Weak perfect graph theorem

Weak perfect graph theorem (Lovász '72): G is perfect <-->$\overline{G}$ is perfect.

Lovász' 2nd perfect graph theorem: G is perfect <--> For any induced subgraph H of G: $|V(H)| \le \alpha(H) \cdot \omega(H)$

Note that is the size of the largest indpendent (sometimes called stable) set.
Remainder: $A \subseteq V(G)$ is an independent set in graph G if no two elements of A are adjacent.

Lovász' 2nd perfect graph theorem implies the weak perfect graph theorem.

Proof: Let G be a perfect graph. Let $\overline{H}$ be an induced subgraph of $\overline{G}$. In this case H is an induced subgr of G. Note that $\alpha(\overline{J}) = \omega(J)$ for any graph J. We utilize the 2nd perfect graph theorem. Since G is perfect:

$$\alpha(\overline{H}) \cdot \omega(\overline{H}) = \omega(H) \cdot \alpha(H) \ge |V(H)| = |V(\overline{H})|$$

This holds for all induced subgraphs of $\overline{G}$, therefore by the 2nd perfect graph theorem, $\overline{G}$ is perfect. $\square$.

Proof of the 2nd perfect graph theorem:
The direction that G is perfect --> $|V(H)| \le \alpha(H) \cdot \omega(H)$ is easy:
Since G is perfect, $\chi(H) = \omega(H)$ so there is a proper coloring of H with $\omega(H)$ colors. Each color class in that coloring is an independent set, therefore its size is at most $\alpha(H)$. The coloring covers all the vertices of H, therefore $\alpha(H)\,\omega(H) \ge |V(H)|$.
The opposite direction requires some definitions and lemmas to prove.

Definition: G is called imperferct if G is not perfect.

Definition: G is called minimal imperfect if G is imperfect but any induced subgraph H of G which is not G is perfect.

Clearly any imperfect graph contains a minimal imperfect induces subgraph.

Lemma 1:
Let G be a minimal imperfect graph and A be an independent set of G. Then $\omega(G \setminus A) = \omega(G)$, where G\A is the induces subgraph of G whose vertex set is V(G)\A.

Proof: Indirectly assume that $\omega(G\setminus A) < \omega(G)$. Then:
G is minimal imperfect, thus G\A is perfect and therefore $\chi(G\setminus A) = \omega(G\setminus A)$
Pick an optimal proper coloring of G\A, so it uses $\omega(G\setminus A)$ colors. Use this coloring for G while we introduce a new color for A. In this way we obtain a proper coloring of G with $\omega(G\setminus A)+1$ colors.

$$\chi(G) \le \omega(G\setminus A)+1 \underset{\substack{\uparrow \\ \text{by the indirect assumption}}}{\le} \omega(G) \le \chi(G) \Rightarrow \underbrace{\chi(G)=\omega(G) \quad \underbrace{\forall \underset{\text{induced}}{H \subset G} \;\; \chi(H)=\omega(H)}_{\text{G is minimal imperfect}}}_{\text{G is perfect}}$$

But G cannot be impefect and perfect at the same time, so this is a contradiction. $\square$.

Lemma 2:

Let G be a minimal imperfect graph. Let $\alpha = \alpha(G)$ and $\omega = \omega(G)$.
Then there is a collection of independent sets $A_0, A_1, A_2, \ldots A_\omega, A_{\omega+1}, \ldots A_{2\omega}, A_{2\omega+1}, \ldots A_{\alpha\omega}$

and cliques $B_0, B_1, B_2, \ldots B_\omega, B_{\omega+1}, \ldots B_{2\omega}, B_{2\omega+1}, \ldots B_{\alpha\omega}$ which satisfy:

1) $\forall i: A_i \cap B_i = \emptyset$
2) $\forall i,j\ i \neq j: |A_i \cap B_j| = 1$

Proof:
Let $A = \{a_1, a_2, \dots a_\alpha\}$ be a maximum independent set, so its size is α. G is minimal imperfect, so $G \setminus \{a_i\}$ is perfect. By lemma 1, $\chi(G\setminus\{a_i\}) = \omega(G\setminus\{a_i\}) = \omega$. Pick an optimal coloring of $G\setminus\{a_i\}$ and denote its color classes by $A_{(i-1)\omega+1}, A_{(i-1)\omega+1}, \dots A_{i\omega}$. All of these are independent sets.

For any i, by lemma 1., $\omega(G\setminus A_i) = \omega(G)$ so for any Ai there is a maximum clique in G which is disjoint from Ai. Choose such a clique to be Bi.
Now we have the collections, we need to verify the 2 properties:
Let Bi be a clique from the clique collection:
Case 1: $A_0 \cap B_i = \emptyset$, Bi is a maximum clique in each $G\setminus\{a_k\}$, which is perfect, therefore it does not matter what optimal coloring of G\{ak} is choosen, each color class intersect Bi. Therefore by the choice rule of Bi we have that i=0.
Case 2: $A_0 \cap B_i \neq \emptyset$ Clearly the size of intersection of a clique and an independent set is either 0 or 1.
So $|A_0 \cap B_i| = 1$. Let $a_k = A_0 \cap B_i$

In G\{aj} where $j \neq k$ Bi is still a clique of size ω, so there is no matter what optimal coloring of G\{aj} is choosen, each color class intersect Bi. So $i \notin \{\omega(j-1)+1, \omega(j-1)+2, \dots, \omega j\}$
In G\{ak} Bi\{ak} is a clique of size $\omega-1$, and an optimal coloring of G\{ak} uses ω colors, so there is exactly one color class which is disjoint from Bi, and that must be Ai. $\square$

Lemma 3:
Let $A_1, A_2, \dots A_m$ and $B_1, B_2, \dots B_m$ be subsets of an n element set which satisfy:
1) $\forall i: |A_i \cap B_i| = 0$
2) $\forall i,j,\ i \neq j: |A_i \cap B_j| = 1$

then $m \leq n$.

Proof:
Let A be an mxn matrix whose ith row is the characteristic vector of Ai and let B be an nxm matrix whose jth row is the characteristic vector of Bj. By the 2 conditions:

$$A \cdot B = C = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & & \vdots \\ 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{bmatrix}$$ where C is an mxm matrix

The deterimnant of C is not zero, therefore the rank of C is m.
We know from linear algebra that min(n,m) $\geq$ rank(B) $\geq$ rank(A B)=rank(C)=m $\Rightarrow n \geq m$. $\square$

End of the proof of the 2nd perfect graph theorem:
Now we show that if for all H induced subgraph of G $|V(H)| \leq \alpha(H) \cdot \omega(H) \Rightarrow$ G is perfect.

Let H be a minimal imperfect graph. By lemma 2, we can pick collections of independent sets and cliques of H: $A_0, A_1, \dots A_{\alpha(H)\omega(H)}$; $B_0, B_1, \dots B_{\alpha(H),\omega(H)}$ which satisfy that: $\forall i\ |A_i \cap B_i| = 0$ and $\forall i,j\ i \neq j: |A_i \cap B_j| = 1$.

So by lemma lemma 3 $\alpha(H)\,\omega(H) + 1 \leq |V(H)|$.

Therefore if for all H induced subgraph of G $\alpha(H)\omega(H) \geq |V(H)|$ then G cannot contain a minimal imperfect subgraph and therefore it must be perfect. $\square$.