Combinatorial optimization

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1. Find the dual of the following linear program. Show that $x_1 = 3,\ x_2 = -1,\ x_3 = 0$ is an optimal solution of the primal program and $y_1 = 4,\ y_2 = 2,\ y_3 = 3,\ y_4 = 0$ is an optimal solution of the dual program.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\leq 1 \\ 2x_1 + 3x_2 + x_3 &\leq 3 \\ 3x_1 + x_2 + x_3 &\leq 8 \\ 2x_1 + 5x_2 &\leq 2 \\ \max\{17x_1 + 17x_2 + 17x_3\} & \end{aligned}$$

2. An electric company believes they will need the amounts of generating capacity shown in the following table during the next five years.

Year	1	2	3	4	5
Generating capacity (10^6 kwh)	80	100	120	140	160

The company has a choice of building (and then operating) power plants with the specifications shown in the following table.

Plant	Generating capacity (10^6 kwh)	Construction cost ($\$10^6$)	Annual Operating Cost ($\$10^6$)
1	70	20	1.5
2	50	16	0.8
3	60	18	1.3
4	40	14	0.6

Any of the four plants can be built such that they can start to operate at the beginning of any of the five years. Operating costs have to be paid even in the first year of operation. The company wants to decide which plants to build and when to build them such that the total costs of meeting the generating capacity requirements are minimized.

Formalize this problem as an LP or IP.

3. Write the following linear programming problem in the $\max\{\underline{c}^T\underline{x} | A\underline{x} \leq \underline{b}\}$ form, then give its dual.

$$\begin{aligned} 2x_1 - 3x_2 - 4x_3 &\leq 12 \\ x_1 + x_2 + 3x_4 &\geq 10 \\ x_2 - 2x_3 + x_4 &= 4 \\ \max 2x_1 + 12x_2 &- 3x_3 \end{aligned}$$

Does the dual program have a solution? Does the primal program have an optimal solution?

4. A company produces T–shirts in three sizes, large, medium, and small, which yield a net unit profit of \$3.50, \$2.80, and \$2.20, respectively. The company has three centers where T–shirts can be manufactured and these centers have a capacity of turning out 230, 280, and 190 pieces of T–shirts per day, respectively, regardless of the size or combination of sizes involved. Manufacturing T–shirts requires cotton yarn and each piece of large, medium, and small sizes produced require 190, 160, and 140 grams of cotton yarn, respectively. The centers 1, 2, and 3 have 50, 42, and 35 kilograms of cotton yarn available per day, respectively. Market studies indicate that there is a market for 310, 240, and 210 pieces of the large, medium, and small sizes, respectively, per day. How many pieces of each of the sizes should be produced at the various centers in order to maximize the profit?

5. "Mama's Kitchen" serves from 5:30 a.m. each morning until 1:30 p.m. in the afternoon. Tables are set and cleared by busers working 4–hour shifts beginning on the hour from 5:00 a.m. through 10:00 a.m. Most are college students who hate to get up in the morning, so Mama's pays \$9 per hour for the 5:00, 6:00, and 7:00 a.m. shifts, and \$7.50 per hour for the others. (That is, a person works a shift consisting of 4 consecutive hours, with the wages equal to 4×\$9 for the three early shifts, and 4×\$7.50 for the 3 later shifts.) The manager seeks a minimum cost staffing plan that will have at least the number of busers on duty each hour as specified below:

	5 am	6 am	7 am	8 am	9 am	10am	11am	Noon	1 pm
required number	2	3	5	5	3	2	4	6	3

6. A premium car manufacturer has 2 distribution centers in Central-Europe. An examination of their shipping department records indicates that, in the upcoming quarter, the distribution centers located in Prague and Budapest will have in inventory 60, 75 of its new supercar, respectively. Quarterly orders submitted by dealerships serviced by the distribution centers require the following numbers of the supercars for the upcoming quarter:

	Number of units		Number of units
Dealer A	25	Dealer C	30
Dealer B	40	Dealer D	35

Transportation costs (in euros per car) between each distribution center and the dealerships are as shown in the table below.

Distribution centers	Dealers A	B	C	D
Prague	750	650	1750	900
Budapest	900	300	450	1200

The manufacturer wants to minimize the transportation costs from the distribution centers to the dealerships.

Formalize this problem as an LP or IP. (You do not have to solve it.)

7. Give the dual of the following LP. Does the dual program have an optimal solution?

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &\leq 1 \\
2x_1 + 3x_2 + x_3 &\geq -3 \\
3x_1 + x_2 + x_3 &\leq 8 \\
2x_1 + 5x_2 &\leq 2 \\
x_1, x_2, x_3 &\geq 0 \\
\max\{x_1 + 3x_2 + 6x_3\}
\end{aligned}
$$