Information Theory—Midterm test, 15 November 2000

Important! Answers are not complete without sufficient reasoning.

Problem 1 *Let $C : \mathcal{X} \mapsto \{0,1\}^*$ be a nonsingular but not uniquely decodable code. Let $X \in \mathcal{X}$ have entropy $H(X)$. Compare (establish inequality or equality)*

(a) $H(C(X))$ and $H(X)$.

(b) $H(C(X^n))$ and $H(X^n)$, where $X^n = X_1 \dots X_n$, $X_i \in \mathcal{X}$ and $C(X^n) = C(X_1) \dots C(X_n)$.

Problem 2 *A fair coin is tossed until two heads or two tails are seen. Let X_1 and X_2 be the outcomes of the first two tosses, Y the outcome of the last (second or third) toss and N the number of tosses. Calculate $H(X_1)$, $H(X_2)$, $H(Y)$, $H(N)$, $I(X_1, Y)$, $I(X_2, Y)$, $I(X_1, X_2; Y)$, $I(Y, N)$, $I(X_1, N)$, $I(X_2, N)$, $I(X_1, X_2; N)$.*

Problem 3 *Let X and Y be random variables wiht five possible values: a, b, c, d, e. The distribution of X is $(1/2, 1/4, 1/8, 1/16, 1/16)$, while that of Y is $(1/4, 1/4, 1/4, 1/8, 1/8)$. We receive a message, a binary codeword of length three. We know that it is the codeword of a value of one of the two random variables. With probabilty 60% the outcome of X and with probabilty 40% the outcome of Y was sent. We also know that in case of both random variables an optimal code is used. We would like to guess which one of the five possible messages: a, b, c, d, e was sent. How to do that with minimal error probability?*

Problem 4 *Let $\mathbf{Z} = Z_1, Z_2, \dots$ be a binary, stationary Markov chain with transition probabilities*

$$\mathbf{P}\{Z_2 = 0 | Z_1 = 0\} = \frac{4}{5},\ \mathbf{P}\{Z_2 = 1 | Z_1 = 0\} = \frac{1}{5},\ \mathbf{P}\{Z_2 = 0 | Z_1 = 1\} = \frac{2}{5},\ \mathbf{P}\{Z_2 = 1 | Z_1 = 1\} = \frac{3}{5}.$$

Find the stationary distribution and the entropy of the chain.

Suppose that we start the chain from the stationary distribution and let $Y_1, Y_2, \dots$ be a sequence of independent, identically distributed binary random variables with $P(Y_i = 0) = \frac{2}{5}$. Define the source $\mathbf{X} = X_1, X_2, \dots$ with $X_i = 3Z_i + 2Y_i$. Find the entropy of source $\mathbf{X}$ if $Z_1, Z_2, \dots$ is independent from $Y_1, Y_2, \dots$.

Problem 5 *Give the definition of the ϵ-typical sequences. Why are they important? How can we use them in source coding?*