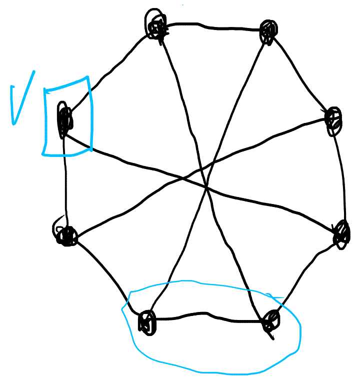
1. Let G be any graph which has at least 10 vertices. Show that $\omega(G) \geq 4$ or $\alpha(G) \geq 3$.

Let v be a vertex. If it is not adjacent to 4 vertices, then these vertices are either connected to each other so they induce a clique of size 4 or two of them are not adjacent which form an independent set of size 3 with v.
Otherwise at least 6 vertices are adjacent to G. The graph which is induced by these six vertices is either contains a clique of size 3 which with v form a clique of size 4, or an independent set of size 3. So we are done.

2. Proove that $R(3,3,3) \leq 17$ and prove that $R(3,4) = 9$.

Let v be a vertex in a 17 vertex complete graph and color its edges by three colors: blue, red, green. There is a color, w.l.o.g blue s. t. v is incident to 6 blue edges. The other endpoints of these edges induce a clique of size at least 6. If there is a blue edge, then that edge with the two adjacent edges going to v form a blue triangle.
If there is no blue edge, then this clique of size >=6 has only green and red edges therefore it either contains a red or a green triangle. So R(3,3,3)<=17.

An 8 vertex graph which neither contains a clique of size 3 nor an independent set of size 4: Clearly it does not contain a triangle. If we choose a vertex v, then 4 vertices are not adjacent to this, but that 4 vertices form a cycle of length 4, therefore no independent set of size 4 can be chosen. Therefore R(3,4)>8.



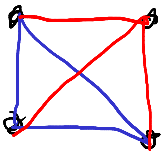
We show that R(4,3)<=9, of course R(4,3)=R(3,4).
Consider an arbitrary 9 vertex graph. If there is a vertex v which has 4 non-neighbours, then these vertices are either connected to each other so they induce a clique of size 4 or two of them are not adjacent which form an independent set of size 3 with v.
Therefore the degree of each vertex is at least 5. Furthermore, because the sum of the degrees must be even, there is a vertex w whose degree is at least six. The neighbours of w induce an at least six vertex graph which either has an independent set of size 3 or a clique of size 3 which can be extended by w to a clique of size 4.

3. a. We have colored the edges of a complete graph by red and blue. Prove that it has a monochromatic spanning tree.

If the blue edges form a connected graph, then it has a spanning tree which is blue. Otherwise, the there are at least two blue connected components. In that case if two vertices are in different blue components, then there is a red edge between them. If two vertices are in the same blue component then there is a two long red path between them which uses a vertex from another blue component. Therefore in that case the red edges form a connected graph which has a red spanning tree.

b. Is it true that a monochromatic Hamiltonian-path also exists?

No, example:



4. Prove, that $R_3(4,4) \leq 21$.

Let S be a 21 element set and color every 3 element subset by either red or blue. We would like to show that there are 4 elements such that any 3 element subset of them have the same color. Let v be an element of S. Consider the 3 element subsets which contain v. Remove v from them, then we end up with two element subsets, so a an edge colored complete graph. Let the color of edge (a,b) be the color of (v,a,b).
This complete graph has 20 vertices and 20= $\binom{4+4+2}{4-1} \geq R(4,4)$ so this graph contains a monochromatic clique of size 4.
W.l.o.g. we have a blue clique of size 4. Then if we put back vertex v we obtain 6 blue triangles (3 element subsets). If any three element of the clique of size 4 forms a blue triangle, then these elements with v form a 4 element subset such that all of its triangles are blue. Otherwise any three element of this size 4 clique form a red triangle, and in that case the vertices of the clique is a 4 element subset such that all of its triangles are red.

5. Prove the following inequality: $R_3(k,l) \leq R_2(R_3(k-1,l), R_3(k,l-1)) + 1$. What kind of upper bound (in magnitude) on $R_3(k,k)$ comes from this inequality?

Consider a set of size $R_3(k,l) \leq R_2(R_3(k-1,l), R_3(k,l-1)) + 1$. and let v be one of its elements.
Let's say that the first color is blue, the second color is red.
If we consider only the triangles which contain v and delete v we obtain an edge colored complete graph over $R_2(R_3(k-1,l), R_3(k,l-1))$ vertices. So it either contains a blue clique of size $R_3(k-1,l)$ or a red clique of size $R_3(k,l-1)$. In the first case either this blue clique contains an l element subset whose all triangles are colored red or it contains a k-1 element subset whose all triangles are colored blue. We can extend this k-1 element subset by v and all of the triangles are still colored blue because this was a blue clique.
We can handle the second case in the exact same way.

6. Prove, that if $c \geq 3$, then $R_t(n_1, n_2, \ldots, n_c) \leq R_t(n_1, n_2, \ldots, n_{c-2}, R_t(n_{c-1}, n_c))$.

We tell it for graphs, the same can be told for hypergraphs.
Assume that there is a graph having R(n1,n2,..nc-2,R(nc-1,nc)) vertices. Color it by n different colors. Assume that the two colors are the same. Then there is a color i such that it contains a clique of size ni whose color is i. If i<c-1 then we ore done.
If i is either c-1 or c, then in the graph there is a clique of size R(nc-1,nc) which is colored by c-1 and c. But then in that there is either a clique of size nc colored by c or a clique of size nc-1 colored by c-1.

7. ~~8~~. Show that for each positive integer k there is a thresshold $N(k)$, such that if $n > N(k)$ and we color the subsets of the set $[n] := \{1, 2, \ldots, n\}$ by k colors, then there are disjoint subsets X_1 and X_2 of $[n]$ such that the colors of X_1, X_2 and $X_1 \cup X_2$ are the same. Is this statement true for 3 disjoint subsets?

Create a complete graph G where the edge ij correspond to the subset {i,i+1,..j-1}.
If G has at least R(3,3,...3) elements where the number of arguments is t, then it does not matter how we color the graph (and the corresponding sets) we always obtain a monochromatic triangle which correspond to X, Y and XUY where X and Y are disjoint. This reasoning works if we have 3 or more disjoint subsets.

8. ~~9~~. Let $H(V, E)$ be a k uniform hypergaph which has less than 2^{k-1} edges. Show that the vertices of H can be colored by red and blue in such a way that no edge is monochromatic. (Each edge have blue and red vertices.)

Color each vertex independently and randomly in such a way that the probability that a vertex is blue is 1/2 and similarly the probability that a vertex is red is 1/2.
Let A be in edge of H. H contains k vertices, therefore:

$$\mathbb{P}(A \text{ is monochromatic}) = \mathbb{P}(\text{all vertices of } A \text{ are red}) + \mathbb{P}(\text{all vertices of } A \text{ are blue}) =$$
$$= \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k-1}$$

Let X_A be the random variable whose value is 1 if A is monochromatic and 0 otherwise.

$$\mathbb{E}X_A = \mathbb{P}(A \text{ is monochromatic}) = \left(\frac{1}{2}\right)^{k-1}$$

Let Y be the random variable whose value is the number of monochromatic edges of H.

$$Y = \sum_{A \in E} X_A$$

expectation is linear

$$\mathbb{E}Y = \mathbb{E}\sum_{A \in E} X_A \overset{\downarrow}{=} \sum_{A \in E} \mathbb{E}X_A < 2^{k-1} \cdot \left(\frac{1}{2}\right)^{k-1} = 1$$

The expected value of number of monochromatic edges is less than one. Therefore there is a coloring when there are 0 monochromatic edges.

9. ~~11~~. Show that if G is an n vertex graph, then $\max(\alpha(G), \omega(G)) \geq$ ~~$\frac{1}{2}$~~ $\log_4 n$.

Let k be the biggest number such that R(k,k)<=n. In that case $\max(\alpha(G), \omega(G)) \geq k$.

and $$n < R(k+1, k+1) < 2^{2k} = 4^k$$

$$\Rightarrow \log_4 n \leq k$$

$$\Rightarrow \log_4 n \leq k \leq \max(\alpha(G), \omega(G))$$