2. Assume that we know that the Van der Waerden's theorem is true when we color by two colors. By using this, show that it is true when we color by three or more colors.

N(3,k)<=N(2,N(2,k)): Because if we have integers 1,2...N(2,N(2,k)) and we color it by three colors then there is either an airthmetic progression of length N(2,k)>k in the first color or there is an arithmetic prorgession of length N(2,k) which uses the second and third color. In the later case lets say that the common difference is d and the first element is a. Now if we color the numbers 1,2...N(2,k) by the same way how we colored this arithmetic progression by the 2nd and 3rd colors, then it has a monochromatic arithmetic progression of length k, starts at B and its common difference is D. This gives an arithmetic progression in the original one, which starts at a+Bd and its difference is dD.

3. Prove that for all $k$ positive integers there is a number $N(k)$ such that it does not matter how we color the numbers $1, 2, 3, \ldots, N(k)$ by $k$ colors, then there are three different numbers $x$, $y$, $z$ which have the same color and $x + y = 2z$.

By Van der Warden's theorem there is a number N(k,3) such that if we color the integers 1,2,3.. N(k,3) by k colors there is a monochromatic arithmetic progression of length 3. Consider such a monochromatic arithmetic progression: a, a+d, a+2d. x=a, y=a+2d z=a+d.

4. Prove that for all $t, k > 0$ there is a number $M = M(t, k)$ such that it does not matter how we color the numbers $1, 2, \ldots, M$ by $t$ colors there is always a monochromatic geometric progression whose length is $k$.

Consider the numbers $1,2...2^{N(k,t)}$.
Color the numbers and consider only the numbers which are powers of 2, so
$2, 2^2, 2^3, \ldots 2^i$.
Now we create a new coloring of the number numbers 1,2....N(k,t) by giving the color of $2^i$ to i. It has a monochromatic arithmetic progression of size k. Clearly there is a corresponding monochromatic geometric prorgression in $1,2,..2^{N(k,t)}$ whose length is k.

5. Let $a_n$ be an infinite monotone increasing sequence of natural numbers. Show that there is either an arbitrary long subsequence of $a_n$ such that any two elements of it are coprimes or there is an arbitrary long subsequence of $a_n$ such that no two elements of it are coprimes.

Let G be the following graph: Its vertices are the numbers and vi and vj are adjacent if and only if ai and aj are coprimes. Let k be an arbitrary number. If we consider the first K(k,k) vertices of G, then they span a subgraph which contains either a clique of size k or an independent set of size k. The clique correspond to a monotone increasing subsequence which all elements are coprimes, the independent set correspond to a monotone increasing subsequence which does not contain two elements which are coprimes. So we are done.

OR

Pick all the elements of the sequence which are even. If its infinite then we are done, since it contains infinitely many pairwise not coprime numbers.
Otherwise pick all the elements of the sequence which are divisible by 3.If its infinite then we are done, since it contains infinitely many pairwise not coprime numbers.
...
Otherwise pick all the elements of the sequence which are divisible by the ith prime.

If it is infinite then we are done.
...

If we have not found an infinite subsequence whose all elements are divisible by the same prime, then do the following:

If the sequence contains an even number, then select an even number and delete other even numbers.
If the remaining sequence contains a number which is divisible by 3, then select such a number and delete other numbers which are divisible by 3.
...
If the remaining sequence contains a number which is divisible by the ith prime, then select such a number and delete other numbers which are divisible by the ith prime.

Since there is no prime which divides infinetly many elements of the original sequence, at each step we delete finitely many elements and the remaining subsequnce still contains infinitely many elements. Therefore this procedure never stops and we select infinitely many coprime elements. So we are done.

6. Prove that for all $t > 0$ there is a number $M = M(t)$ such that it does not matter how we color the numbers $1, 2, \ldots, M$ by $t$ colors there are numbers $x, y, z$ which have the same color and $x + y = z$ and $x \neq y$.

This is basicly Schur's thm with an additional requirement. Int the proof of Schur's thm we end up with a graph whose triangle a,b,c is monochromatic and x=c-b, y=b-a, z=c-a.
If c-b=b-a, then x=y. But if we take R(4,4,4...4) numbers, then there is also a monochromatic K4 which contains 4 monochromatic triangles. Lets say that the vertices are a, b, c and d.
If a, b and c are not good because c-b=b-a, then d,b,a is good, because d is not a solution of c=2b-a.

7. Can we color the integers by two colors such that there is no monochromatic infinite arithmetic progression? What about infinite monochromatic geometric progressions?

Let's color the first number by blue, the next two numbers by red, the next three by blue the next four by red and continue this pattern. Clearly, if we have an arithmetic progression with difference d, then it cannot be monochromatic, bexause after a while the number of consecutive red numbers is bigger then d and the same holds for the number of consecutive blue numbers.

A similar construction works for avoiding a monochromatic geometric progression.
Color the first element blue, the next 2! elements red, the next 3! elements blue, the next 4! elements red, etc.

8. Show a $(k-1)^2$ vertex graph which neither contains a clique of size $k$ nor contains an independent set of size $k$.

A graph which contains k-1 connected components and each component is a $K_{k-1}$ is good.
It has (k-1)^2 vertices and clearly does neither
contains a clique of size k, nor an independent set of size k, because an independent set can contain only one element from each clique and there are only k-1 of them.

9. Prove that for all $n$ there is a number $K(n)$, such that any $K(n)$ distinct points on the plane determines at least n different distances.

Clearly if we have 4 points, then it is not possible that their pairwise distances are the same.

We show that $K(n) \leq R(\underbrace{4,4,\ldots 4}_{n-1}) = R$

Create a complete graph from the R points and assign the distance of the points as the colors of the edges. We know, that it does not contain a monochromatic K4.
Indirectly assume that there are only n-1 colors. By the def of R(4,4,..4), there is a monochromatic K4. This is a contradiction, therefore there are at least n colors, so there are at least n different distances.