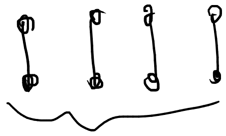
Turán type theorems

In this lecture we are going to talk about simple graphs.

Question: What is the maximum number of edges of an n vertex simple graph if it does not contain  as a subgraph? 0

What is the maximum number of edges of an n vertex simple graph if it does not contain  as a subgraph? $\lfloor \frac{n}{2} \rfloor$

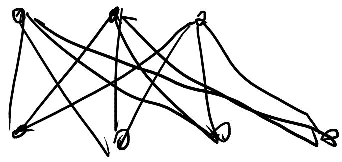


$\lfloor \frac{n}{2} \rfloor$

Question: What is the maximum number of edges of an n vertex simple graph if it does not contain a triangle?

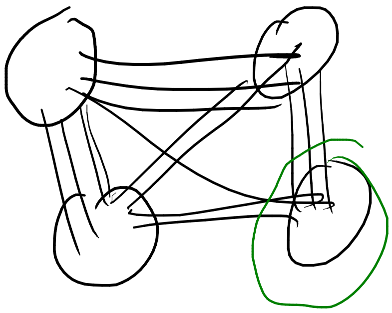
Theorem(Mantel 1907): $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$

Example: $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$:



Proof: Later

definition: A graph G is k-partite, if V(G) is partitioned to k independent sets, which are call the classes of the vertices. Example: a 4-partite graph:



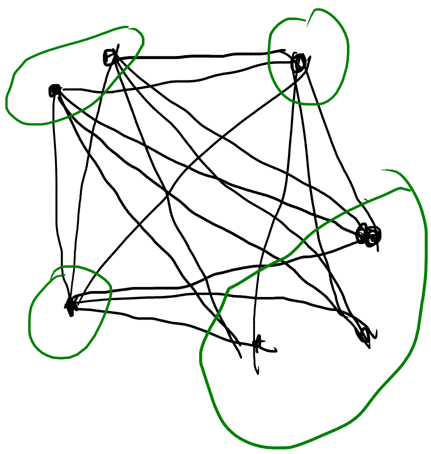
a vertex class

Note: If G is k-partite <==> $\chi(G) \leq k$

because the independent sets are the color classes when we color the graph with k colors.

definition: A graph G is a complete k-partite graph, if V(G) is partitioned to k independent sets and if vertices u an v are from different classes, then they are adjacent.

Example: A complete 4-partite graph:



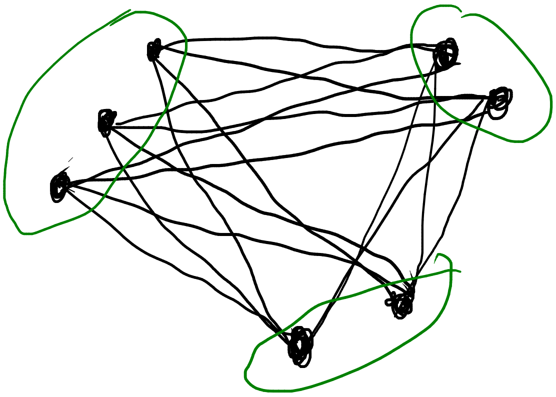
Note: If G is a complete k-partite graph, then

$\chi(G) = k.$

definition: The Turán graph $T_{n,m}$ is a complete m-partite n-vertex graph whose classes are as equal as possible, so if n=km+r, then there are m-r classes which contain k vertices and r classes which contain k+1 vertices.

Example: $T_{7,3}$:

$$T_{n,2} = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$$



Definition: ex(n;H) denotes the maximum number of edges of an n vertex simple graph which does not contain H as a subgraph. Ex(n,H) denotes the set of n vertex simple graphs which do not contain H as a subgraph and each of them has ex(n,H) edges. These graphs are called extremal graphs.

Theorem Turán '41: $ex(n, K_{m+1}) = |E(T_{n,m})|$, $Ex(n, K_{m+1}) = \{T_{n,m}\}$.

In other words: The n vertex graph which does not contain a K_{m+1} subgraph and has the most edges is $T_{n,m}$. and there is no other extremal graph.

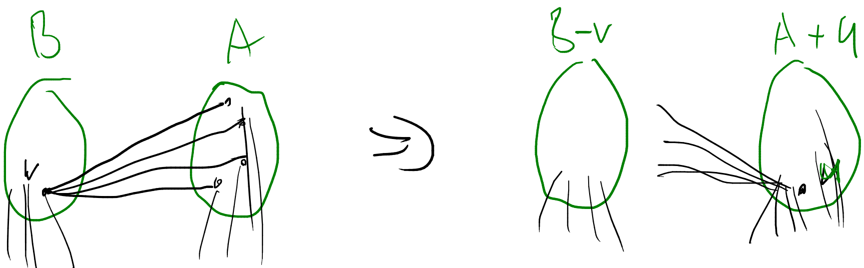
Proof: $T_{n,m}$ does not contain K_{m+1} as a subgraph, therefore $ex(n, K_{m+1}) \geq |E(T_{n,m})|$.

If we have an n vertex m-partite graph, then it has at most as many edges as $T_{n,m}$ has:

Clearly it is enough to consider complete n vertex m-partite graphs, because we can obtain such a graph by edge addition.

Let G be a complete n vertex m-partite graph which is not a Turan graph, therefore there are two classes of G such that the difference between their sizes is at least 2.

So lets say in class A there are a vertices, in class B there are b vertices and b>=a+2.



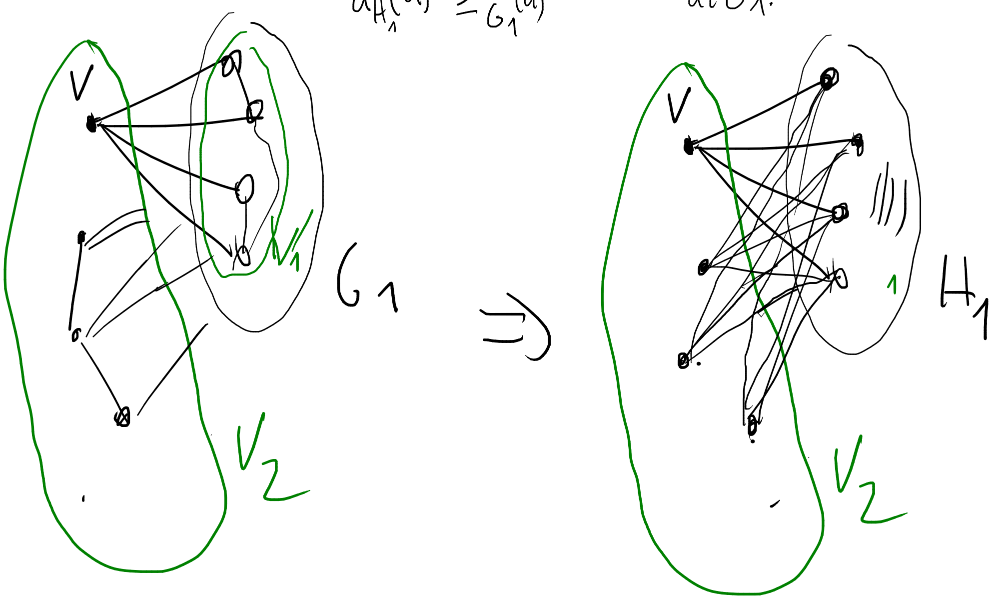
We delete a vertex v from B and add a vertex u to A and connect u to every vertex outside of A. v and u are adjacent to the same amount of vertices outside of AUB, but v had a neighbors in AUB and u has b-1 neighbors in AUB. The degree of each vertex of B increased by one, and the degree of each vertex of A decreased by one. So the sum of degrees increased by b-1-a+b-1-a=2b-2-2a>=2(a+2)-2-2a=2. So the new graph has more edges.

Therefore G does not have the maximum amount of edges among the n vertex m-partite graphs, therefore the only n vertex m-partite graph which has the maximum amount of edges among the n vertex m-partite graphs is $T_{n,m}$

Claim: If G is an n vertex graph and $K_{m+1} \notin G$ then there is an n vertex complete m-partite graph H, such that $d_G(v) \le d_H(v)$ for all $v \in V(G)$, and there is an equality at all vertices if and only if G is m-partite.
(Which means that |E(G)|<=|E(H)| and if G is not an m-partite graph, then |E(G)|<|E(H)|)

Proof: We use induction on m. Clearly if m=1, then the graph is the empty graph which is a complete 1-partite. So lets assume that the statment is true for m-1 and now let G bet an n vertex graph s.t. $K_{m+1} \notin G$.

Let v be a vertex of G whose degree is maximum. Let V1 be the neighbors of v and V2 be the rest of the vertices of G. V1 induces a graph G_1 and $G_1 \not\supseteq K_m$ because otherwise a clique of size m of G1 would be extended to a clique of size m+1 of G by v.
Therefore we can use the induction hypothesis, so G1 can be replaced by an m-1 partite graph H1 such that $d_{H_1}(u) \ge d_{G_1}(u)$ for all $u \in G_1$.



We delete all the edges in V2 but we add all the edges between V1 and V2. Denote the obtained graph by H. Since H1 is an m-1-partite graph, H is m-partite.
The degrees:

$d_G(v) = d_H(v)$

If $u \in V_1$, then: $d_G(u) \le d_{G_1}(u) + |V_2| \le d_{H_1}(u) + |V_2| = d_H(u)$

If $u \in V_2$, then $d_G(u) \le d_G(v) = |V_1| = d_H(u)$

because v is a max degree vertex in G

So H satisfies the desired properties. If equality holds at each vertex, $d_{G_1}(u) = d_H(u)\ \forall u \in V(G_1)$ andby induction G1 is m-1-partite and each element of V1 is incident to each element of V2 and sincev is a max degree vertex G is an m-partite graph.

The end of Turán's thm's proof:
If G is an n vertex graph which does not contain K_{m+1} as a subgraph, then there is an n vertex m-partite graph which has at least as many edges, and if it is not the Turán graph $T_{n,m}$, then $T_{n,m}$ has even more edges. Therefore $ex(n, K_{m+1}) = |E(T_{n,m})|$

Furthermore if G was not an m-partite graph, then the obtained m-partite graph has more edges than G, therefore the only extremel graph is $T_{n,m}$. □

It can be counted, that in the Turán graph $T_{n,m}$ $\quad d(v) \approx \left(1-\frac{1}{m}\right)n$

So approximately the 1/m part of the edges are missing.

Also: $\quad |E(T_{n,m})| \approx \left(1-\frac{1}{m}\right)\binom{n}{2}$

More precisely: $\quad \lim_{n\to\infty} \frac{|E(T_{n,m})|}{\binom{n}{2}} = 1-\frac{1}{m}$

Corrolary of Turán's them: If G is an n vertex graph and $|E(G)| > |E(T_{n,m})|$

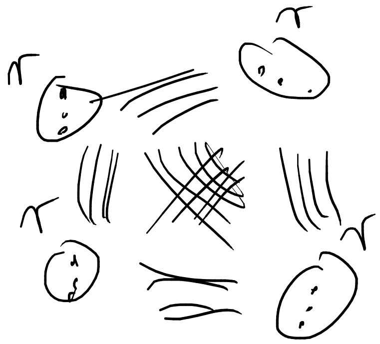
$$\Downarrow$$

$$K_{m+1} \subseteq G$$

Theorem: Erdős-Stone: For all $\varepsilon > 0$ and $r > 0$ there is a number $n_0(\varepsilon, r)$ s.t. if G has n vertices

$$n > n_0(\varepsilon, r) \text{ and } |E(G)| \geq |E(T_{n,m})| + \varepsilon \cdot n^2 \Rightarrow T_{(m+1)r,\, m+1} \subseteq G$$

So there is a blown-up K_{m+1} in G where every vertex of K_{m+1} has been replaced by r twins. This is the Turán graph $T_{(m+1)r,\, m+1}$.



An equivalent form of Erdős-Stone: For $\varepsilon > 0$ and $r > 0$ there is a number $n_0(\varepsilon, r)$ s.t. G has n vertice

$$n > n_0(\varepsilon, r) \text{ and } |E(G)| \geq \binom{n}{2}\left(1-\frac{1}{m}+\varepsilon\right) \Rightarrow T_{(m+1)r,\, m+1} \subseteq G.$$

Note: Let H be an arbitrary graph which satisfies $\chi(H) \leq m+1$. If r is sufficiently large, then $H \subseteq T_{(m+1)r,\, m+1}$

So any m+1 chromatic graph is a subgraph of a blown-up K_{m+1}.

Theorem: Erdős-Simonovits:

$$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} = 1-\frac{1}{\chi(H)-1}$$

Proof: $T_{n,\chi(H)-1}$ does not contain H as a subgraph, therefore

$$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} \geq \lim \frac{|E(T_{n,\chi(H)-1})|}{\binom{n}{2}} = 1-\frac{1}{\chi(H)-1}$$

By the Erdős-Stone thm, for any $\varepsilon > 0$ if n is big enough and G is an n-vertex graph:

$|E(G)| \geq \left(1 - \frac{1}{\chi(H)-1} + \varepsilon\right)\binom{n}{2} \Rightarrow G$ contains a blow-up $K_{\chi(H)}$ which contains H as a subgraph.

$$\Rightarrow \lim_{n \to \infty} \frac{ex(n,H)}{\binom{n}{2}} \leq 1 - \frac{1}{\chi(H)-1} \qquad \square.$$

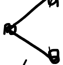
So the asymptotics of ex(n,H) is determined by the chromatic number of H.
However, when H is bipartite, the Erdős-Stone theorem only implies that ex(n,H)=o(n^2).
Not even the asymptotics of ex(n,H) is determined for any bipartite H, but we know it for some bipartite graphs.
Examples:

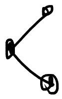
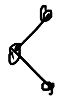
$ex(n, C_4) \approx n^{3/2}$ $\qquad ex(n, K_{2,2}) \approx n^{3/2}$

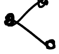
$ex(n, C_6) \approx n^{4/3}$ $\qquad ex(n, K_{3,3}) \approx n^{5/3}$

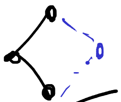
$ex(n, C_{10}) \approx n^{6/5}$ $\qquad c \cdot n^{2-\frac{2}{r}} \leq ex(n, K_{r,s}) \leq c \cdot n^{2-\frac{1}{r}}$

Theorem: Erdős-Kővári-Sós-Turán: $ex(n, C_4) \leq c \cdot n^{3/2}$

Proof: Let G be an n vertex graph which does not contain C4 as a subgraph. We count the number of  . If we calculate these at the middle vertices, then their numbers are:

$\#$  $= \sum_{i=1}^{n} \binom{d_i}{2}$ because any vertex together with its two neighbors form a  .

If we try to calculate the $\#$  at its endpoints, then what we see is that if we fix two vertices, there can ~~be~~ at most one  whose endpoints are the two fixed vertices, otherwise there would be a C4:



Let $\bar{d}$ denote the average degree, so $\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n}$

The function $f(n) = \binom{n}{2} = \frac{n(n-1)}{2}$ is convex, therefore we can apply Jensen's inequality:

$f\left(\sum_{i=1}^{n} \frac{1}{n} d_i\right) \leq \sum_{i=1}^{n} \frac{1}{n} f(d_i)$ so:

$$\binom{\bar{d}}{2} \leq \frac{\binom{d_1}{2} + \binom{d_2}{2} + \dots + \binom{d_n}{2}}{n}$$

$\Rightarrow \binom{\bar{d}}{2} \cdot n \leq \sum_{i=1}^{n} \binom{d_i}{2} = \#$ $\leq \binom{n}{2}$

$$\frac{\bar{d}(\bar{d}-1)}{2}n = \binom{\bar{d}}{2}n \le \binom{n}{2} = \frac{n(n-1)}{2}$$

$$\bar{d}(\bar{d}-1) \le n-1$$

$$\bar{d} \le \sqrt{n}$$

$$\bar{d} = \frac{\Sigma d_i}{n} = \frac{2e}{n} \le \sqrt{n} \Rightarrow e \le \frac{n^{3/2}}{2}$$

$$\Rightarrow ex(n, C_4) \le \frac{n^{3/2}}{2} \quad \square.$$