Combinatorial optimization

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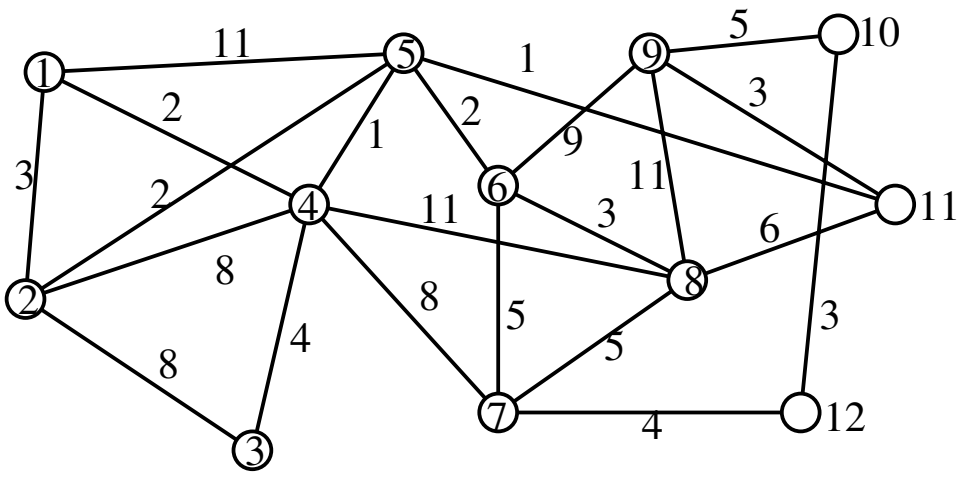
1. Let $G = (V(G), E(G))$ be the following graph: $V(G) = \{1, 2, 3, 4, 5, 6, 7, 8\}$
   $E(G) = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{7, 8\}$
   Let $H = (V(H), E(H))$ be the following graph: $V(H) = \{1, 2, 3, 4\}$
   $E(H) = \{\{1, 2\}, \{1, 4\}, \{2, 3\}$.
   (a) Draw a diagram of G.
   (b) Determine the degree of vertex 3.
   (c) Does G contain an isolated vertex?
   (d) Is G simple?
   (e) Is G connected?
   (f) Determine the number of connected components of G.
   (g) How many cycles does G contain?
   (h) Draw a diagram of H.
   (i) Is H a subgraph of G?
   (j) Is H a spanning subgraph of G?
   (k) Is H an induced subgraph of G?

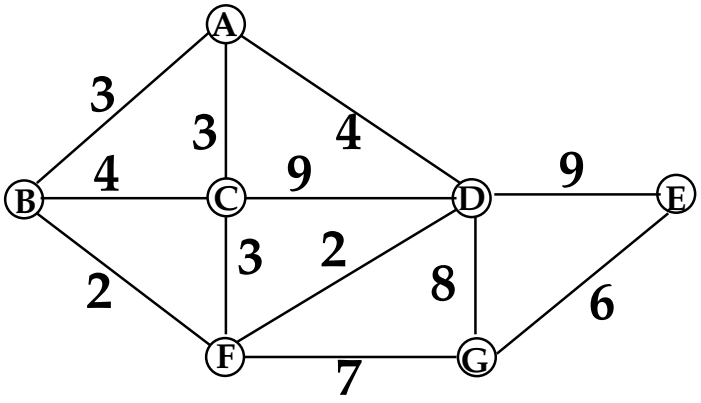
2. Draw a tree which contains exactly two vertices of degree three, one vertex of degree two, four vertices of degree one and nothing else.

3. An n vertex graph does not contain a cycle and the number of its connected components is k. How many edges does this graph have?

4. Find a minimum weight spanning tree in the following graph. (Use Kruskal's algorithm.)

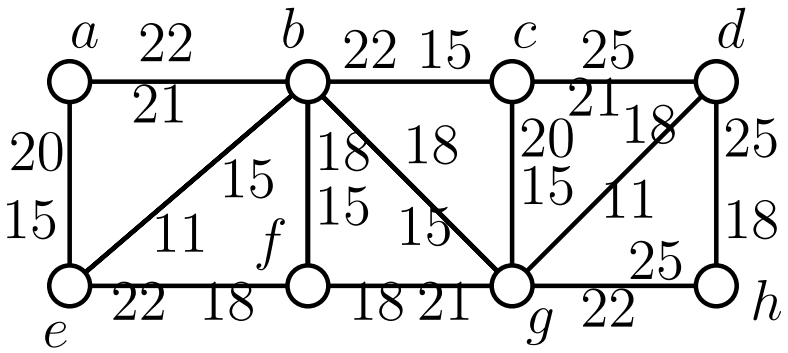


5. Find a minimum weight spanning tree in the following graph.



6. We own a construction company. A county want to build a sewage system which connects all the towns and the wastewater treatment facility. The above graph represents the county. The wastewater treatment facility is located at vertex E and the rest of the vertices are towns. The numbers are the building price of the correspponding sewage sections proposed by some of our competitors. (Fortunately we have a friend at the country's council who sent us this important data.) We know that the county will pay for the cheapest sewage system. We can build a direct sewege pipeline between vertices A and E (this is not shown in the graph). How much shall we ask for this section if we want to maximize our profit?

7. Here is a map of a country's road system which is obsolete. The country also does not have a motorway. The vertices are the towns and the edges are roads.



The government wants to improve the infrastructure. It plans to create a motorway network by upgrading existing roads in such a way that the created motorway network connects all the towns. The roads which will not be upgraded have to be reconstructed due to their bad condition. Each edge has two cost, the smaller one is the reconstruction cost and the other one is the upgrade cost (both are in million dollars). Of course the government wants to spend as few money on this project as possible. Suggest a plan! How much is the cost of the cheapest project?

8. Let G be a simple connected graph and let e and f be two edges of G. Show that there is a spanning tree of G which contains e and f simultaneously.