Duality, Integer Programming

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BME

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These statements are equivalent

1. $c^T x$ is bounded above on the set of solutions of $Ax \leq b$.
2. The system $Az \leq \underline{0}$, $c^T z > 0$ does not have a solution.
3. The system $y^T A = c^T$, $y \geq 0$ has a solution.

Proof: 1.$\rightarrow$ 2.: Indirectly assume that there is a solution z_0 of $Az \leq \underline{0}$, $cz > 0$. Consider a solution x_0 of $Ax \leq b$.

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Then $x_0 + \lambda z_0$ is also a solution of $Ax \leq b$ if $\lambda > 0$:
$A(x_0 + \lambda z_0) = Ax_0 + \lambda Az_0 \leq b + \lambda \underline{0} = b$.

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3.$\rightarrow$ 1.: Let x_0 be any solution of $Ax \leq b$ and let y_0 be a solution of $y^T A = c^T, y \geq 0$. Then:
$c^T x_0 = y_0^T Ax_0 \leq y_0^T b = s$, where s is a number and it is an upper bound on the objective function $c^T x$, because the value of s does not depend on the choice of x_0. $\square$

Corollary of the proof:
The objective function of the linear program $\max\{c^T x \mid Ax \leq b\}$ is bounded above by $y^T b$, where y is a solution of the system $y^T A = c^T$, $y \geq \underline{0}$. (Or equivalently written: $A^T y = c$, $y \geq \underline{0}$).

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If we want the best upper bound on the objective function $c^T x$, then we are looking for the vector y which minimizes $b^T y$ respect to $A^T y = c$, $y \geq \underline{0}$. This is another linear program.

This linear program is called as the **dual**, while the original linear program is called as the **primal** program.

Dual of the soft drink problem:

Primal problem:

$$2x_1 + 3x_2 \leq 18$$
$$4x_1 + 1x_2 \leq 16$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$\max \; x_1 + x_2$$

Matrix representation:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 18 \\ 16 \\ 0 \\ 0 \end{bmatrix}$$

$$\max \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Dual Problem:

$$\begin{bmatrix} 2 & 4 & -1 & 0 \\ 3 & 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{y} \geq \underline{0}$$

$$\min \begin{bmatrix} 18 & 16 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$2y_1 + 4y_2 - y_3 = 1$$
$$3y_1 + y_2 - y_4 = 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$y_3 \geq 0$$
$$y_4 \geq 0$$
$$\min \; 18y_1 + 16y_2$$

Equivalent form of this problem:

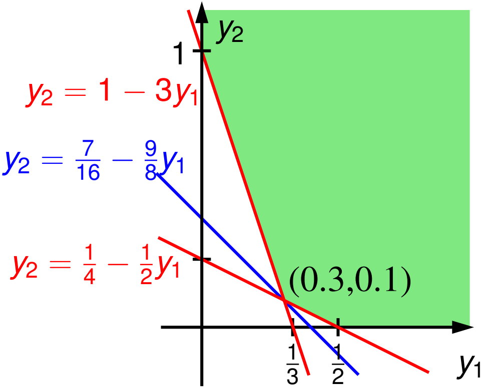
Each of the variables y_3 and y_4 appear in exactly one equality, they are non-negative and they do not appear in the objective function. They are called “slack” variables in the terminology and their addition is a method to convert inequalities to equalities by keeping solvability and the optimum value.

$$\begin{aligned} 2y_1 + 4y_2 - y_3 &= 1 \\ 3y_1 + y_2 - y_4 &= 1 \\ y_1, y_2, y_3, y_4 &\geq 0 \\ \min\ 18y_1 + 16y_2 & \end{aligned} \qquad\qquad \begin{aligned} 2y_1 + 4y_2 &\geq 1 \\ 3y_1 + y_2 &\geq 1 \\ y_1, y_2 &\geq 0 \\ \min\ 18y_1 + 16y_2 & \end{aligned}$$

If we omit the slack variables, then we obtain two inequalities instead of equalities and the obtained linear program has the same optimum value.
Furthermore a solution of the obtained problem can be easily converted to the solution of the original problem and vice versa.

Solving the dual by the graphical method

$$2y_1 + 4y_2 \geq 1$$
$$3y_1 + y_2 \geq 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$\min\ 18y_1 + 16y_2$$



The lines which correspond to the objective function is $y_2 = \frac{s}{16} - \frac{9}{8}y_1$. The smallest s which gives intersection with the solution set is $s = 7$.
So the optimal solution is $y_1 = \frac{3}{10}$, $y_2 = \frac{1}{10}$ and the value of this solution is 7.

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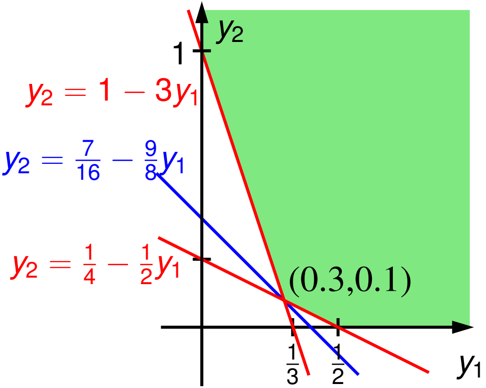

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This was the optimum value of the primal linear program!
It is not a coincidence!

Duality Theorem

Theorem

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$ If the objective function $c^T x$ on the solutions of $Ax \leq b$ is bounded above, then:

1. The dual program $\min\{b^T y \mid A^T y = c, y \geq \underline{0}\}$ has a solution and it is is bounded below.
2. $c^T x$ has a maximum on the set $\{x \mid Ax \leq b\}$ and similarly $b^T y$ has a minimum on the set $\{y \mid A^T y = c, y \geq \underline{0}\}$.
3. $\max\{c^T x \mid Ax \leq b\} = \min\{b^T y \mid A^T y = c, y \geq \underline{0}\}$

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We have proved **(1)**. We are going to prove **(2)** and **(3)** by a lemma stated in the next slide.
The statement of **(3)** is called the Duality Theorem.

Lemma

If $Ax \leq b$ has a solution, t is a real number and the system $Ax \leq b$, $c^T x \geq t$ does not have a solution, then the system $y^T A = c,\ y \geq \underline{0},\ y^T b < t$ has a solution.

Proof:

We can write the system $Ax \leq b,\ c^T x \geq t$ in the following matrix form:

$$\left[\frac{A}{-c^T}\right] \cdot x \leq \left[\frac{b}{-t}\right]$$

Since it does not have a solution, by the 1st version of Farkas' Lemma the following system does have a solution:

$$\left[\frac{y}{\lambda}\right]^T \cdot \left[\frac{A}{-c^T}\right] = \underline{0},\ \left[\frac{y}{\lambda}\right]^T \geq \underline{0},\ \left[\frac{y}{\lambda}\right]^T \cdot \left[\frac{b}{-t}\right] < 0$$

Which is equivalent to: $y^T A = \lambda c^T$, $y \geq \underline{0},\ \lambda \geq 0$, $y^T b < \lambda t$.
If $\lambda = 0$, then $y^T A = 0, y \geq 0, y^T b < 0$ has a solution and the 1st version of Farkas' Lemma implies that $Ax \leq b$ does not have a solution, which is is a contradiction, so $\lambda \neq 0$.
Therefore for $y' = y/\lambda$ we have that $y'^T A = c,\ y' \geq \underline{0}, y'^T b < t$.

□

Proof of the Theorem

Claim: The existence of the optimum solution

If $\sup\{c^T x | Ax \leq b\}$ exists, so the objective function is bounded by above, then $\sup\{c^T x | Ax \leq b\} = \max\{c^T x | Ax \leq b\}$.

Proof: Let $t = \sup\{c^T x | Ax \leq b\}$. Indirectly assume that there is no x_0 such that $Ax_0 \leq b$ and $c^T x_0 = t$. Then we apply the previous lemma, so we have an y_0 which satisfies that $y_0 \geq \underline{0}$, $y_0^T A = c^T$ and $y_0^T b < t$. Therefore: $t = \sup\{c^T x | Ax \leq b\} = \sup\{y_0^T Ax | Ax \leq b\} \leq \sup\{y_0^T b | Ax \leq b\} = y_0^T b < t$, which is a contradiction. □

We can use this result to show that $\min\{b^T y \mid A^T y = c, y \geq \underline{0}\}$ has an optimum solution. It can be written as:

$$\min \left\{ b^T y \;\middle|\; \begin{bmatrix} A^T \\ \hline -A^T \\ \hline \text{-I} \end{bmatrix} y \leq \begin{bmatrix} \text{c} \\ \hline \text{-c} \\ \hline \underline{0} \end{bmatrix} \right\} = \min\{b^T y | By \leq d\} =$$

$-\max\{-b^T y | By \leq d\}$ and this has an optimal solution, which is an optimal solution of $\min\{b^T y \mid A^T y = c, y \geq \underline{0}\}$.

Proof of the Duality theorem:

Claim: The optimal values are the same

If $\max\{c^T x | Ax \leq b\}$ exists, then
$\max\{c^T x \mid Ax \leq b\} = \min\{b^T y \mid A^T y = c, y \geq \underline{0}\}$.

Proof: Let $t = \min\{b^T y \mid A^T y = c, y \geq \underline{0}\}$, so t is the optimum value of the dual program. We have seen that t exists. Indirectly assume that $t > \max\{c^T x \mid Ax \leq b\}$. This means that the system $Ax \leq b$, $c^T x \geq t$ does not have a solution. The lemma implies that the system $y \geq \underline{0}$, $y^T A = c$ and $y^T b < t$ has a solution. Let y_0 be such a solution. y_0 is also a solution of the dual, and $y_0^T b$ is less than the optimaum value of the dual, which is a contradiction.

Other forms of duality

In many problems, the variables must be non-negative. The dual of these problems have the same non-negativity criteria.

Primal Program:

$$\begin{aligned} Ax &\le b \\ x &\ge 0 \\ \max\ c^T x \end{aligned}$$

Dual Program:

$$\begin{aligned} A^T y &\ge c \\ y &\ge 0 \\ \min\ b^T y \end{aligned}$$

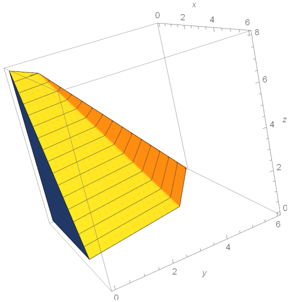
For these programs the duality theorem gives that $\max\{c^T x \mid Ax \le b,\ x \ge 0\} = \min\{b^T y \mid A^T y \ge c,\ y \ge 0\}$

Remark: The dual of the dual program is the primal program.

Recall: Softdrink problem with 3 drinks:

Assume that we also sell a third drink for 1€and it contains 2 deciliter of juice and 2 deciliter of water.
The variables x_1, x_2, x_3 encodes that how many euros do we earn from selling each drink. We obtain the following set of inequalities:

$$
\begin{aligned}
2x_1 + 3x_2 + 2x_3 &\leq 18 \\
4x_1 + x_2 + 2x_3 &\leq 16 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
x_3 &\geq 0
\end{aligned}
$$



The feasible region is a polyhedron. The objective function $s = x_1 + x_2 + x_3$ is a plane. Therefore we are looking for a plane, which intersects this polyhedron and where s is maximal.

An application of duality

We have seen that the graphical method works well when there are only two variables. If our linear program contains at most two inequalities, except the nonnegativity criteria, then the dual program has an equivalent form which has at most two variables!
Now we can handle our earlier problem which leaded to a 3 dimensional problem:

Primal:

$$
\begin{aligned}
2x_1 + 3x_2 + 2x_3 &\leq 18 \\
4x_1 + x_2 + 2x_3 &\leq 16 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
x_3 &\geq 0 \\
\max\ x_1 + x_2 + x_3 &
\end{aligned}
$$

Dual:

$$
\begin{aligned}
2y_1 + 4y_2 &\geq 1 \\
3y_1 + y_2 &\geq 1 \\
2y_1 + 2y_2 &\geq 1 \\
y_1 &\geq 0 \\
y_2 &\geq 0 \\
\min\ 18y_1 + 16y_2 &
\end{aligned}
$$

Solving the 3 drink softdrink problem's dual

Dual:

$$2y_1 + 4y_2 \geq 1$$
$$3y_1 + y_2 \geq 1$$
$$2y_1 + 2y_2 \geq 1$$
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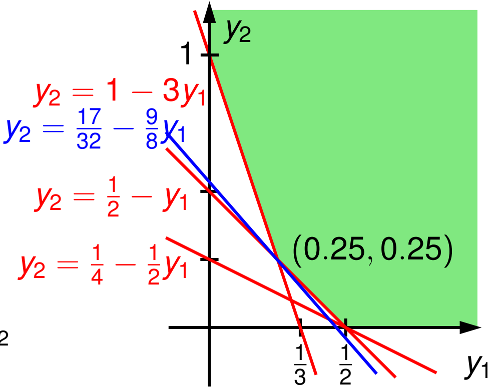

So the optimal solution is $y_1 = 1/4$, $y_2 = 1/4$ with optimum value $\frac{17}{2}$. This is also the optimum value for the Primal program, but we do not know an optimal solution, yet.

Solving the 3 drink softdrink problem's dual

Dual:

$$2y_1 + 4y_2 \geq 1$$
$$3y_1 + y_2 \geq 1$$
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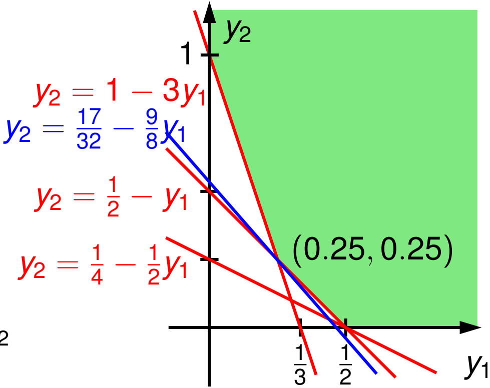

So the optimal solution is $y_1 = 1/4$, $y_2 = 1/4$ with optimum value $\frac{17}{2}$. This is also the optimum value for the Primal program, but we do not know an optimal solution, yet.

If somebody give a solution of the primal whose value is this, then it is optimal.

$x_1 = 0$ $x_2 = 1$ $x_3 = 7.5$ is a solution and $x_1 + x_2 + x_3 = 8.5$, so it is an optimal solution of the Primal program.

Complexity of Linear Programming

The classes P, NP and $NP - Complete$ contain decision problems. Therefore we define the decision version of Linear Programming.

Decision version of LINEAR PROGRAMMING:
Input: A matrix $A \in \mathbb{R}^{m \times n}$, a vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ and a number k.
Question: Is there a vector $x \in \mathbb{R}^n$ which satisfies that $Ax \leq b$ and $c^T x \geq k$?

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Theorem
LINEAR PROGRAMMING is in P.

There are many polynomial time algorithms which solves a linear program. However in the applications usually Simplex method is used, which is not a polynomial time algorithm, but it runs pretty well on inputs which come from the real world.

A modification of the soft drink problem:

We have two kind of products which we serve in cups: Cup A contains 2 deciliters of juice and 4 deciliters of water for 1€. Cup B contains 3 deciliters of juice and 1 deciliter of water for 1€. We sell cups of drinks which can not be divided. Now we have 20 deciliters of juice and 15 deciliters of water. How many cups of A and B should be produced to maximize our profit?

$$
\begin{aligned}
2x + 3y &\leq 20 \\
4x + 1y &\leq 15 \\
x, y &\geq 0 \\
x, y &\in \mathbb{Z} \\
\max x + y &
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$$2x + 3y \leq 20$$
$$4x + 1y \leq 15$$
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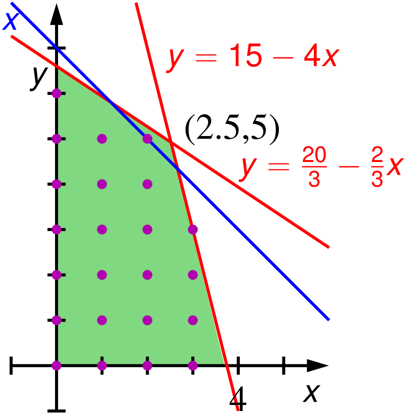
$$x, y \in \mathbb{Z}$$
$$\max x + y$$

The optimal solution is $x = 2, y = 5$, while the optimal solution of the corresponding LP is (2.5,5).



Integer Programming

An integer programming problem is the following:
Input: A matrix $A \in \mathbb{R}^{m \times n}$ and vectors $\underline{b} \in \mathbb{R}^m$ and $\underline{c} \in \mathbb{R}^n$.
Task: Find an integer vector $\underline{x} \in \mathbb{Z}^n$ which satisfies $A\underline{x} \leq \underline{b}$ and maximizes $\underline{c}^T \underline{x}$.

Some other forms of integer programming problems:

- Minimize $\underline{c}^T \underline{x}$, respect to $A\underline{x} \leq \underline{b}$, $\underline{x} \in \mathbb{Z}^n$.
- Maximize/minimize $\underline{c}^T \underline{x}$, respect to $A\underline{x} \leq \underline{b}$,$\underline{x} \geq 0$, $\underline{x} \in \mathbb{Z}^n$.
- Maximize/minimize $\underline{c}^T \underline{x}$, respect to $A\underline{x} = \underline{b}$, $\underline{x} \geq 0$, $\underline{x} \in \mathbb{Z}^n$.

Do we have a duality theorem for integer programs?

Assume that an integer program $\max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\}$ is given. We have four problems:

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- The integer program $\max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\}$, denote its optimum value by $\max_{IP}$.
- The dual integer prog. $\min\{y^T b \mid A^T y = c, y \geq 0, y \in \mathbb{Z}^m\}$, denote its optimum value by $\min_{DIP}$.
- The linear program $\max\{c^T x \mid Ax \leq b\}$, denote its optimum value by $\max_{LP}$.
- The dual linear program, $\min\{y^T b \mid A^T y = c, y \geq 0\}$ denote its optimum value by $\min_{DLP}$.

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- The dual integer prog. $\min\{y^T b \mid A^T y = c, y \geq 0, y \in \mathbb{Z}^m\}$, denote its optimum value by $\min_{DIP}$.
- The linear program $\max\{c^T x \mid Ax \leq b\}$, denote its optimum value by $\max_{LP}$.
- The dual linear program, $\min\{y^T b \mid A^T y = c, y \geq 0\}$ denote its optimum value by $\min_{DLP}$.

If x is the solution of the IP, then it is also a solution of the corresponding LP, but it is not true in the opposite direction. Assume that all of these programs have optimal solutions. In this case, we have that:

$$\max_{IP} \leq \max_{LP} = \min_{DLP} \leq \min_{DIP}$$

Conclusion: $\max_{IP} \leq \min_{DIP}$. Equality usually does not hold.

Complexity of the decision version of Integer Programming

Decision version of INTEGER PROGRAMMING (IP for short):
Input: A matrix $A \in \mathbb{R}^{m \times n}$, vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ and a real number k.
Question: Is there an integer vector $x \in \mathbb{Z}^n$ which satisfies that $Ax \leq b$ and $c^T x \geq k$?

Theorem
INTEGER PROGRAMMING is NP-Complete.