Information Theory—Midterm test, 4 November 1999

Important! Answers are not complete without sufficient reasoning.

Problem 1 *Let X be a random variable with probability distribution*

$$\mathbf{P}\{X = i\} = p_i, \qquad i = 0, 1, \ldots, 255,$$

and Y be a random variable uniformly distributed on the integers $0, 1, \ldots, 255$. Assuming that X and Y are independent, calculate the entropy of $(X + Y)$ mod 256 .

Problem 2 *Let $X_1, \ldots, X_n$ be binary random variables. Denote by $\mathbf{R} = (R_1, R_2, \ldots)$ the run lengths of this sequence. For example the sequence 1110010001111 yields run lengths $\mathbf{R} = (3, 2, 1, 3, 4)$. Compare $H(X_1, \ldots, X_n), H(\mathbf{R})$ and $H(\mathbf{R}, X_n)$. Show all equalities and inequalities.*

Problem 3 *The outcome of an experiment is a random variable with seven possible values and probability distribution $(1/3, 1/3, 1/9, 1/9, 1/27, 1/27, 1/27)$. We want to transmit the result. We can transmit it either in binary form for 200 Ft per bit or in ternary form for 325 Ft per ternary symbol. Which one should we choose and what code should we use to minimize the expected cost? Which one should we choose if the experiment is repeated many times independently and we can code the results jointly? ($\log 3 \approx 1.585$)*

Problem 4 *Let $\mathbf{Z} = Z_1, Z_2, \ldots$ be a binary, stationary Markov chain with transition probabilities*

$$\mathbf{P}\{Z_2 = 0|Z_1 = 0\} = \frac{1}{4}, \ \mathbf{P}\{Z_2 = 1|Z_1 = 0\} = \frac{3}{4}, \ \mathbf{P}\{Z_2 = 0|Z_1 = 1\} = \frac{3}{8}, \ \mathbf{P}\{Z_2 = 1|Z_1 = 1\} = \frac{5}{8}.$$

Find the stationary distribution and the entropy of the chain.
Suppose that we start the chain from the stationary distribution and let $Y_1, Y_2, \ldots$ be a sequence of independent, identically distributed binary random variables with $P(Y_i = 0) = \frac{1}{3}$. Define the source $\mathbf{X} = X_1, X_2, \ldots$ with $X_i = 2Z_i + Y_i$. Find the entropy of source $\mathbf{X}$ if $Z_1, Z_2, \ldots$ is independent from $Y_1, Y_2, \ldots$.

Problem 5 *Give the precise definition of uniquely decodable codes.*
What is the difference between McMillan's and Kraft's inequality?