}) K_{i+1}(\bar{X}_j, \bar{X}_{i+1}) - \mathbf{E}[h_{i+1}(b, \bar{X}_{i+2})|\mathcal{F}_{i+1}] \right| \\
&\qquad + \left| \sum_{j=-d+1}^{i} (\hat{h}_{i+1}(...) - h_{i+1}(...) + c_{i+1}) K_{i+1}(...) \right. \\
&\qquad\qquad \left. - \mathbf{E}[\hat{h}_{i+1}(...) - h_{i+1}(...) + c_{i+1}|\mathcal{F}_{i+1}] \right| \\
&\quad \leq \sup_{b \in S, \bar{x} \in [A,B]^{dm}} \left| \sum_{j=-d+1}^{i} h_{i+1}(b, \bar{X}_{j+1}) K_{i+1}(\bar{X}_j, \bar{x}) \right. \\
&\qquad\qquad \left. - \mathbf{E}[h_{i+1}(b, \bar{X}_{i+2})|\bar{X}_{i+1} = \bar{x}] \right| + 2\|\hat{h}_{i+1} - h_{i+1} + c_{i+1}\|_\infty \\
&\quad \leq \sup_{b \in S, \bar{x} \in [A,B]^{dm}} \left| \sum_{j=-d+1}^{i} h_{i+1}(b, \bar{X}_{j+1}) K_{i+1}(\bar{X}_j, \bar{x}) \right.
\end{aligned}$$

$$-\mathbf{E}[h_{i+1}(b,\bar{X}_{i+2})|\bar{X}_{i+1}=\bar{x}]\Big| + \|\hat{h}_{i+1}-h_{i+1}\|. \tag{5.3.3}$$

For this, recall the norm $\|\cdot\| = 2\inf_{c\in\mathbb{R}}\|\cdot+c\|_\infty$ on $C(S\times[A,B]^{dm})$. By the contraction property of $\hat{M}_i$,

$$\begin{aligned}\|\hat{h}_i-h_i\| &= \|\hat{M}_i\hat{h}_i-M_ih_i\|\\ &\le \|\hat{M}_i\hat{h}_i-\hat{M}_ih_i\|+\|\hat{M}_ih_i-M_ih_i\|\\ &\le (1-\delta_i)\|\hat{h}_i-h_i\|+2\|\hat{M}_ih_i-M_ih_i\|_\infty,\end{aligned}$$

and hence

$$\|\hat{h}_i-h_i\| \le \frac{2}{\delta_i}\|\hat{M}_ih_i-M_ih_i\|_\infty. \tag{5.3.4}$$

It is easily established that

$$\begin{aligned}&\|\hat{M}_ih_i-M_ih_i\|_\infty \qquad (5.3.5)\\ &\le \sup_{b\in S,\bar{x}\in[A,B]^{dm}}\left|\sum_{j=-d+1}^{i-1} f(b,\bar{X}_{j+1})K_i(\bar{X}_j,\bar{x})-m(b,\bar{x})\right|\\ &\quad+(1-\delta_i)\sup_{b\in S,\bar{x}\in[A,B]^{dm}}\left|\sum_{j=-d+1}^{i-1} h_i(b,\bar{X}_{j+1})K_i(\bar{X}_j,\bar{x})-\mathbf{E}[h_i(b,\bar{X}_{d+1})|\bar{X}_d=\bar{x}]\right|.\end{aligned}$$

As a consequence of (5.3.2)-(5.3.5) we obtain

$$\begin{aligned}&\mathbf{E}\sup_{b\in S}\left|H_i^{(1)}+H_i^{(2)}\right|\\ &\le \left(1+\frac{2}{\delta_{i+1}}\right)\mathbf{E}\sup_{b\in S,\bar{x}\in[A,B]^{dm}}\left|\sum_{j=-d+1}^{i} f(b,\bar{X}_{j+1})K_{i+1}(\bar{X}_j,\bar{x})-m(b,\bar{x})\right|\\ &\quad+\left(1+\frac{2}{\delta_{i+1}}(1-\delta_{i+1})\right)\mathbf{E}\sup_{b\in S,\bar{x}\in[A,B]^{dm}}\Bigg|\sum_{j=-d+1}^{i} h_{i+1}(b,\bar{X}_{j+1})K_{i+1}(\bar{X}_j,\bar{x})\\ &\qquad\qquad -\mathbf{E}[h_{i+1}(b,\bar{X}_{i+2})|\bar{X}_{i+1}=\bar{x}]\Bigg|\\ &\le \frac{const.}{\delta_{i+1}^2}\sup_{g\in\mathcal{G}}\mathbf{E}\sup_{b\in S,\bar{x}\in[A,B]^{dm}}\Bigg|\sum_{j=-d+1}^{i} g(b,\bar{X}_{j+1})K_{i+1}(\bar{X}_j,\bar{x})\\ &\qquad\qquad -\mathbf{E}[g(b,\bar{X}_{i+2})|\bar{X}_{i+1}=\bar{x}]\Bigg|,\end{aligned}$$

and under the assumption (5.1.6) of the theorem we find that

$$\mathbf{E}\sup_{b\in S}\left|H_i^{(1)}+H_i^{(2)}\right|\to 0\quad (i\to\infty).$$

Calculating expectations, summation $\frac{1}{n}\sum_{i=-1}^{n-2}\ldots$ and taking $\liminf_{n\to\infty}$ in (5.3.1) we end up with

$$\begin{aligned}
&\liminf_{n\to\infty}\mathbf{E}\left(\frac{1}{n}\sum_{i=0}^{n-1}m(\hat{b}_i,\bar{X}_i)\right)-\mathbf{E}\left(\frac{1}{n}\sum_{i=0}^{n-1}m(\tilde{b}_i,\bar{X}_i)\right)\\
&\quad\geq\liminf_{n\to\infty}\left\{\mathbf{E}\frac{1}{n}\sum_{i=-1}^{n-2}\left(\hat{h}_{i+1}(\tilde{b}_{i+1},\bar{X}_{i+2})-\hat{h}_{i+1}(\tilde{b}_i,\bar{X}_{i+1})\right)\right.\\
&\quad\quad-\mathbf{E}\frac{1}{n}\sum_{i=-1}^{n-2}\left(\hat{h}_{i+1}(\hat{b}_{i+1},\bar{X}_{i+2})-\hat{h}_{i+1}(\hat{b}_i,\bar{X}_{i+1})\right)\\
&\quad\quad\left.+\mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1}\delta_i\left(\hat{h}_i(\hat{b}_i,\bar{X}_{i+1})-\hat{h}_i(\tilde{b}_i,\bar{X}_{i+1})\right)\right\} \qquad (5.3.6)\\
&\quad=:\liminf_{n\to\infty}\{D_i\},
\end{aligned}$$

$\{\tilde{b}_i\}_i$ being the periodic strategy from Lemma 4.2.6. This is the analogue of (4.2.12) we were looking for.

Finally, we observe that

$$\begin{aligned}
\hat{h}_{i+1}(b_{i+1},\bar{X}_{i+2})-\hat{h}_{i+1}(b_i,\bar{X}_{i+1})&=\left(h_{i+1}(b_{i+1},\bar{X}_{i+2})-h_{i+1}(b_i,\bar{X}_{i+1})\right)\\
&\quad+\left(\hat{h}_{i+1}(b_{i+1},\bar{X}_{i+2})-h_{i+1}(b_{i+1},\bar{X}_{i+2})+c_{i+1}\right)\\
&\quad+\left(h_{i+1}(b_i,\bar{X}_{i+1})-\hat{h}_{i+1}(b_i,\bar{X}_{i+1})-c_{i+1}\right), \qquad (5.3.7)
\end{aligned}$$

where the expectation of the absolute value of the sum of the last two brackets is bounded from above by

$$2\mathbf{E}\|\hat{h}_{i+1}-h_{i+1}+c_{i+1}\|_\infty=\mathbf{E}\|\hat{h}_{i+1}-h_{i+1}\|\to 0\quad (i\to\infty), \qquad (5.3.8)$$

using (5.3.4), (5.3.5) and the assumption (5.1.6) of the theorem. (5.3.7) and (5.3.8) show that for the purpose of the asymptotical inference in (5.3.6), we can replace $\hat{h}_{i+1}$ by h_{i+1} in the definition of D_i. We can then argue exactly in the same way as in the proof of Theorem 4.2.1 (starting from (4.2.12)) to obtain the optimality relation

$$\liminf_{n\to\infty}\left(\mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1}m(\hat{b}_i,\bar{X}_i)-\mathbf{E}\frac{1}{n}\sum_{i=0}^{n-1}m(b_i,\bar{X}_i)\right)\geq 0. \qquad \square$$

It remains to prove Theorem 5.1.2:

Proof of Theorem 5.1.2. This will be done by application of Corollary 5.2.2 for the process $\{\bar{X}_i\}_i$. Clearly, the GSM-property of $\{X_i\}_i$ makes $\{\bar{X}_i\}_i$ a GSM-process, too.

Set $\mathcal{G} := \{\delta_i \cdot h_i, \delta_i \cdot f | i = 1, 2, ...\}$. Lemma 4.2.3 implies that $\|g\|_\infty \leq \|f\|_\infty$ for all $g \in \mathcal{G}$. By assumption 3 of the theorem the requirements of Proposition 4.3.1 are met so that we can find a constant $C > 0$ with

$$|g(s, \bar{x}) - g(t, \bar{x})| \leq C\|s - t\|_\infty \tag{5.3.9}$$

for all $g = \delta_i \cdot h_i \in \mathcal{G}, \bar{x} \in [A, B]^{dm}$ and $s, t \in S$. Increasing C to at least the Lipschitz constant of f, (5.3.9) holds for $\delta_i \cdot f$ as well. Thus, the conditions (5.2.2)-(5.2.4) of Theorem 5.2.1 are fulfilled for the class $\mathcal{G}$. (5.2.5) holds because of $f_{X_0|\bar{X}_0}$ being Lipschitz continuous, and we conclude

$$\frac{1}{\delta_{i+1}^2} \sup_{g \in \mathcal{G}} \mathbf{E} \sup_{b \in S, \bar{x} \in [A,B]^{dm}} \left| \sum_{j=-d+1}^{i} g(b, \bar{X}_{j+1}) K_{i+1}(\bar{X}_j, \bar{x}) \right. \\ \left. - \mathbf{E}[g(b, \bar{X}_{i+2}) | \bar{X}_{i+1} = \bar{x}] \right| \leq const. \cdot \frac{1}{\delta_{i+1}^2} \cdot \frac{\log i}{i^\alpha}$$

for some $\alpha > 0$. By assumption 4 on $\{\delta_i\}_i$ in Theorem 5.1.2, we find that

$$\frac{1}{\delta_{i+1}^2} \cdot \frac{\log i}{i^\alpha} \longrightarrow 0 \quad (i \to \infty),$$

which finally yields the assertion. $\square$

CHAPTER 6

Portfolio selection functions in stationary return processes

In Chapter 5 we considered d-stage Markov processes in which portfolio selection could be done on the basis of the returns on the last d market days. By the Markov property, the d-past completely characterized the stochastic regime of the next market day. However, more general return processes $\{X_i\}_i$, such as merely stationary and ergodic processes will fail to have the Markov property. Then, in principle, the investor is forced to evaluate the conditional log-optimal portfolio $b^*(X_n, ..., X_0)$ given the past returns $X_0, ..., X_n$ starting from day zero. As we will explain in Section 6.1, this is not always feasible. Choosing a portfolio as a function of the d-past works well in Markov return processes. Hence it is a natural modification of the conditional log-optimal strategy to consider log-optimal *portfolio selection functions* $f_{opt}(\cdot) : \mathbb{R}_0^{dm} \to S$ in stationary ergodic return processes as well. These choose a portfolio from the portfolio simplex S on the basis of the asset returns during the last d market periods (Section 6.1). Log-optimal portfolio selection functions can only be calculated if one happens to know the underlying return distribution. Otherwise the investor has to rely on estimates.

Section 6.2 describes an estimator $Z_n^L(\cdot)$ for a log-optimal portfolio selection function, with strong consistency results given in Lemma 6.2.1 and Theorem 6.2.2. The estimator works sequentially: the return data of the stocks is included in the estimation process as it emerges. The central question is how an investor using the estimated log-optimal portfolio selection function $Z_n^L(\cdot)$ competes with other investors using different portfolio selection functions. As we will see, repeated investment according to the estimate is optimal among all investment strategies based on portfolio selection functions of the last d market periods. In particular, it performs no worse than the unknown log-optimal portfolio selection function f_{opt} itself (Corollary 6.2.3).

In all this, $L > 0$ is a parameter of the underlying stock market characterizing its regularity properties beyond stationarity and ergodicity. L is unknown to the investor. To avoid this drawback in the cases relevant for practical application, an adaptive estimator $Z_n(\cdot)$ is constructed that does not require explicit knowledge of L. This estimator features the same convergence properties as $Z_n^L(\cdot)$ (Theorem 6.2.4), making it most appealing for application.

In Section 6.3 we prove the results of the preceding sections, and the chapter is concluded by several simulations and examples (Section 6.4).

6.1 Portfolio selection functions

Let $T \in \mathbb{N}$ be fixed and $\{X_i\}_{i=-T}^{\infty}$ an $[a,b]^m$-valued ($0 < a \leq b < \infty$), stationary and ergodic process of return vectors in a market of m shares. As usual, at time i, the return process up to and including X_i has been observed.

It is natural for the investor to choose his investment portfolio on the basis of recently observed returns, say on the basis of the last $d \in \mathbb{N}$ market periods ($d \leq T$ fixed throughout). If investment performance is assessed on the basis of logarithmic utility, the investor's aim is to find a **log-optimal portfolio selection function** of the d-past, i.e., a measurable function

$$b : [a,b]^{dm} \longrightarrow S := \{s \in \mathbb{R}^m : \sum_{j=1}^{m} s_j = 1, s_j \geq 0\}$$

such that ($< \cdot, \cdot >$ denoting the Euclidean scalar product)

$$\mathbf{E}\left(\log < b(X_{-d}, ..., X_{-1}), X_0 >\right) \geq \mathbf{E}\left(\log < f(X_{-d}, ..., X_{-1}), X_0 >\right) \qquad (6.1.1)$$

for all measurable $f : [a,b]^{dm} \longrightarrow S$. At time $n \in \mathbb{N}_0$, b advises the investor to allocate his wealth to the single shares according to the portfolio $b(X_{n-d+1}, ..., X_n) = b(\bar{X}_{n+1})$, where $\bar{X}_n$ is a shorthand notation for the d-past $(X_{n-d}, ..., X_{n-1}) \in \mathbb{R}_+^{dm}$ of X_n.

In contrast to the conditional log-optimal portfolio b^*, which is a function of all past return data, we only include the last d observations. This brings about several advantages:

– It is plausible that one should drop observations from the far-away past if the stationarity of the market is not clear. Outdated observations (under non-stationarity they appear to be drawn from a "wrong" distribution) may endanger the performance of the conditional log-optimal portfolio. We conjecture that portfolio selection functions are less sensitive to deviations from stationarity of the return process. Finding empirical evidence or counterevidence for this, however, is beyond the scope of this thesis.

– Each market has a specific log-optimal portfolio selection function, which always remains the same in stationary markets. Log-optimal portfolio selection functions are therefore much easier to interpret than conditional log-optimal portfolios (which are not much of a single market characteristic, but a sequence of functions). The shape of a log-optimal portfolio selection function allows us to find structures in the stock quote chart that should be interpreted as "buy", "hold" and "sell" signals for the single stocks. This makes log-optimal portfolio selection functions a theoretically well-founded counterpart of heuristic chart analysis (as presented, e.g., in Möller, 1998).

– As already noted in Chapter 1, estimation of b^* from market data is highly problematic. As we shall see, estimation of log-optimal portfolio selection functions, however, can exploit recent, powerful nonparametric regression estimation algorithms in stationary ergodic processes.

In order to find a log-optimal portfolio selection function, we observe that

$$\mathbf{E}\left(\log < b(X_{-d}, ..., X_{-1}), X_0 >\right) = \int \mathbf{E}\left[\log < b(\bar{x}), X_0 > |\bar{X}_0 = \bar{x}\right] \mathbf{P}_{\bar{X}_0}(d\bar{x}).$$

Hence, it suffices to consider pointwise maximization of

$$R(s, \bar{x}) := \mathbf{E}\left[\log < s, X_0 > |\bar{X}_0 = \bar{x}\right]$$

for fixed $\bar{x}$. Here and in the sequel, the quantities s and $\bar{x}$ are to be implicitly understood as $s \in S$ and $\bar{x} \in \mathbb{R}_+^{dm}$.

Let $KT(\bar{x}) \subseteq S$ denote the set of Kuhn-Tucker-points (cf. Foulds, 1981), i.e. the set of solutions of the convex maximization problem

$$R(s, \bar{x}) \longrightarrow \max_{s \in S}! \tag{6.1.2}$$

Because of continuity of $R(\cdot, \bar{x})$ and because of S being a compact set, $KT(\bar{x}) \neq \emptyset$, and the existence of solutions to (6.1.2) as well as to (6.1.1) is guaranteed.

$$R^*(\bar{x}) := \max_{s \in S} R(s, \bar{x})$$

is the maximum of the target function and

$$R_{max} := \int R^*(\bar{x}) \mathbf{P}_{\bar{X}_0}(d\bar{x}) = \max_{b:[a,b]^{dm} \to S} \mathbf{E} \left(\log < b(\bar{X}_0), X_0 > \right)$$

denotes the maximal expected logarithmic return.

To solve this maximization problem with historical return data (the return distribution being unknown), Walk and Yakowitz (1999) and Walk (2000) have suggested recursive estimation of log-optimal portfolio selection functions. For this, we will use a nonparametric, strongly consistent regression estimation scheme (i.e., with probability one, the estimates converge to the true regression function in the pointwise sense). For a detailed overview of nonparametric regression estimation (for non-i.i.d., i.e. dependent data), we refer the reader to Bosq (1996), Härdle (1990) and Härdle et al. (1998).

Until recently, for non-i.i.d. data, only nonparametric regression estimators were available for which strong consistency was linked up with appropriate mixing conditions (Györfi, Härdle et al., 1989, and Bosq, 1996). These ensure a suitable decay of dependency in the data. Others made regularity assumptions on conditional densities of the process variables (Laib, 1999, Laib and Ould-Said, 1996). On the other hand, the examples in Györfi et al. (1998) show that we actually have to impose some regularity conditions on the process in order to be able to obtain strong consistency. Correspondingly, the consistency proofs for the estimated log-optimal portfolio selection functions in Walk and Yakowitz (1999) and Walk (2000) also rely on mixing conditons.

However, mixing conditions can hardly be verified from observational data using some statistcal testing procedure. Now, with the work of Yakowitz, Györfi et al. (1999) an algorithm has been proposed that achieves strong consistency under other conditions than mixing. The mixing requirement, ensuring a suitable decay of dependency in the data, was replaced by a condition on the finite dimensional (more precisely the d-dimensional) distribution of the process, namely a Lipschitz condition on the regression function. In particular, processes featuring long-range dependence (which must be expected in financial data, see e.g. Ding et al., 1993; Peters, 1997) are not precluded from consideration as they would have been under mixing conditions.

In this chapter, the estimator of Yakowitz, Györfi et al. (1999) is combined with a stochastic projection algorithm (Kushner and Clark, 1978) to obtain a strongly consistent sequential estimator for a log-optimal portfolio selection function of the d-past of the return process. The mixing conditions in Walk and Yakowitz (1999) and Walk (2000) are replaced by a Lipschitz condition on the gradient of the target function.

6.2 Estimation of log-optimal portfolio selection functions

Throughout this chapter we assume that the following regularity conditions V1 and V2 hold:

V1: $\{X_i\}_{i=-T}^{\infty}$ ($T \in \mathbb{N}$) is an $[a,b]^m$-valued stationary ergodic stochastic process on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ ($0 < a \leq b < \infty$ need not be known explicitly). Some $d \in \mathbb{N}$ ($d \leq T$) is fixed.

V2: The gradient of the target function $R(s, \bar{x})$,

$$m(s,\bar{x}) := \mathbf{E}\left[\frac{X_0}{< s, X_0 >}\middle|\bar{X}_0 = \bar{x}\right]$$

(which we already know from the Kuhn-Tucker conditions in Theorem 1.3.3), is a Lipschitz continuous function of $\bar{x}$ with Lipschitz constant $L/\sqrt{md}$, i.e.

$$|m(s,\bar{x}) - m(s,\bar{y})| \leq \frac{L}{\sqrt{md}}|\bar{x} - \bar{y}| \qquad \text{for all } \bar{x}, \bar{y} \in \mathbb{R}_+^{dm}, s \in S.$$

Condition V2 is fulfilled if the conditional distribution $\mathbf{P}_{X_0|\bar{X}_0}$ has a density $f_{X_0|\bar{X}_0}(x_0, \bar{x})$ with respect to some measure μ, such that

$$|f_{X_0|\bar{X}_0}(x_0,\bar{x}) - f_{X_0|\bar{X}_0}(x_0,\bar{y})| \leq \frac{La}{m\sqrt{d} \cdot b\,\mu([a,b]^m)}|\bar{x} - \bar{y}|$$

(note the similarity to the Lipschitz conditions of Theorem 5.1.2). In particular, this holds if $f_{X_0|\bar{X}_0}$ is continuously differentiable. Hence, V2 is a condition on the variability of the return vectors and as such a condition on the risk inherent in the market.

At time n, the investor's task is to produce an estimate $Z_n^L(\bar{x})$ of the value of a log-optimal portfolio selection function given the last d observed return vectors are $\bar{x} \in \mathbb{R}^{dm}$. This can be done by the following **projection algorithm**:

1. Before we start the estimation process, we fix a partition $\mathcal{P}_k$ of $\mathbb{R}_+^{dm}$ into cubes of volume $(2^{-k-2})^{dm}$ for each positive integer k. For $\bar{x} \in \mathbb{R}_+^{dm}$ the element of $\mathcal{P}_k$ containing $\bar{x}$ is denoted by $A_k(\bar{x})$. We also fix some sequence $\alpha_n > 0$ $(n \in \mathbb{N})$.

2. Then, at time n, we calculate a partitioning regression estimate of the gradient $m(s, \bar{x})$ of the target function (Yakowitz, Györfi et al., 1999): More precisely, for $N_n \in \mathbb{N}$ with $\lim_{n\to\infty} N_n = \infty$ and $\bar{X}_j := (X_{j-d}, ..., X_{j-1})$ we construct the gradient estimate by

$$\hat{m}_{n,L}(s, \bar{x}) = \hat{M}_{1,n}(s, \bar{x}) + \sum_{k=1}^{N_n} \hat{\Delta}_{k,n,L}(s, \bar{x}),$$

   using

$$\hat{M}_{k,n}(s, \bar{x}) := \left(\sum_{j=-M}^{n} \frac{X_j \cdot \mathbf{1}_{A_k(\bar{x})}(\bar{X}_j)}{< s, X_j >} \right) \Big/ \left(\sum_{j=-M}^{n} \mathbf{1}_{A_k(\bar{x})}(\bar{X}_j) \right), \tag{6.2.1}$$
$$\hat{\Delta}_{k,n,L}(s, \bar{x}) := T_{L2^{-k}}(\hat{M}_{k,n}(s, \bar{x}) - \hat{M}_{k-1,n}(s, \bar{x})).$$

   Here, $M := T - d \in \mathbb{N}_0$ is the length of the training period of the algorithm before the first estimate is produced. $T_{L2^{-k}}$ denotes the truncation operator, defined for $z = (z_1, ...z_m) \in \mathbb{R}^m$ by $T_{L2^{-k}}z = (w_1, ...w_m)$ with $w_i := \operatorname{sgn} z_i \cdot \min\{|z_i|, L2^{-k}\}$.

3. Having obtained an estimate of the gradient of the target function, we apply the classical projection algorithm to estimate the maximum in (6.1.2) (Kushner and Clark, 1978, Sec. 5.3, also used in different form by Walk and Yakowitz, 1999): From the previous estimate $Z_{n-1}^L(\bar{x})$ we calculate an updated estimate by

$$Z_n^L(\bar{x}) := \Pi\left(Z_{n-1}^L(\bar{x}) + \alpha_n \hat{m}_{n,L}(Z_{n-1}^L(\bar{x}), \bar{x})\right). \tag{6.2.2}$$

Here, for $x \in \mathbb{R}^m$, $\Pi(x)$ denotes the best approximating (in the Euclidean norm) element of x in the simplex S, i.e. the projection of x on S. To start the iteration at time $n = 0$, we use an arbitrary starting estimate $Z^L_{-1}(\bar{x}) \in S$.

Note that we assume L to be known. At a later stage the algorithm is modified so as to comprise an adaptive choice of the market parameter L, which then allows estimation without knowledge of the precise value of L.

The following lemma featuring the basic convergence properties of the estimate will be crucial to the main results of this chapter. It shows that our estimates approach the set $KT(\bar{x})$ of Kuhn-Tucker points of (6.1.2), i.e., the collection of values of log-optimal portfolio-selection functions at $\bar{x}$.

Lemma 6.2.1. *Let $\rho\left(Z^L_n(\bar{x}), KT(\bar{x})\right) := \inf_{y \in KT(\bar{x})} \|Z^L_n(\bar{x}) - y\|$ denote the Euclidean distance of $Z^L_n(\bar{x})$ from the set $KT(\bar{x})$. Then, under the assumptions*

1. *V1 and V2,*
2. *$\alpha_n \longrightarrow 0$ $(n \to \infty)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,*

we have that for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$:

$$\lim_{n \to \infty} \rho\left(Z^L_n(\bar{x}), KT(\bar{x})\right) = 0 \quad \mathbf{P} - a.s..$$

To formulate this result more neatly, let $Z^{L*}_n(\bar{x})$ denote the best approximating (in the Euclidean metric) element of $Z^L_n(\bar{x})$ in $KT(\bar{x})$ (observe that $KT(\bar{x})$ is compact). Note that $Z^{L*}_n(\bar{x})$ is a log-optimal portfolio selection function as $\bar{x}$ varies. Then Lemma 6.2.1 can be rephrased more explicitly as

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Theorem 6.2.2. *Under the assumptions of Lemma 6.2.1 one has*

1. *Pointwise strong consistency for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$:*

$$|Z^L_n(\bar{x}) - Z^{L*}_n(\bar{x})| \longrightarrow 0 \quad (n \to \infty) \quad \mathbf{P} - a.s.$$

and

$$R(Z^L_n(\bar{x}), \bar{x}) \longrightarrow R^*(\bar{x}) \quad (n \to \infty) \quad \mathbf{P} - a.s.. \tag{6.2.3}$$

2. *Strong L_1-consistency:*

$$\int |Z_n^L(\bar{x}) - Z_n^{L*}(\bar{x})| \mathbf{P}_{\bar{X}_0}(d\bar{x}) \longrightarrow 0 \quad (n \to \infty) \quad \mathbf{P} - a.s.$$

and

$$\int R(Z_n^L(\bar{x}), \bar{x}) \mathbf{P}_{\bar{X}_0}(d\bar{x}) \longrightarrow R_{max} \quad (n \to \infty) \quad \mathbf{P} - a.s., \tag{6.2.4}$$

hence also in $L_r(\mathbf{P})$ for any $r \in \mathbb{N}$.

The limit relations (6.2.3) and (6.2.4) are the central results for the proposed estimation procedure. They demonstrate that in the long run $Z_n^L(\bar{x})$ almost surely achieves the optimal expected growth of wealth among all strategies based on portfolio selection functions of the d-past.

Remark concerning Lemma 6.2.1 and Theorem 6.2.2. The limit relations in Lemma 6.2.1 and Theorem 6.2.2, part 1 are true even in the stronger sense that a fixed exceptional null set of $\omega \in \Omega$ and a fixed exceptional null set of $\bar{x} \in [a,b]^{dm}$ exist, outside which for all ω and $\bar{x}$

$$\begin{aligned} &\rho\left(Z_n^L(\bar{x}), KT(\bar{x})\right) \longrightarrow 0, \\ &|Z_n^L(\bar{x}) - Z_n^{L*}(\bar{x})| \longrightarrow 0 \quad \text{and} \\ &R(Z_n^L(\bar{x}), \bar{x}) \longrightarrow R^*(\bar{x}) \end{aligned}$$

as $n \to \infty$ (cf. proof of Theorem 6.2.2).

Until now, we have merely considered the problem how to *estimate* a log-optimal portfolio selection function. Of course this is not the primary task of the investor. He would like to actually *use* a log-optimal portfolio selection function or – in case such a function is unknown – the estimates Z_n^L to rebalance his investment porfolio. At time $i \in \mathbb{N}_0$, $Z_i^L(\bar{x})$ is the most recent estimate of a log-optimal portfolio selection function $b(\bar{x})$ in (6.1.1). The investor therefore takes $Z_i^L(\bar{X}_{i+1}) = Z_i^L(X_{i+1-d}, ..., X_i)$ as the investment scheme to be used at time $i \in \mathbb{N}_0$. The accumulated investment returns up to time $n \in \mathbb{N}$ are

$$R_n := \prod_{i=0}^{n-1} < Z_i^L(\bar{X}_{i+1}), X_{i+1} > .$$

The following corollary shows that, asymptotically, the investment strategy $Z_i^L(\bar{X}_{i+1})$ is superior to any other strategy using a portfolio selection function of the last d market periods (**pathwise competitive optimality**).

Corollary 6.2.3. *Suppose the support of the distribution* $\mathbf{P}_{X_0}$ *is not confined to a hyperplane in* $\mathbb{R}^m$ *containing the diagonal* $\{(d,...,d)^T \in \mathbb{R}^m | d \in \mathbb{R}\}$. *For any measurable portfolio selection function* $f : \mathbb{R}_+^{dm} \longrightarrow S$ *with accumulated returns*

$$V_n := \prod_{i=0}^{n-1} < f(\bar{X}_{i+1}), X_{i+1} >$$

we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{V_n}{R_n} \leq 0 \quad \mathbf{P}-a.s..$$

Lemma 6.2.1 and Theorem 6.2.2 are statements corresponding to Theorem 1 in Walk and Yakowitz (1999), Corollary 6.2.3 is a generalisation of Corollary 1 in Walk (2000). However, the statements are valid under fundamentally different (in fact, considerably weaker) assumptions.

As already mentioned, in practical applications the exact value of L is not disclosed to the investor. On the other hand, one can assume that, as the share prices take on rational, i.e. countably many values only, so does the return process $\{X_i\}_{i=-T}^{\infty}$ as a process of price ratios. In this situation an adaptive choice of L can be carried out in the following way:

Having fixed a sequence $\gamma_n \in \mathbb{N}, \gamma_n \longrightarrow \infty$, for the nth investment step a random variable

$$L_n := \arg \max_{K\in\{1,\dots,\gamma_n\}} \frac{1}{n+M+1} \sum_{j=-M}^{n} \log < Z_n^K(\bar{X}_j), X_j > \tag{6.2.5}$$

is defined, and the estimate of a log-optimal portfolio selection function $b(\bar{x})$ is

$$Z_n(\bar{x}) := Z_n^{L_n}(\bar{x}).$$

For this procedure, we have

Theorem 6.2.4. *Assume the distribution* $\mathbf{P}_{X_0}$ *is supported on a denumerable set and the support of the distribution is not confined to a hyperplane in* $\mathbb{R}^m$ *containing the diagonal* $\{(d,...,d)^T \in \mathbb{R}^m | d \in \mathbb{R}\}$. *Then Lemma 6.2.1, Theo-*

rem 6.2.2 and Corollary 6.2.3 remain valid if Z_n^L is replaced by Z_n.

For the regression estimator of Yakowitz, Györfi et al. (1999) it is not yet known whether there exists an adaptive rule for the choice of the Lipschitz constant L generating a procedure that is strongly consistent for arbitrary stationary ergodic processes with Lipschitz continuous regression function. Theorem 6.2.4 is remarkable because it asserts that for the application of the regression estimate to the portfolio optimization problem such adaptation can be achieved.

We finish this section with two remarks about extensions of the stated results.

Remark concerning (6.2.1). Lemma 6.2.1, the first part of Theorem 6.2.2 and Theorem 6.2.4 still hold if we use kernel estimates

$$\hat{M}_{k,n}(s,\bar{x}) := \left(\sum_{j=-M}^{n} \frac{X_j}{< s, X_j >} K\left(\frac{\bar{X}_j - \bar{x}}{h_k}\right)\right) \Big/ \left(\sum_{j=-M}^{n} K\left(\frac{\bar{X}_j - \bar{x}}{h_k}\right)\right)$$

instead of the partitioning estimates in (6.2.1). For this, we choose a continuous kernel function $K : \mathbb{R}_+^{dm} \longrightarrow \mathbb{R}_+$ having compact support, $K(0) > 0$, and bandwidths $h_k := 2^{-k-2}$ (Yakowitz, Györfi et al., 1999, Sec. 3). However, as we shall see, the argument in the proof of the second part of Theorem 6.2.2 breaks down unless the distribution of $\bar{X}_0$ is supported on a denumerable set.

Remark concerning (6.2.5). (6.2.5) can be seen as an application of the principle of empirical risk minimization. By (6.2.2) random classes of functions $Z_n^L : [a,b]^{dm} \to S$ are defined, parametrized by admissible step widths $\alpha_0, ..., \alpha_n$ and potential Lipschitz constants L. An estimator is picked out minimizing the empirical risk, here the negative empirical mean return. This can also be used to choose suitable step widths. In fact, the same arguments as used in the proof of Theorem 6.2.4 show: If $Z_n = Z_n^{(k,\alpha_0,...,\alpha_n)}$ is an estimator constructed with step widths $\alpha_0, ..., \alpha_n$ and Lipschitz parameter k such that

$$\frac{1}{n+M+1} \sum_{j=-M}^{n} \log < Z_n^{(k,\alpha_0,...,\alpha_n)}(\bar{X}_j), X_j > \tag{6.2.6}$$
$$\geq \frac{1}{n+M+1} \sum_{j=-M}^{n} \log < Z_n^{(L,1,1,1/2,...,1/n)}(\bar{X}_j), X_j >$$

(for sufficiently large n), then Theorem 6.2.4 remains valid for this Z_n.

6.3 Checking the properties of the estimation algorithm

We now move on to the proof of the statements of the preceding section.

6.3.1 Proof of the convergence Lemma 6.2.1

The algorithm (6.2.2) is an application of a classical projection algorithm for problem (6.1.2), using an estimate of the gradient of the target function,

$$m(s,\bar{x}) = \frac{\partial}{\partial s}\mathbf{E}\left[\log < s, X_0 > |\bar{X}_0 = \bar{x}\right] = \mathbf{E}\left[\frac{X_0}{< s, X_0 >}\middle|\bar{X}_0 = \bar{x}\right],$$

for $s \in S$. The gradient estimate is obtained via a partitioning method in (6.2.1). For this reason, before we can turn to the proof of the crucial convergence Lemma 6.2.1, we have to formulate some preliminary results on consistency properties of the gradient estimate.

The data the statistician can access at time $n \in \mathbb{N}_0$,

$$\left(\bar{X}_i := (X_{i-d}, ..., X_{i-1}), \quad Y_i^{(s)} := \frac{X_i}{< s, X_i >}\right)_{i=-M}^{n},$$

are drawn from a stationary and ergodic process. Indeed, referring to Stout (1974), Theorem 3.5.8, as $\{X_i\}_{i=-d-M}^{\infty}$ is stationary and ergodic, so is the stochastic process $\{(X_{i-d}, ..., X_i)\}_{i=-M}^{\infty}$. This follows from the cited theorem, observing that $< s, X_i >> 0$ for all $s \in S$, so that $f_s : \mathbb{R}_+^{n(d+1)} \longrightarrow \mathbb{R}_+^{n(d+1)} : (x_1, ..., x_{1+d}) \longmapsto (x_1, ..., x_d, x_{1+d} / < s, x_{1+d} >)$ is a well-defined measurable mapping.

Moreover, $Y_i^{(s)}$ is bounded and Lipschitz continuous in s because of

$$\left|\frac{X_i}{< s, X_i >}\right| = \frac{\sqrt{\sum_{j=1}^m X_{i,j}^2}}{\sum_{j=1}^m s_j X_{i,j}} \le \frac{\sqrt{mb^2}}{a\sum_{j=1}^m s_j} = \sqrt{m}\frac{b}{a} \tag{6.3.1}$$

and (applying the Cauchy-Schwarz inequality)

$$\begin{aligned}&\left|\frac{X_i}{< s, X_i >} - \frac{X_i}{< t, X_i >}\right| \\ &\quad \le \left|\frac{1}{< s, X_i >} - \frac{1}{< t, X_i >}\right| \cdot |X_i| = \frac{|< t-s, X_i >|}{|< s, X_i >< t, X_i >|} \cdot |X_i|\end{aligned}$$

$$\begin{aligned}
&\leq \frac{|X_i|^2}{|<s, X_i><t, X_i>|} \cdot |s-t| \leq \frac{mb^2}{\sum_{j=1}^m s_j X_{i,j} \cdot \sum_{j=1}^m t_j X_{i,j}} \cdot |s-t| \\
&\leq \frac{mb^2}{\sum_{j=1}^m s_j a \cdot \sum_{j=1}^m t_j a} \cdot |s-t| = \frac{mb^2}{a^2} \cdot |s-t| \quad (s,t \in S). \qquad (6.3.2)
\end{aligned}$$

Yakowitz, Györfi et al. (1999) propose a strongly consistent estimator for a Lipschitz continuous regression function based on stationary ergodic data. The gradient $m(s,\bar{x})$ is the regression function of $Y_i^{(s)}$ on $\bar{X}_i$, so that we may use this estimator to obtain a gradient estimate. To this end, let $\mathcal{P}_k$ be a partition of $\mathbb{R}_+^{dm}$ into cubes of volume $(2^{-k-2})^{dm}$. $A_k(\bar{x})$ denotes the element of $\mathcal{P}_k$ in which $\bar{x} \in \mathbb{R}_+^{dm}$ comes to lie. Then $m(s,\bar{x})$ is estimated by

$$\hat{m}_{n,L}(s,\bar{x}) := \hat{M}_{1,n}(s,\bar{x}) + \sum_{k=1}^{N_n} \hat{\Delta}_{k,n,L}(s,\bar{x})$$

with

$$\hat{M}_{k,n}(s,\bar{x}) := \left(\sum_{j=-M}^{n} \frac{X_j}{<s, X_j>} \mathbf{1}_{A_k(\bar{x})}(\bar{X}_j) \right) \Big/ \left(\sum_{j=-M}^{n} \mathbf{1}_{A_k(\bar{x})}(\bar{X}_j) \right) \qquad (6.3.3)$$

$$\hat{\Delta}_{k,n,L}(s,\bar{x}) := T_{L2^{-k}}(\hat{M}_{k,n}(s,\bar{x}) - \hat{M}_{k-1,n}(s,\bar{x})) \qquad (6.3.4)$$

and some fixed sequence $N_n \in \mathbb{N}$ satisfying $\lim_{n\to\infty} N_n = \infty$.

The estimator is motivated by the telescopic expansion

$$m(s,\bar{x}) = \lim_{k\to\infty} M_k(s,\bar{x}) = M_1(s,\bar{x}) + \sum_{k=2}^{\infty} \Delta_k(s,\bar{x}) \qquad (6.3.5)$$

of the limit relation

$$M_k(s,\bar{x}) := \mathbf{E}\left[\frac{X_0}{<s, X_0>} \middle| \bar{X}_0 \in A_k(\bar{x}) \right] \longrightarrow m(s,\bar{x})$$

for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$, where $\Delta_k(s,\bar{x}) := M_k(s,\bar{x}) - M_{k-1}(s,\bar{x})$ (Yakowitz, Györfi et al., 1999, eq. (4)). In addition, a truncated version of the expansion is defined by

$$m_L(s,\bar{x}) := M_1(s,\bar{x}) + \sum_{k=2}^{\infty} \Delta_{k,L}(s,\bar{x})$$

with $\Delta_{k,L}(s,\bar{x}) := T_{L2^{-k}} \Delta_k(s,\bar{x})$.

The convergence of the components (6.3.3) and (6.3.4) in the definition of the estimator to the corresponding components in expansion (6.3.5) is given by the following

Lemma 6.3.1. *Under the assumption V1 one has for* $\mathbf{P}_{\bar{X}_0}$*-a.a.* $\bar{x}$ *and any fixed* $s \in S$

1. $$\left(\hat{M}_{1,n}(s,\bar{x}) - M_1(s,\bar{x})\right) \longrightarrow 0 \quad (n \to \infty) \quad \mathbf{P} - a.s.,$$
2. $$\left(\hat{\Delta}_{k,n,L}(s,\bar{x}) - \Delta_{k,L}(s,\bar{x})\right) \longrightarrow 0 \quad (n \to \infty) \quad \mathbf{P} - a.s..$$

Proof. Straightforward application of the ergodic theorem (Stout, 1974, Theorem 3.5.7, Yakowitz, Györfi et al., 1999, eq. (16)) yields

$$\hat{M}_{k,n}(s,\bar{x}) \longrightarrow M_k(s,\bar{x}) \quad (n \to \infty) \quad \mathbf{P} - a.s.,$$

in particular the first part of the lemma. Since the truncation operator is itself Lipschitz continuous with Lipschitz constant 1, the second part of the lemma is obtained by

$$\begin{aligned}
&\left|\hat{\Delta}_{k,n,L}(s,\bar{x}) - \Delta_{k,L}(s,\bar{x})\right| \\
&\quad \le \left|T_{L2^{-k}}(\hat{M}_{k,n}(s,\bar{x}) - \hat{M}_{k-1,n}(s,\bar{x})) - T_{L2^{-k}}(M_k(s,\bar{x}) - M_{k-1}(s,\bar{x}))\right| \\
&\quad \le \left|\hat{M}_{k,n}(s,\bar{x}) - M_k(s,\bar{x})\right| + \left|M_{k-1}(s,\bar{x}) - \hat{M}_{k-1,n}(s,\bar{x})\right| \\
&\quad \longrightarrow 0 \quad \mathbf{P} - a.s. \quad (n \to \infty).
\end{aligned}$$

□

In Yakowitz, Györfi et al. (1999) a lemma analogous to the above one is used to obtain pointwise strong consistency of $\hat{m}_{n,L}(s,\bar{x})$. However, the proof of Lemma 6.2.1 requires convergence to hold uniformly in S, which will be derived from the following lemma.

Lemma 6.3.2. *Let* $S \subseteq \mathbb{R}^d$ *be some compact set,* $K > 0$ *and* $(f_n)_{n\in\mathbb{N}}$ *a class of functions* $f_n : S \longrightarrow \mathbb{R}^d$ *with* $|f_n(s) - f_n(t)| \le K|s-t|$ *for all* $s, t \in S$.

Then $\lim_{n\to\infty} f_n(s) = 0$ *for all* $s \in S$ *implies that* $\lim_{n\to\infty} \sup_{s\in S} |f_n(s)| = 0$.

Proof. Let $\delta > 0$ be arbitrary but fixed. Choose a finite δ-net $\mathcal{N}$ in S. Then for all $s \in S$ there exists some $t_s \in \mathcal{N}$ with $|s - t_s| \le \delta$. This yields

$$|f_n(s)| \le |f_n(s) - f_n(t_s)| + |f_n(t_s)| \le K|s - t_s| + |f_n(t_s)| \le K \cdot \delta + |f_n(t_s)|$$

and

$$\sup_{s\in S}|f_n(s)| \le K\cdot\delta + \sup_{t\in\mathcal{N}}|f_n(t)|.$$

Since $\mathcal{N}$ is finite, one has

$$\limsup_{n\to\infty}\sup_{s\in S}|f_n(s)| \le K\cdot\delta + \limsup_{n\to\infty}\sup_{t\in\mathcal{N}}|f_n(t)| = K\cdot\delta + 0 = K\cdot\delta.$$

The assertion follows from δ being arbitrary. □

There are two consequences of Lemma 6.3.2 which we are going to need.
Consequence 1: For $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$ we have

$$\lim_{n\to\infty}\sup_{s\in S}\left|\hat{M}_{1,n}(s,\bar{x}) - M_1(s,\bar{x})\right| = 0 \quad \mathbf{P}-a.s.. \tag{6.3.6}$$

This is a consequence of Lemma 6.3.2 with $f_n(s) := \hat{M}_{1,n}(s,\bar{x}) - M_1(s,\bar{x})$. According to Lemma 6.3.1 $\lim_{n\to\infty} f_n(s) = 0$ $\mathbf{P}$-a.s. and $|f_n(s) - f_n(t)| \le |\hat{M}_{1,n}(s,\bar{x}) - \hat{M}_{1,n}(t,\bar{x})| + |M_1(s,\bar{x}) - M_1(t,\bar{x})|$ hold for any $s\in S$ and for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$. To bound the terms in the latter expression, one uses

$$\begin{aligned}
&|\hat{M}_{k,n}(s,\bar{x}) - \hat{M}_{k,n}(t,\bar{x})| \\
&\quad\le \left|\sum_{j=-M}^{n}\left(\frac{X_j}{< s, X_j >} - \frac{X_j}{< t, X_j >}\right)\mathbf{1}_{A_k(\bar{x})}(\bar{X}_j)\right| \Big/ \left(\sum_{j=-M}^{n}\mathbf{1}_{A_k(\bar{x})}(\bar{X}_j)\right) \\
&\quad\le \max_{-M\le j\le n}\left|\frac{X_j}{< s, X_j >} - \frac{X_j}{< t, X_j >}\right| \le \frac{mb^2}{a^2}|s-t|
\end{aligned} \tag{6.3.7}$$

(the latter inequality from (6.3.2)) and

$$\begin{aligned}
&|M_k(s,\bar{x}) - M_k(t,\bar{x})| \\
&\quad\le \mathbf{E}\left[\left|\frac{X_0}{< s, X_0 >} - \frac{X_0}{< t, X_0 >}\right|\middle|\bar{X}_0 = \bar{x}\right] \le \mathbf{E}\left[\frac{mb^2}{a^2}|s-t|\middle|\bar{X}_0 = \bar{x}\right] \\
&\quad= \frac{mb^2}{a^2}|s-t|.
\end{aligned} \tag{6.3.8}$$

Altogether this yields

$$|f_n(s) - f_n(t)| \le \frac{mb^2}{a^2}|s-t| + \frac{mb^2}{a^2}|s-t| = 2\frac{mb^2}{a^2}|s-t|$$

for all $s,t\in S$, the situation as required in Lemma 6.3.2, and (6.3.6) is valid.

Consequence 2: Let $R \in \{2, 3, ...\}$ be fixed. Then for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$

$$\lim_{n\to\infty} \sup_{s\in S} \left| \sum_{k=2}^{R} \left(\hat{\Delta}_{k,n,L}(s,\bar{x}) - \Delta_{k,L}(s,\bar{x}) \right) \right| = 0 \quad \mathbf{P} - a.s. \tag{6.3.9}$$

holds.

Here, put $f_n(s) := \sum_{k=2}^{R} \left(\hat{\Delta}_{k,n,L}(s,\bar{x}) - \Delta_{k,L}(s,\bar{x}) \right)$. For any fixed $s \in S$ and for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$ one has $\lim_{n\to\infty} \left(\hat{\Delta}_{k,n,L}(s,\bar{x}) - \Delta_{k,L}(s,\bar{x}) \right) = 0$ $\mathbf{P}$-a.s. according to Lemma 6.3.1 and hence $\lim_{n\to\infty} f_n(s) = 0$ $\mathbf{P}$-a.s. for all $s \in S$. Moreover,

$$|f_n(s) - f_n(t)| \le \sum_{k=2}^{R} \left| \hat{\Delta}_{k,n,L}(s,\bar{x}) - \hat{\Delta}_{k,n,L}(t,\bar{x}) \right| + \sum_{k=2}^{R} \left| \Delta_{k,L}(s,\bar{x}) - \Delta_{k,L}(t,\bar{x}) \right|.$$

The first term can be bounded using

$$\begin{aligned}
&\left| \hat{\Delta}_{k,n,L}(s,\bar{x}) - \hat{\Delta}_{k,n,L}(t,\bar{x}) \right| \\
&\quad = \left| T_{L2^{-k}}(\hat{M}_{k,n}(s,\bar{x}) - \hat{M}_{k-1,n}(s,\bar{x})) - T_{L2^{-k}}(\hat{M}_{k,n}(t,\bar{x}) - \hat{M}_{k-1,n}(t,\bar{x})) \right| \\
&\quad \le \left| \hat{M}_{k,n}(s,\bar{x}) - \hat{M}_{k,n}(t,\bar{x}) \right| + \left| \hat{M}_{k-1,n}(s,\bar{x}) - \hat{M}_{k-1,n}(t,\bar{x}) \right| \\
&\quad \le 2\frac{mb^2}{a^2}|s-t|
\end{aligned}$$

(with (6.3.7)), and we obtain

$$\begin{aligned}
|f_n(s) - f_n(t)| \;\le\; & 2(R-1)\frac{mb^2}{a^2}|s-t| \\
& + \sum_{k=2}^{R} |M_k(s,\bar{x}) - M_k(t,\bar{x})| + \sum_{k=2}^{R} |M_{k-1}(s,\bar{x}) - M_{k-1}(t,\bar{x})|
\end{aligned}$$

for all $s, t \in S$. Appealing to inequality (6.3.8) this becomes

$$|f_n(s) - f_n(t)| \le 4(R-1)\frac{mb^2}{a^2}|s-t|,$$

and the requirements of Lemma 6.3.2 are met. The assertion (6.3.9) follows.

Finally we establish the desired strong consistency of the gradient estimate $\hat{m}_{n,L}(s,\bar{x})$ uniformly in S.

Lemma 6.3.3. *Under the assumption V1*

$$\lim_{n\to\infty} \sup_{s\in S} |\hat{m}_{n,L}(s,\bar{x}) - m(s,\bar{x})| = 0 \quad \mathbf{P} - a.s.$$

holds for $\mathbf{P}_{\bar{X}_0}$*-a.a.* $\bar{x}$*.*

Proof. In view of (6.3.6) it suffices to show that for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$

$$\lim_{n\to\infty} \sup_{s\in S} \left| \sum_{k=2}^{N_n} \hat{\Delta}_{k,n,L}(s,\bar{x}) - \sum_{k=2}^{\infty} \Delta_{k,L}(s,\bar{x}) \right| = 0 \quad \mathbf{P} - a.s..$$

Let $R \in \{2, 3, ...\}$ be arbitrary. For sufficiently large n we have $N_n > R$ and as in Yakowitz, Györfi et al. (1999), proof of Theorem 1, we obtain

$$\begin{aligned} &\left| \sum_{k=2}^{N_n} \hat{\Delta}_{k,n,L}(s,\bar{x}) - \sum_{k=2}^{\infty} \Delta_{k,L}(s,\bar{x}) \right| \\ &\leq \left| \sum_{k=2}^{R} \left(\hat{\Delta}_{k,n,L}(s,\bar{x}) - \Delta_{k,L}(s,\bar{x}) \right) \right| + \sum_{k=R+1}^{N_n} |\hat{\Delta}_{k,n,L}(s,\bar{x})| + \sum_{k=R+1}^{\infty} |\Delta_{k,L}(s,\bar{x})| \end{aligned}$$

The last two terms are bounded by

$$\begin{aligned} &\sum_{k=R+1}^{N_n} |\hat{\Delta}_{k,n,L}(s,\bar{x})| + \sum_{k=R+1}^{\infty} |\Delta_{k,L}(s,\bar{x})| \\ &\qquad \leq \sum_{k=R+1}^{N_n} L \cdot 2^{-k} + \sum_{k=R+1}^{\infty} L \cdot 2^{-k} \leq 2L \sum_{k=R+1}^{\infty} 2^{-k} = 2^{-(R-1)} \cdot L. \end{aligned}$$

Hence

$$\begin{aligned} &\sup_{s\in S} \left| \sum_{k=2}^{N_n} \hat{\Delta}_{k,n,L}(s,\bar{x}) - \sum_{k=2}^{\infty} \Delta_{k,L}(s,\bar{x}) \right| \\ &\qquad \leq \sup_{s\in S} \left| \sum_{k=2}^{R} \left(\hat{\Delta}_{k,n,L}(s,\bar{x}) - \Delta_{k,L}(s,\bar{x}) \right) \right| + 2^{-(R-1)} \cdot L. \end{aligned}$$

Using (6.3.9) we derive

$$\limsup_{n\to\infty} \sup_{s\in S} \left| \sum_{k=2}^{N_n} \hat{\Delta}_{k,n,L}(s,\bar{x}) - \sum_{k=2}^{\infty} \Delta_{k,L}(s,\bar{x}) \right| \leq 2^{-(R-1)} \cdot L$$

$\mathbf{P}$-a.s., and the assertion follows letting $R \to \infty$. □

Having obtained these preliminary results, we can now move on to proving Lemma 6.2.1. First, recall that the projection algorithm (6.2.2) is given by the recurrence relation ($n \in \mathbb{N}_0$)

$$Z_{-1}^L(\bar{x}) \in S \quad \text{arbitrary and} \quad Z_n^L(\bar{x}) := \Pi\left(Z_{n-1}^L(\bar{x}) + \alpha_n \hat{m}_{n,L}(Z_{n-1}^L(\bar{x}), \bar{x})\right), \tag{6.3.10}$$

where

$$\hat{m}_{n,L}(s, \bar{x}) = \hat{M}_{1,n}(s, \bar{x}) + \sum_{k=1}^{N_n} \hat{\Delta}_{k,n,L}(s, \bar{x}).$$

Posing this as

$$Z_n^L(\bar{x}) = \Pi\left(Z_{n-1}^L(\bar{x}) + \alpha_n m(Z_{n-1}^L, \bar{x}) + \alpha_n \beta_n(\bar{x})\right)$$

with

$$\beta_n(\bar{x}) := \hat{m}_{n,L}(Z_{n-1}^L(\bar{x}), \bar{x}) - m(Z_{n-1}^L(\bar{x}), \bar{x}), \tag{6.3.11}$$

the projection algorithm (6.3.10) is a special case of the projection algorithm

$$W_n = \Pi\left(W_{n-1} + \alpha_n\left(m(W_{n-1}) + \xi_n + \beta_n\right)\right)$$

in Kushner and Clark (1978, eq. 5.3.1) for $\xi_n := 0$. Their Theorem 5.3.1 adapted to the case $\xi_n := 0$ reads

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Lemma 6.3.4. *Under the assumptions*

1. $m(\cdot, \cdot)$ *is continuous,*
2. $\alpha_n > 0$ *with* $\alpha_n \longrightarrow 0$ *for* $n \to \infty$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
3. *there exists a* $c \geq 0$ *with* $|\beta_n(\bar{x})| \leq c < \infty$ *for all* $n \in \mathbb{N}_0$,
4. $\beta_n(\bar{x}) \longrightarrow 0$ $\mathbf{P}$-a.s. *for* $n \to \infty$

the projection algorithm converges,

$$\lim_{n\to\infty} \rho(Z_n^L(\bar{x}), KT(\bar{x})) = 0 \quad \mathbf{P} - a.s..$$

At last, this lemma allows us to give a

Proof of Lemma 6.2.1. The crucial point is to show that under the assumptions V1 and V2 and for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$ parts 3 and 4 of Lemma 6.3.4 hold when we choose the β_n defined by (6.3.11).

To this end, we first note that as in Yakowitz, Györfi et al. (1999, Corollary 1) assumption V2, i.e., the existence of a constant L with

$$|m(s,\bar{x}) - m(s,\bar{y})| \leq \frac{L}{\sqrt{md}}|x-y| \quad \text{for all } \bar{x},\bar{y} \in \mathbb{R}_+^{dm}, s \in S,$$

implies

$$m(s,\bar{x}) = m_L(s,\bar{x}).$$

Concerning part 3, one has

$$\begin{aligned}
&|\hat{m}_{n,L}(Z_{n-1}^L,\bar{x}) - m(Z_{n-1}^L,\bar{x})| = |\hat{m}_{n,L}(Z_{n-1}^L,\bar{x}) - m_L(Z_{n-1}^L,\bar{x})| \\
&\leq |\hat{M}_{1,n}(Z_{n-1}^L,\bar{x}) - M_1(Z_{n-1}^L,\bar{x})| + \sum_{k=2}^{N_n} \left|\hat{\Delta}_{k,n,L}(Z_{n-1}^L,\bar{x}) - \Delta_{k,L}(Z_{n-1}^L,\bar{x})\right| \\
&\leq |\hat{M}_{1,n}(Z_{n-1}^L,\bar{x})| + |M_1(Z_{n-1}^L,\bar{x})| + \sum_{k=2}^{\infty} \left|\hat{\Delta}_{k,n,L}(Z_{n-1}^L,\bar{x})\right| + \sum_{k=2}^{\infty} \left|\Delta_{k,L}(Z_{n-1}^L,\bar{x})\right| \\
&\leq |\hat{M}_{1,n}(Z_{n-1}^L,\bar{x})| + |M_1(Z_{n-1}^L,\bar{x})| + 2L\sum_{k=2}^{\infty} 2^{-k}.
\end{aligned}$$

Combining this with the inequalites

$$\begin{aligned}
|\hat{M}_{1,n}(Z_{n-1}^L,\bar{x})| &= \left|\sum_{j=-M}^{n} \frac{X_j}{< Z_{n-1}^L, X_j >} \mathbf{1}_{A_1(\bar{x})}(\bar{X}_j)\right| \Big/ \left(\sum_{j=-M}^{n} \mathbf{1}_{A_1(\bar{x})}(\bar{X}_j)\right) \\
&\leq \max_{-M\leq j\leq n} \left|\frac{X_j}{< Z_{n-1}^L, X_j >}\right|, \\
|M_1(Z_{n-1}^L,\bar{x})| &= \mathbf{E}\left[\left|\frac{X_0}{< Z_{n-1}^L, X_0 >}\right| \Big| \bar{X}_0 \in A_1(\bar{x})\right] \leq \sup_{\omega\in\Omega} \left|\frac{X_0}{< Z_{n-1}^L, X_0 >}\right|
\end{aligned}$$

we get

$$\begin{aligned}
&|\hat{m}_{n,L}(Z_{n-1}^L,\bar{x}) - m(Z_{n-1}^L,\bar{x})| \\
&\qquad \leq \max_{-M\leq j\leq n} \left|\frac{X_j}{< Z_{n-1}^L, X_j >}\right| + \sup_{\omega\in\Omega} \left|\frac{X_0}{< Z_{n-1}^L, X_0 >}\right| + 2\cdot\frac{L}{2} \\
&\qquad \leq 2\sqrt{m}\frac{b}{a} + L =: c,
\end{aligned}$$

the latter inequality due to (6.3.1).

Condition 4. is readily verified from

$$|\hat{m}_{n,L}(Z_{n-1}^L,\bar{x}) - m(Z_{n-1}^L,\bar{x})| \leq \sup_{s\in S} |\hat{m}_{n,L}(s,\bar{x}) - m_L(s,\bar{x})|$$

and Lemma 6.3.3. This finishes the proof of Lemma 6.2.1. □

6.3.2 Proof of the related Theorems 6.2.2 - 6.2.4

Proof of Theorem 6.2.2. For any fixed $\bar{x}$, $R(\cdot, \bar{x})$ is a continuous function on the compact set S, thus uniformly continuous. Part 1 of the theorem directly follows from Lemma 6.2.1.

To prove part 2, observe that (6.3.3) involves only denumerably many functions $\mathbf{1}_{A_k(\bar{x})}$ (for fixed k and all possible values of $\bar{x}$). Thus, the exceptional set of $\omega \in \Omega$ in Lemma 6.3.1 can be made independent from the chosen $\bar{x}$. This continues throughout the proof of Lemma 6.2.1. Hence as $n \to \infty$ we have for $\mathbf{P} \otimes \mathbf{P}_{\bar{X}_0}$-a.a. $(\omega, \bar{x})$

$$\begin{aligned} &\rho\left(Z_n^L(\bar{x}), KT(\bar{x})\right) \longrightarrow 0, \\ &|Z_n^L(\bar{x}) - Z_n^{L*}(\bar{x})| \longrightarrow 0, \\ &R(Z_n^L(\bar{x}), \bar{x}) \longrightarrow R^*(\bar{x}). \end{aligned} \tag{6.3.12}$$

From part 1, the Lebesgue dominated convergence theorem yields the assertions of the second part of the theorem. The limit relation (6.3.12), now valid **P**-a.s. for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$, and

$$|Z_n^L(\bar{x}) - Z_n^{L*}(\bar{x})| \leq \max_{s,t \in S} |s - t| \leq \sqrt{m}$$

imply

$$\int |Z_n^L(\bar{x}) - Z_n^{L*}(\bar{x})| \mathbf{P}_{\bar{X}_0}(d\bar{x}) \longrightarrow 0 \quad (n \to \infty)$$

$\mathbf{P} - a.s.$ and in $L_r(\mathbf{P})$ $(r \in \mathbb{N})$. The same arguments, starting from

$$\begin{aligned} &|R(Z_n^L(\bar{x}), \bar{x}) - R^*(\bar{x})| \\ &\qquad \leq |R(Z_n^L(\bar{x}), \bar{x})| + |R^*(\bar{x})| \leq 2 \max_{s \in S, y \in [a,b]^m} |\log < s, y > | < \infty, \end{aligned}$$

yield the remaining parts of the proof. $\square$

Proof of Corollary 6.2.3. The assumption on the support of $\mathbf{P}_{X_0}$ implies that an essentially, i.e., $\mathbf{P}_{\bar{X}_0}$-a.s. unique log-optimal portfolio selection function

$$w(\bar{x}) := \arg\max_{s \in S} \mathbf{E}[\log < s, X_0 > | \bar{X}_0 = \bar{x}]$$

exists (Algoet and Cover, 1988, p. 877, corrected in Österreicher and Vajda, 1993, and Vajda and Österreicher, 1994). The accumulated return using the log-optimal portfolio selection function for investment is denoted by

$$R_n^* := \prod_{i=0}^{n-1} < w(\bar{X}_{i+1}), X_{i+1} > .$$

It follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{V_n}{R_n} \le \limsup_{n\to\infty} \frac{1}{n} \log \frac{V_n}{R_n^*} + \limsup_{n\to\infty} \frac{1}{n} \log \frac{R_n^*}{R_n}. \tag{6.3.13}$$

For the first term on the right hand side,the ergodic theorem and the optimality of w imply

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \frac{V_n}{R_n^*} \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \frac{< f(\bar{X}_{i+1}), X_{i+1} >}{< w(\bar{X}_{i+1}), X_{i+1} >} \\
&= \mathbf{E} \left(\log \frac{< f(\bar{X}_0), X_0 >}{< w(\bar{X}_0), X_0 >} \right) \\
&= \int \big(\mathbf{E}[\log < f(\bar{x}), X_0 > | \bar{X}_0 = \bar{x}] - \mathbf{E}[\log < w(\bar{x}), X_0 > | \bar{X}_0 = \bar{x}] \big) \, \mathbf{P}_{\bar{X}_0}(d\bar{x}) \\
&\le 0.
\end{aligned} \tag{6.3.14}$$

Arguing along the lines of Walk (2000, Corollary 1), the second term has limiting behaviour

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \frac{R_n^*}{R_n} \\
&\quad = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \big(\log < w(\bar{X}_{i+1}), X_{i+1} > - \log < Z_i^L(\bar{X}_{i+1}), X_{i+1} > \big) \\
&\quad = 0.
\end{aligned} \tag{6.3.15}$$

This is seen as follows: (6.3.12) combined with Egorov's theorem shows that for each $\epsilon > 0$ we can find sets $\tilde{\Omega} \subseteq \Omega$ and $\tilde{I} \subseteq [a, b]^{dm}$ such that $\mathbf{P}(\tilde{\Omega}) \ge 1 - \epsilon$, $\mathbf{P}_{\bar{X}_0}(\tilde{I}) \ge 1 - \epsilon$ and

$$Z_n^L \to w \quad \text{uniformly on} \quad \tilde{\Omega} \times \tilde{I}. \tag{6.3.16}$$

Then

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \big(\log < w(\bar{X}_{i+1}), X_{i+1} > - \log < Z_i^L(\bar{X}_{i+1}), X_{i+1} > \big) \right|$$

$$\le \frac{1}{n}\sum_{i=0}^{n-1}\left|\left(\log < w(\bar{X}_{i+1}), X_{i+1} > - \log < Z_i^L(\bar{X}_{i+1}), X_{i+1} >\right)\right| \cdot \mathbf{1}_{\tilde{I}}(\bar{X}_{i+1})$$
$$+\frac{1}{n}\sum_{i=0}^{n-1}\left|\left(\log < w(\bar{X}_{i+1}), X_{i+1} > - \log < Z_i^L(\bar{X}_{i+1}), X_{i+1} >\right)\right| \cdot \mathbf{1}_{\tilde{I}^c}(\bar{X}_{i+1})$$
$$\le \frac{c}{n}\sum_{i=0}^{n-1}\left|w(\bar{X}_{i+1}) - Z_i^L(\bar{X}_{i+1})\right| \cdot \mathbf{1}_{\tilde{I}}(\bar{X}_{i+1}) + \frac{c}{n}\sum_{i=0}^{n-1}\mathbf{1}_{\tilde{I}^c}(\bar{X}_{i+1})$$

for some constant $c > 0$. The first term tends to zero by (6.3.16), the second term to $c \cdot \mathbf{P}_{\bar{X}_0}(\tilde{I}^C) \le c \cdot \epsilon$. Now, let ϵ go to zero.

Finally, (6.3.14) and (6.3.15) plugged into (6.3.13) finish the proof. □

Proof of Theorem 6.2.4. It suffices to prove Lemma 6.2.1 for Z_n instead of Z_n^L. Without loss of generality one can set $M = 0$. In the following we assume n to be sufficiently large such that $\gamma_n \ge L$. Because $\bar{X}_0$ takes on values in a denumerable set $\mathcal{X}$, for any $\epsilon > 0$ there exists a finite subset $\bar{\mathcal{X}} \subseteq \mathcal{X}$ with $\mathbf{P}(\bar{X}_0 \in \bar{\mathcal{X}}^C) \le \epsilon$. As in the proof of Corollary 6.2.3, $w(\cdot)$ denotes the essentially unique log-optimal portfolio selection function.

For $\omega \in \Omega, \bar{x} \in \mathcal{X}$, we consider the sequence

$$Z_n^{L_n}(\bar{x}) = Z_n^{L_n}(\bar{x}, \omega) \in S$$

and show that for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$ a $\mathbf{P}$-a.s. limit relation $Z_n^{L_n}(\bar{x}, \omega) \longrightarrow w(\bar{x})$ holds.

To establish this, we work through the following: Consider an accumulation point $f_\omega(\bar{x})$ of the sequence, say

$$Z_{n'}^{L_{n'}}(\bar{x}, \omega) \longrightarrow f_\omega(\bar{x})$$

for a subsequence n', and show that $\mathbf{P}$-a.s.

$$f_\omega(\bar{x}) = w(\bar{x}) \quad \text{for } \mathbf{P}_{\bar{X}_0} - a.a.\ \bar{x}. \tag{6.3.17}$$

Indeed, (6.3.17) implies the existence of a set $A \in \mathcal{A}$, $\mathbf{P}(A) = 1$, such that for any $\omega \in A$

$$Z_n^{L_n}(\bar{x}, \omega) \longrightarrow w(\bar{x}) \tag{6.3.18}$$

for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$. This is seen as follows: Assume we had an $\bar{x}$ with $\mathbf{P}_{\bar{X}_0}(\{\bar{x}\}) > 0$ and $\mathbf{P}(A(\bar{x})) > 0$, where $A(\bar{x}) := \{\omega | Z_n^{L_n}(\bar{x}, \omega) \nrightarrow w(\bar{x})\}$. $\mathbf{P}(A(\bar{x})) > 0$ implies $A(\bar{x}) \cap A \neq \emptyset$, hence an $\omega \in A(\bar{x}) \cap A$ exists with $Z_n^{L_n}(\bar{x}, \omega) \nrightarrow w(\bar{x})$ on the one hand (according to the construction of $A(\bar{x})$), and $Z_n^{L_n}(\bar{x},\omega) \to w(\bar{x})$ on the other hand (according to (6.3.18) and $\mathbf{P}_{\bar{X}_0}(\{\bar{x}\}) > 0$). This is a contradiction. Hence, for $\mathbf{P}_{\bar{X}_0}$-a.a. $\bar{x}$, we have $\mathbf{P}(A(\bar{x})) = 0$, i.e.

$$Z_n^{L_n}(\bar{x},\omega) \longrightarrow w(\bar{x}) \quad \mathbf{P}\text{-a.s.}.$$

We now tackle the proof of (6.3.17).
For any $\bar{x} \in \bar{\mathcal{X}}$ the sequence $Z_{n'}^{L_{n'}}(\bar{x},\omega)$ takes on values in the compact set S, i.e. there exists an accumulation point $f_\omega(\bar{x})$. Since $\bar{\mathcal{X}}$ is finite, repeatedly taking subsequences gives an index sequence n'' for which

$$Z_{n''}^{L_{n''}}(\bar{x},\omega) \longrightarrow f_\omega(\bar{x}) \tag{6.3.19}$$

uniformly for all $\bar{x} \in \bar{\mathcal{X}}$. n'' will be denoted by n again in the following.

For any fixed ω

$$\begin{aligned}
&\Big|\frac{1}{n+1}\sum_{i=0}^{n} \log < Z_n^{L_n}(\bar{X}_i,\omega), X_i > -\mathbf{E}\Big(\log < f_\omega(\bar{X}_0), X_0 > \Big)\Big| \\
&\le \ \Big|\frac{1}{n+1}\sum_{i=0}^{n} \log < Z_n^{L_n}(\bar{X}_i,\omega), X_i > \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_i) \\
&\qquad\qquad -\mathbf{E}\Big(\log < f_\omega(\bar{X}_0), X_0 > \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_0)\Big)\Big| \\
&\quad + \frac{1}{n+1}\sum_{i=0}^{n} \Big| \log < Z_n^{L_n}(\bar{X}_i,\omega), X_i > \Big| \mathbf{1}_{\bar{\mathcal{X}}^C}(\bar{X}_i) \\
&\quad + \mathbf{E}\Big(\Big| \log < f_\omega(\bar{X}_0), X_0 > \Big| \mathbf{1}_{\bar{\mathcal{X}}^C}(\bar{X}_0)\Big) \\
&\le \ \frac{1}{n+1}\sum_{i=0}^{n} \Big| \log < Z_n^{L_n}(\bar{X}_i,\omega), X_i > - \log < f_\omega(\bar{X}_i), X_i > \Big| \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_i) \\
&\quad + \Big|\frac{1}{n+1}\sum_{i=0}^{n} \log < f_\omega(\bar{X}_i), X_i > \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_i) \\
&\qquad\qquad -\mathbf{E}\Big(\log < f_\omega(\bar{X}_0), X_0 > \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_0)\Big)\Big| \\
&\quad + c \cdot \Big(\frac{1}{n+1}\sum_{i=0}^{n} \mathbf{1}_{\bar{\mathcal{X}}^C}(\bar{X}_i) + \mathbf{P}(\bar{X}_0 \in \bar{\mathcal{X}}^C)\Big)
\end{aligned} \tag{6.3.20}$$

with a constant $c = c(d, m, a, b) \in \mathbb{R}_+$.

Because of the uniform convergence in (6.3.19) the first term of (6.3.20) satisfies (for n sufficiently large)

$$\Big|\log < Z_n^{L_n}(\bar{X}_i,\omega), X_i > - \log < f_\omega(\bar{X}_i), X_i > \Big| \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_i)$$

$$\leq c \cdot \left| Z_n^{L_n}(\bar{X}_i, \omega) - f_\omega(\bar{X}_i) \right| \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_i) \leq c \cdot \epsilon \qquad (6.3.21)$$

for all $i = 0, ..., n$. Without loss of generality we may use the same constant c as above.

For the second term it will be shown at the end of this proof that in $\mathbf{P}$-a.a. ω

$$\limsup_{n\to\infty} \Big| \frac{1}{n+1} \sum_{i=0}^{n} \log < f_\omega(\bar{X}_i), X_i > \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_i) \\ -\mathbf{E}\Big(\log < f_\omega(\bar{X}_0), X_0 > \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_0) \Big) \Big| = 0. \qquad (6.3.22)$$

For the third term the ergodic theorem yields that $\mathbf{P}$-a.s.

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{1}_{\bar{\mathcal{X}}^C}(\bar{X}_i) = \mathbf{P}(\bar{X}_0 \in \bar{\mathcal{X}}^C) \leq \epsilon. \qquad (6.3.23)$$

(6.3.21) to (6.3.23) plugged into (6.3.20) yield

$$\limsup_{n\to\infty} \left| \frac{1}{n+1} \sum_{i=0}^{n} \log < Z_n^{L_n}(\bar{X}_i, \omega), X_i > -\mathbf{E}\left(\log < f_\omega(\bar{X}_0), X_0 > \right) \right| \leq 3 \cdot c \cdot \epsilon,$$

and from ϵ being arbitrary it follows that

$$\frac{1}{n+1} \sum_{i=0}^{n} \log < Z_n^{L_n}(\bar{X}_i, \omega), X_i > \longrightarrow \mathbf{E}\left(\log < f_\omega(\bar{X}_0), X_0 > \right) \qquad (6.3.24)$$

for $\mathbf{P}$-a.a. ω.

On the other hand, using the definition of the random variable L_n, we obtain $\mathbf{P}$-a.s.

$$\frac{1}{n+1} \sum_{i=0}^{n} \left\{ \log < Z_n^{L_n}(\bar{X}_i, \omega), X_i > - \log < w(\bar{X}_i), X_i > \right\} \\ \geq \frac{1}{n+1} \sum_{i=0}^{n} \left\{ \log < Z_n^{L}(\bar{X}_i, \omega), X_i > - \log < w(\bar{X}_i), X_i > \right\} \longrightarrow 0, \qquad (6.3.25)$$

where a limit relation analogous to (6.3.15) can be used.

Plugging (6.3.24) into the first line of (6.3.25) and observing

$$\frac{1}{n+1} \sum_{i=0}^{n} \log < w(\bar{X}_i), X_i > \longrightarrow \mathbf{E}\left(\log < w(\bar{X}_0), X_0 > \right)$$

we obtain that (again for $\mathbf{P}$-a.a. ω)

$$\mathbf{E}\left(\log < f_\omega(\bar{X}_0), X_0 >\right) - \mathbf{E}\left(\log < w(\bar{X}_0), X_0 >\right) \geq 0.$$

Because of the essential uniqueness of the optimum $w(\cdot)$, for $\mathbf{P}$-a.a. ω we infer (6.3.17), namely that $f_\omega(\bar{x}) = w(\bar{x}) \quad \mathbf{P}_{\bar{X}_0} - a.s.$.

So it only remains to demonstrate (6.3.22). To this end, let $C := C\left([a,b]^{dm}, S\right)$ be the space of continuous functions $f : [a,b]^{dm} \longrightarrow S$, equipped with the supremum norm $\sup_{\bar{x}\in[a,b]^{dm}} |f(\bar{x})|$. For f_ω we can find a continuation $\bar{f}_\omega$ contained in C, which coincides with f_ω on $\bar{\mathcal{X}}$. Due to the separability of C (Megginson, 1998, Sec. 1.12) a denumerable set $\mathcal{G} \subseteq C$ can be found, such that for any given $\epsilon > 0$ and any f_ω there exists a function $g_\omega \in \mathcal{G}$ satisfying $\sup_{\bar{x}\in\bar{\mathcal{X}}} |f_\omega(\bar{x}) - g_\omega(\bar{x})| \leq \epsilon$.

For given $f : [a,b]^{dm} \longrightarrow S$ use the shorthand notation $H_n(\omega, f)$ for

$$\frac{1}{n+1}\sum_{i=0}^{n} \log < f(\bar{X}_i), X_i > \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_i) - \mathbf{E}\left(\log < f(\bar{X}_0), X_0 > \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_0)\right).$$

Then

$$\begin{aligned}
&|H_n(\omega, f_\omega)| = |H_n(\omega, g_\omega) + H_n(\omega, f_\omega) - H_n(\omega, g_\omega)| \\
&\quad\leq |H_n(\omega, g_\omega)| + \frac{1}{n+1}\sum_{i=0}^{n} \left|\log < f_\omega(\bar{X}_i), X_i > - \log < g_\omega(\bar{X}_i), X_i >\right| \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_i) \\
&\qquad + \mathbf{E}\Big(\left|\log < f_\omega(\bar{X}_0), X_0 > - \log < g_\omega(\bar{X}_0), X_0 >\right| \mathbf{1}_{\bar{\mathcal{X}}}(\bar{X}_0)\Big) \\
&\quad\leq |H_n(\omega, g_\omega)| + 2\cdot c\cdot \sup_{\bar{x}\in\bar{\mathcal{X}}} |f_\omega(\bar{x}) - g_\omega(\bar{x})| \\
&\quad\leq |H_n(\omega, g_\omega)| + 2\cdot c\cdot \epsilon.
\end{aligned}$$

Because of ϵ being arbitrary, it suffices to convince ourselves of

$$H(\omega, g_\omega) := \limsup_{n\to\infty} |H_n(\omega, g_\omega)| = 0$$

for $\mathbf{P}$-a.a. ω. As to this, observe that (because of $\mathcal{G}$ being denumerable) the set

$$\{\omega | \exists g \in \mathcal{G} \quad H(\omega, g) > 0\} = \bigcup_{g\in\mathcal{G}} \{\omega | H(\omega, g) > 0\}$$

is measurable. Using the ergodic theorem, the left hand side is a countable union of null sets, i.e. null set itself. Hence for $\mathbf{P}$-a.a. ω

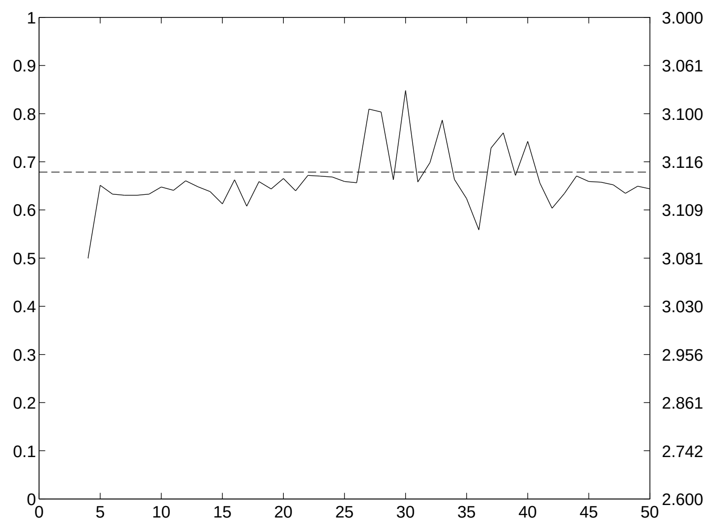
$$H(\omega, g) = 0 \quad \text{for all} \quad g \in \mathcal{G}$$

and in particular $H(\omega, g_\omega) = 0$, which completes the proof. □

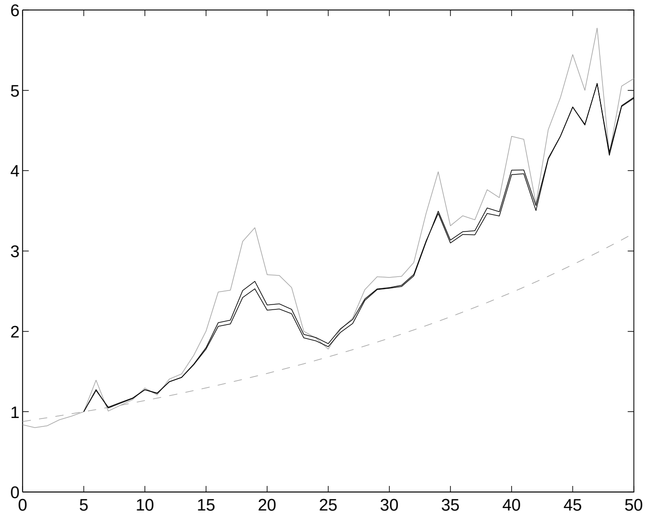
6.4 Simulations and examples

We conclude this chapter by simulations and examples in which we apply the estimated log-optimal portfolio selection functions of Section 6.2 to simulated and real markets. Throughout we select portfolios on the basis of the last $d = 5$ observed return data.

Example 6.1: The market consists of a riskless bond with return of 2.6% per market period and a share that follows a geometrical Brownian motion (Luenberger, 1998, Sec. 11.7; Korn and Korn, 1999, Ch. 2) with a mean return $\mu = 3\%$ per market period and and a volatility $\sigma = 15\%$ per market period. Investment starts after 5 market periods and ends after 50 market periods. In this model, due to the independence of the share's log-returns, the log-optimal portfolio selection function coincides with the log-optimal portfolio, which suggests to invest 67.86% of the current wealth in each market period into



6.1 a) Proportion of wealth invested in the share (left vertical axis), expected portfolio log-return (right vertical axis, in %). Results for the true log-optimal strategy (dashed) and the estimated log-optimal strategy (solid).



6.1 b) Value of a \$1 investment in the share (grey, solid) or in the bond (grey, dashed), respectively. We compare the value of a \$1 investment in the true log-optimal strategy (upper black curve) and the value of a \$1 investment in the estimated log-optimal strategy (lower black curve).

Figure 6.1: Sample path of an investment in a share following a geometrical brownian motion and a bond during 50 market periods.

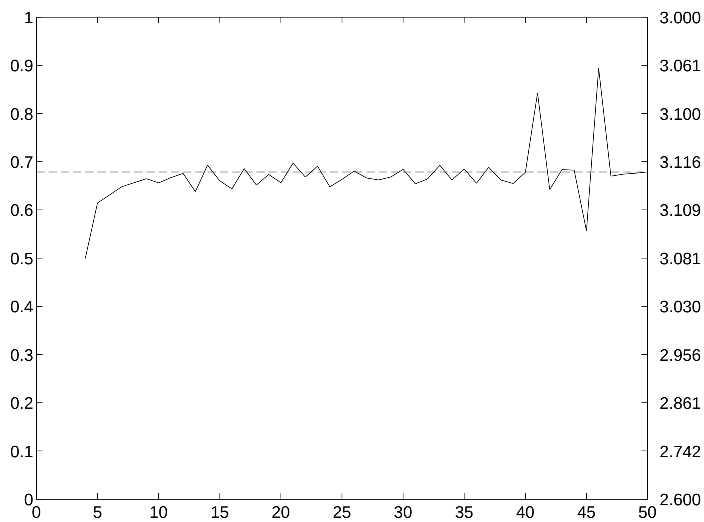
the share (calculated by Cover's algorithm, Theorem 1.3.2). Figures 6.1 and 6.2 show sample paths in the market together with estimation results. Throughout this section we use the kernel variant of the projection algorithm (6.2.2) with a cosine kernel $K(\bar{x}) = \cos(\min\{\|\bar{x}\|_F/100, 1\}) + 1$ (the Frobenius norm $\|\bar{x}\|_F$ of $\bar{x} = (x_{i,j})_{1\le i\le 2, 1\le j\le d}$ being defined as the square root of the sum of the diagonal elements of $\bar{x}^T\bar{x}$) and $L = 100$.

Subgraphs a) of Figures 6.1 and 6.2 show the estimated log-optimal portfolio weight for the share (solid line), i.e. the coordinate of $Z_n^L(X_{n+1-d}, ..., X_n)$ that corresponds to the share. The results can be compared with the true log-optimal portfolio (dashed line) and the expected portfolio returns per market period given on the right vertical axis (in %).

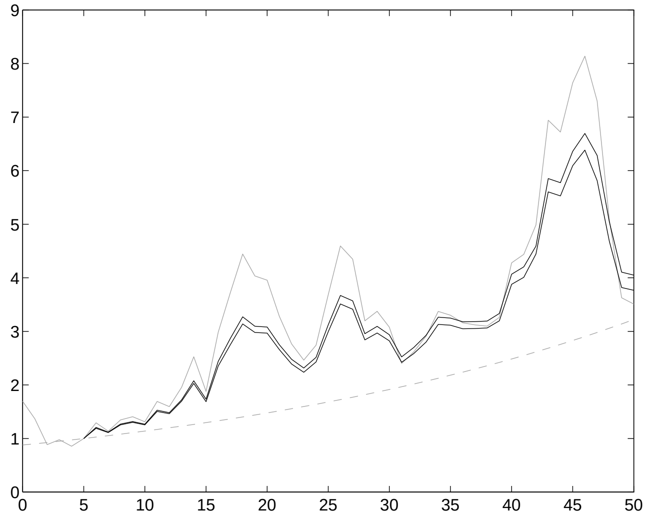
In subgraphs b) we follow the value of a \$1 investment in the share (grey, solid) or in the bond (grey, dashed), respectively. We compare the value of a \$1 investment in the true log-optimal strategy (upper black curve) and the value of a \$1 investment in the strategy using the estimated log-optimal portfolio weights (lower black curve). The results are convincing: As we expect from the competitive optimality result (Corollary 6.2.3), the estimated strategy allows to track the value evolution of the log-optimal strategy.

Example 6.2: In this example we run the projection algorithm for the estimation of log-optimal portfolio selection functions on real market data from NYSE, 22/4/1998-6/7/1998 (daily closing price data from www.wallstreetcity.com). We use the same stocks (YELL, JBHT, UNP) as in Example 2.2, Section 2.3.

Here, we do not know the true log-optimal portfolio selection function. As a substitute reference strategy we use the strategy with the constant weights estimated in Example 2.2, Section 2.3 (i.e., (0.523951, 0.476049) for (JBHT,YELL),



6.2 a) Proportion of wealth invested in the share (left vertical axis), expected portfolio log-return (right vertical axis, in %). Results for the true log-optimal strategy (dashed) and the estimated log-optimal strategy (solid).

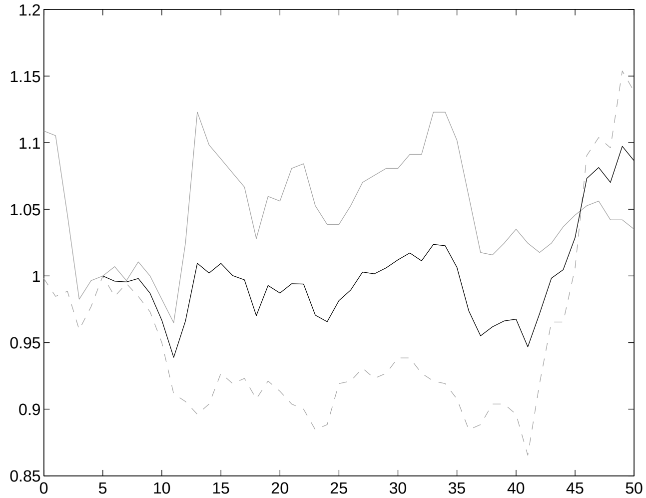


6.2 b) Value of a $1 investment in the share (grey, solid) or in the bond (grey, dashed), respectively. We compare the value of a $1 investment in the true log-optimal strategy (upper black curve) and the value of a $1 investment in the estimated log-optimal strategy (lower black curve).

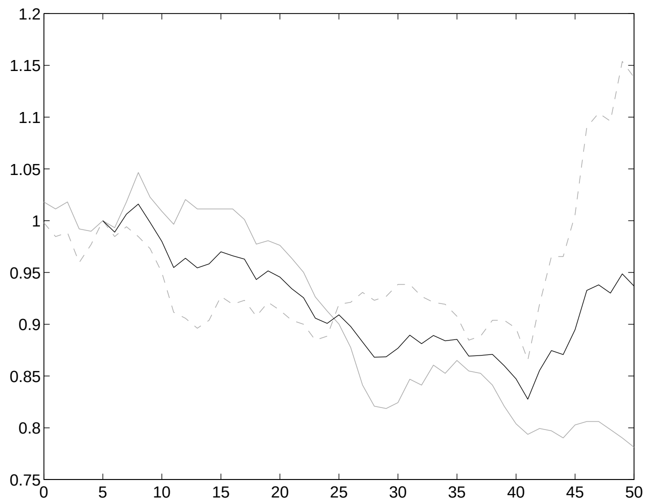
Figure 6.2: Sample path of an investment in a share following a geometrical brownian motion and a bond during 50 market periods.

and (0.465490, 0.53451) for (UNP,YELL)). We then compare the value of a $1 investment in the strategy using the estimated log-optimal portfolio selection function with the value of the reference strategy (Figure 6.3). Investment starts on the 5th day of trading. The value of the estimated log-optimal portfolio strategy virtually coincides with the value of what was believed to be the true log-optimal strategy in Example 2.2, Section 2.3 (we therefore plotted the value for the log-optimal portfolio selection function only). This suggests that in either case we are close to the log-optimal portfolio selection strategy.

– One might be tempted to argue that compared with Section 2.3 not much has been gained. This is not the case. On the contrary, in Section 2.3 we considered a very restrictive model involving independent, log-normally distributed daily



6.3 a) JBHT (grey, solid) and YELL (grey, dashed).



6.3 b) UNP (grey, solid) and YELL (grey, dashed).

Figure 6.3: Value of a \$1 investment in two single stocks (grey), and in the estimated log-optimal portfolio of the two (black, solid) at NYSE 22/4-6/7/1998.

returns. We just sketched and, in fact, then skipped most of the huge effort that should have been put into diagnostic testing of these assumptions in Section 2.3. Much of this effort is superfluous here. Indeed, with the model of Chapter 6 we gained considerable flexibility with respect to the underlying market model, assuming not much more than stationarity and ergodicity. Considering there is no such thing as absolute certainty about what the true stochastic regime in the market looks like, we come to appreciate nonparametric algorithms that work well under very weak assumptions and hence may be applied in many real markets.

L'ENVOI

Clearly, we were only able to cover a small subsample of the problems the investor faces in real markets. In the course of this thesis we derived several algorithms for these selected problems, and the reader might (and hopefully will) find some of them helpful to decide on practical investment problems. Beyond this algorithmics, we hope we have conveyed the key message of this thesis: The insight that nonparametric statistical forecasting and estimation techniques are a valuable tool in portfolio selection and, in fact, in all mathematical finance.

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