1. At most how many edges can an n-vertex graph have if it does not contain

- a cycle??

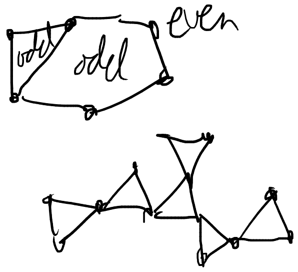
It is a tree, therefore n-1.

- an odd cycle?

It does not contain a triangle, therefore it is at most ex(n,K3)$=\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ and the complete bipartite graph $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ does not contain an odd cycle, so the answer is $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$.

- an even cycle?

In an extremal graph two cycles cannot have a common edge, because that would mean an even cycle: But they can have common vertices. So the cycles are placed over a tree.



To have the most number of edges, we need to have as many cycles as possible. The smallest cycle is the triangle, therefore the extremal graph is several triangles in a tree like structure, so if we delete an edge from each triangle, we obtain a tree. Therefore this graph have $\lfloor \frac{3}{2}(n-1) \rfloor$ edges.

- a path of length 3 nor a cycle of length 3?

Because a path of length 3 is forbidden and any cycle which is longer than 3 contains a path of length 3 as a subgraph, all of the cycles are forbidden. So the extremal graph is a tree and the answer is n-1.

- a spanning tree?

a graph has a spanning tree if and only if it is connected. So the graphs which does not have a spanning tree have more than one connected components. Clearly an extremal graph has only 2 connected components because if a graph has more than two, then we can add an edge between two connected components.
In an extremal graph both connected components are cliques. Lets say the size of the first clique is k, then the size of the second one is n-k. Assume that k>=n-k. If $k \neq n$ then remove a vertex from the smaller clique and add a vertex to the bigger one. In the bigger clique we obtain k new edges and lose n-k-1 edges in the smaller one. So the number of edges is changed by k-(n-k-1)=2k-n+1 which is bigger than 0.
So the extremal graph is a Kn-1 plus an isolated vertex.

2. In a university class there are 90 students. Some students have private conversations in Teams. It does not matter how we choose 10 students, there are at least two of them which have a private conversation. Prove that the number of private conversations are at least 405.

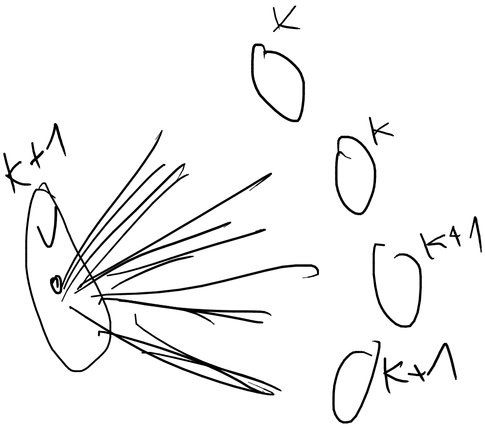
Lets G be the graph whose vertices are the students are two vertices are adjacent if the corresponding students have a private conversation.
The condition says that G does not have an independent set of size 10.
Which is equivalent to that $\overline{G}$ does not contain a K_{10}, therefore $\overline{G}$ has at most $|E(T_{90,9})|$ edges. $|E(T_{90,9})| = \frac{90 \cdot 80}{2} = 3600$.

$|E(G)| = |E(K_{90})| - |E(\overline{G})| \geq |E(K_{90})| - |E(T_{90,9})| = \frac{90 \cdot 89}{2} - 3600 =$

$= 4005 - 3600 = 405$.

3. Prove that the Turán graph $T_{n,m}$ does not contain a Hamiltonian cycle if and only if $m = 2$ and n is odd.

Let n=km+r where 0<=r<m. Then the degree of each vertex in $T_{n,m}$ is at least n-(k+1)>=n(1-1/m)-1 which is bigger than n/2 if m>2, so by Dirac's theorem it contains a Hamiltonian cycle.

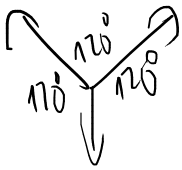


When m=2 and n is even, both classes contain n/2 vertices, therefore the degree of each vertex is n/2 and we can use Dirac's theorem again: $T_{n,2}$ has a Hamiltonian cycle.

When m=2 and n=2k+1, then the smaller class contain k vertices. If we delete those k vertices from the graph we obtain k+1 isolated vertices, so a graph which has k+1 connected components, therefore this Turán graph does not have a Hamiltonian cycle. Remainder: If a graph has a Hamiltonian cycle and we delete k vertices, the obtained graph cannot have more connected components than k.

4. Let $v_1, v_2, \ldots, v_n$ be vectors from a plane, $|v_i| \geq 1$. At least how many pairs satisfy $|v_i + v_j| \geq 1$?

In the plane we can choose at most 2 vectors of unit lenght such that the sum of any two of them has length smaller than 1.



If we have 3 vectors, then there are two of them, such that the angle between them is at most 120 degrees and therefore their sum has length at least 1.

So if we create a graph G whose vertices are the vectors and two vertices are adjacent if and only if their distance is less than one, than G does not contain a K3 as a subgraph. Therefore G has atmost as many edges as $T_{n,2}$ so $|E(G)| \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$.

The quantity which we are looking for is $|E(\overline{G})| = |E(K_n)| - |E(G)| \geq \binom{n}{2} - \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$

and this can be attained if half of the vectors are (0,-1) and the other half are (0,1).

5. At least how many vertices have a simple graph G which does not contain a triangle and $|E(G)| \geq 2|E(K_k)|$?

$$2|E(K_k)| = k(k-1)$$

Let n be the number of the vertices of G which is the smallest which satisfy the properties.

$T_{n-1,2}$ also does not have a triangle, therefore it must have less edges than k(k-1),

but $T_{n,2}$ has at least k(k-1) edges.

$$k(k-1) \leq |E(T_{n,2})| = \lfloor \tfrac{n}{2} \rfloor \cdot \lceil \tfrac{n}{2} \rceil \text{ and } k(k-1) \not\leq |E(T_{n-1,2})| = \lfloor \tfrac{n}{2} \rfloor \cdot \lceil \tfrac{n}{2} \rceil$$

$$\Rightarrow n = 2k-1.$$

11. ~~9~~. n not necesearily different points are given on the plane. At most how many pairs of vertices can be choosen from them such that the distance between the two vertices of a pair is exactly one?

Let G be a graph whose vertices are the points and two vertices are adjacent if and only if the the distance of the corresponding points is one.
Clearly, on the plane there are no 4 points such that the distance between any two of them is exactly one. So K4 is forbidden in G. The answer for the question is at most |E(G)|.

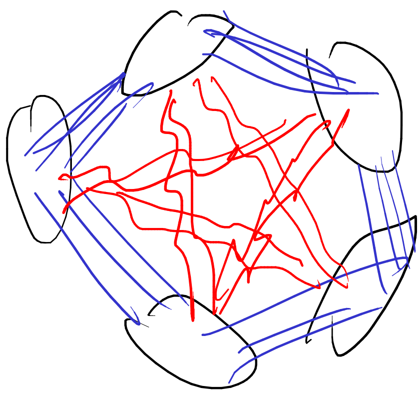
$$|E(G)| \leq |E(T_{n,3})| \approx \frac{1}{3}n^2$$

This can be attained if we put n/3 points to each vertex of an equilateral triangle having edge length of 1.

6. What is the maximum number of edges that an n vertex graph can have if its edges can be colored by two colors such that there is no monochromatic triangle.

The number of red edges is at most $\left(\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil\right)$ and we can say the same for the number of blue edges, however this is not the best upper bound which we can obtain, because it gives approximately $\binom{n}{2}$ which is a trivial bound.

Since R(3,3)=6, if we have a K6, then we obtain a monochromatic triangle. Therefore the graphs which can be colored by two colors such that there is no monochromatic triangle do not contain K6. The graph with most edges among them is $T_{n,5}$ which has approx $\frac{2}{5}n^2$ edges. We can color $T_{n,5}$ this way: and this coloring does not contain a monochromatic triangle.



7. There are n people in a party where there are no two people who knows each other. What is the minimum number of introductions (of two people to each other) which are needed to obtain the following properties:

   1. In any group of three people there are two of them who have been introduced to each other.
   2. Anybody can send a message to anybody else in such a way that the message is handed between people who know each other.

Let G be the graph where the vertices are the people and two vertices are adjacent if the two people have been introduced to each other. The conditions are:
G does not contain an independent set of size 3. G is connected.
What is the minimum number of edges of G?
G does not contain a K3, so it has at most $|E(T_{n,2})|$ edges. However $\overline{T_{n,2}}$ is not connected, therefore G cannot be $T_{n,2}$. $\overline{T_{n,2}}$ is two cliques, therefore adding an edge to it makes it connected, and since $T_{n,2}$ is the only extremal graph which does not contain a K3 $\overline{T_{n,2}}$ + an edge has the least amount of eges among the graphs which does not contain an independent set of size 3 and are connected.

8. A graph has 49 vertices and 1030 edges. Show that the chromatic number of this graph is at least 8 and it can be exactly 8.

Indirectly assume that its chromatic number is at most seven. In this case, it cannot contain a clique of size 8. Therefore the maximum number of its edges is $|E(T_{49,7})| = \frac{7 \cdot 6 \cdot 49}{2} = 1029$. But the given graph contains 1030 edges, therefore its chromatic number is at least 8.

An example: Pick $T_{49,7}$ and add an edge to it. $\chi(T_{49,7}) = 7$ and one endpoint of the new edge requires an 8th color.