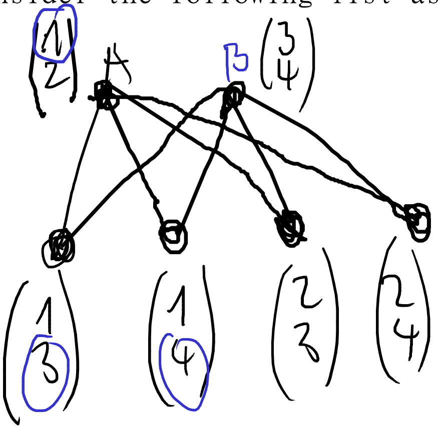
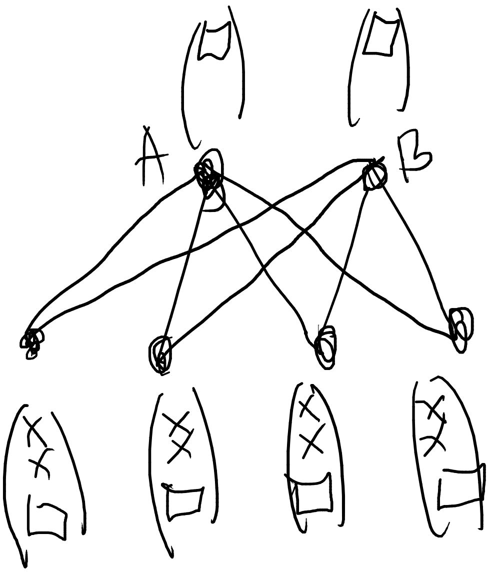
1. Determine the value of $ch(K_{2,4})$. ($K_{2,4}$ is the complete bipartite graph, where the two color classes contain 2 and 4 vertices.)

Its chromatic number is 2, so therefore ch(K2,4)>=2.
Consider the following list assignment:



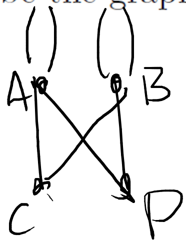
Due tue simmetry it does not matter whether we choose 1 or 2 at vertex A. Choose 1. Then there will be no choice for B. So K4,2 cannot be colored from these lists. Therefore ch(K2,4)>2.



If each list has lenght 3, then it does not matter how we choose from the lists of A and B, there is a color in the remaining lists which is not used for A or B. Pick such a color at each remaining vertex. This gives a proper coloring in case of any set of lists whose length is 3. Therefore ch(K2,4)<=3.
So all in all ch(K2,4)=3.

2. Let $K_{2,2,\dots,2}$ be the graph whose complement is n disjoint edges. Determine $ch\left(K_{2,2,\dots,2}\right)$.

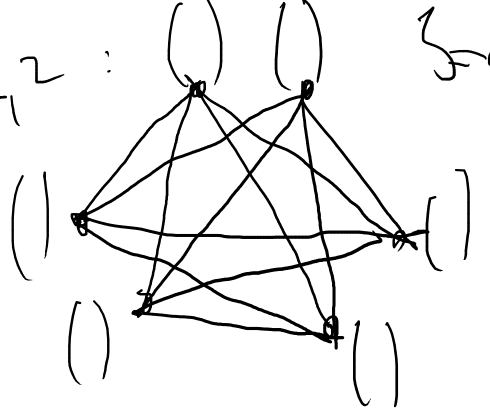
$K_{2,2}$:



$2=\chi(K_{2,2})\le ch(K_{2,2})\le 2$

Consider two long lists. If L(A) and L(B) both contains a color c, then assign c to both A and B and at least one color remains for C and D.
Otherwise A and B have different colors. Color C with an arbitrary color. Color D with a color to make sure that the color of C and D not covers the whole L(A) or L(B).
In this way we can complete the coloring.

$K_{2,2,2}$:



$3=\chi(K_{2,2,2})\le ch(K_{2,2,2})$

$\underbrace{K_{2,2,2..2}}_{n}$

$n=\omega(K_{2,2..2})\le\chi(K_{2,2})\le ch(K_{2,2..2})$

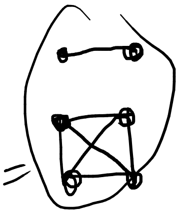
Consider lists of length n.
If there is an independent set of size two whose list contains the same color, then chose that color for them, remove that color from the other lists and we can use induction on the number of vertices.

If there is no independent set which contains the same color, then at each independent set there are 2n different colors.
Create a bipartite graph, where A={set of colors used at the lists} B={vertices of K2,2..,2}.
a color is adjacent to a vertex if that color appears at the list of the vertex. We show that each vertex can receive a different color, so there is a matching in this bipartite graph which covers B. We check the Hall condition: If we pick at most n vertex from B, then since each list has length n, they are adjacent to at least n colors. If we pick more than n vertex, then at least one independent set of size 2 of K2,2...2 is contained in the picked vertices, their color lists are disjoint, so these vertices are adjacent to at least 2n colors. |B|=2n, so we are done, the Hall condition is satisfied and this bipartite graph contains a matching which covers B, so each vertex of K_2,2...2 can be colored by a different color chosen from its list.

So we have shown that K2,2..2 can be colored from any set of lists whose length is n, so ch(K2,2..2)<=n and since we have seen the opposite ineqality, it is n.

3. Is it true that if $\chi(G) = ch(G)$, then $\chi(\overline{G}) = ch(\overline{G})$?

No. We have seen that $ch(K_{4,2}) \neq \chi(K_{4,2})$ but $\overline{K_{4,2}}$:

$G=$ 

$ch(K_2)=2$ $\quad ch(K_4)\le\Delta(K_4)+1=4$

$4=\chi(K_4)$ $\quad$ So $G=\overline{K_{4,2}}$ is a counterexample.

4. Let G be a graph. Is it true, that if for each vertex of G an at least $ch(G)$ long list is given, then there is an order of the vertices such that we can color the vertices greedily according to that order by selecting the smallest color which has not been used at the neighbors at each vertex?

Yes. Assume that we have a good choice of colors. Then the order should be the following: first color the vertices whose color is 1, then color the vertices whose color is 2, etc. each vertex receives a color whose order is not bigger then its previous color and when we want to color a vertex all of the previously colored neighbors of it have received smaller colors, so it can be colored.

5. Let G be a finite simple graph. Prove that if for each vertex v of G $|L(v)| > d(v)$ is satisfied, then G is L-colorable. ($d(v)$ denotes the degree of v)

We simply greedycolor. At each step a color which is not used at the neighbors is available at the list.

6. Show that if T is a tree which has at least two vertices, then $ch(T) = 2$.

$2\le\chi(T)\le ch(T)$

Do a depth first search and greedy color the vertices from two long lists
in the order how the DFS traverses the tree. In such a way when we try to color a vertex only one of its neighbors is already colored
so there is always a usable color at a two long list, so ch(T)<=2. All in all ch(T)=2.

7. Prove that if C is an odd cycle, then $ch(C) = 3$.

$$3=\chi(C)\le ch(C)\le \Delta(C)+1=3$$

$$\Rightarrow ch(C)=3$$

8. Let G be an arbitrary graph. We construct the graph $3G$ in the following way: We pick 3 disjoint copies of G and for each vertex v of G we connect the three copies of v to each other by 3 edges.

Prove that if G is planar, then the list chromatic number of $3G$ is at most 7. (So $ch(3G) \le 7$.)

By thomassen's thm ch(G)<=5. Consider 7 long lists. Denote the copies of G by G1, G2, G3. First chose a proper coloring for G1 from its lists. Since 7>ch(G) it can be done. remove the chosen color of v1 from the lists of its copies v2 and v3.
So In G2 each list has length at least 6. Pick a proper coloring from its list and remove the chosen color of v2 from the list of v3. The lists of G3 has length at least 5, so we can pick a proper coloring for G3 from those lists. By the construciton v1, v2, v3 received different colors, so we have constructed a proper coloring of 3G. This can be done to any list assignment where the length of the lists is 7, so ch(3G)<=7.

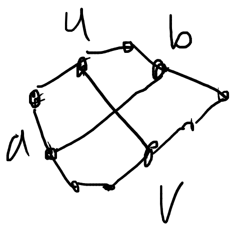
9. We were able to draw G into the plane in such a way that there is only one edge crossing (a point which is not a vertex and two edge cross each other at that point). Prove that:

a. $ch(G) \le 6$,

b. $ch(G) \le 5$.

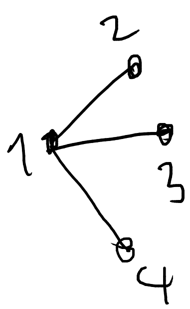
a) Consider 6 long lists. Lets say that the crossing edges are {u,v} and {a,b}. Pick a color of L(u) and remove it from other lists. So each list has length at least 5. Delete edge {u,v}. The obtained graph is planar so by Thomassen's thm it can be colored from lists of length at least five. Pick such a coloring. If u and v received different colors then it is a proper coloring of G. Otherwise use the deleted color to color u. That color is not used anywhere else so it is a proper coloring of G.

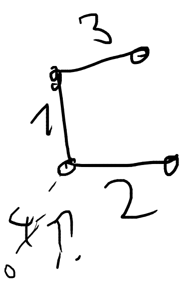
b)



Delete (u,v) and (a,b) and make the face containing u,b,a,v, be the outer face. If u and a are not adjacent, then put an edge between them and create an almost triangulated graph by adding edges. Then we can use the lemma which we used to prove thomasen's thm. Condiser 5 long lists.
Make L(u) and L(a) 1 long and L(v) and L(v) 3 long by deleting some colors in such a way that L(u) and L(v) are disjoint and L(a) and L(b) are disjoint. By the lemma a coloring can be choosen from these lists. This coloring is good for the original graph as well.

10. Show a graph which is not a line graph of a graph.





edge 4 should be adjacent to edge 1 but not to edge 2 and 3 which cannot be.