Information Theory—Repeated midterm test, 6 December 2001

Important! Answers are not complete without sufficient reasoning.

Problem 1 *A playoff consists of a three-game series that terminates as soon as either team wins two games. Let X be the random variable that represents the outcome of a playoff between teams A and B; examples of possible values of X are AA, BAB etc. Let Y be the number of games played, which ranges from 2 to 3.*

(a) Assuming that A and B are equally matched and that the games are independent, calculate $H(X)$, $H(Y)$, $H(Y|X)$ and $H(X|Y)$.

(b) Let Z denote the winning team. Find $H(X|Z)$. Compare to $H(X)$. Find $H(Z|X)$.

(c) Find $I(Y;Z)$.

Problem 2 *Find a possible Huffman code and the Lempel-Ziv-Welch code of the word DECODER. Concerning the latter, assume that the letters C,D,E,O and R are already in the dictionary and their codes are 1,2,3 and 4 respectively. Describe also the final state of the dictionary (words and their codes).*

Problem 3 *Show that the entropy of the distribution*

$$(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_n)$$

cannot be greater than the entropy of the distribution

$$(p_1, \ldots, \frac{p_i + p_j}{2}, \ldots, \frac{p_i + p_j}{2}, \ldots, p_n).$$

Problem 4 *Let $\mathbf{Z} = Z_1, Z_2, \ldots$ be a binary, stationary Markov chain with transition probabilities*

$$\mathbf{P}\{Z_2 = 0|Z_1 = 0\} = \frac{1}{4},\ \mathbf{P}\{Z_2 = 1|Z_1 = 0\} = \frac{3}{4},\ \mathbf{P}\{Z_2 = 0|Z_1 = 1\} = \frac{3}{8},\ \mathbf{P}\{Z_2 = 1|Z_1 = 1\} = \frac{5}{8}.$$

Find the stationary distribution and the entropy of the chain.
Suppose that we start the chain from the stationary distribution and let $Y_1, Y_2, \ldots$ be a sequence of independent, identically distributed binary random variables with $P(Y_i = 0) = \frac{1}{3}$. Define the source $\mathbf{X} = X_1, X_2, \ldots$ with $X_i = 2Z_i + Y_i$. Find the entropy of source $\mathbf{X}$ if $Z_1, Z_2, \ldots$ is independent from $Y_1, Y_2, \ldots$.

Problem 5 *Give the precise definition of uniquely decodable codes. State McMillan's inequality.*