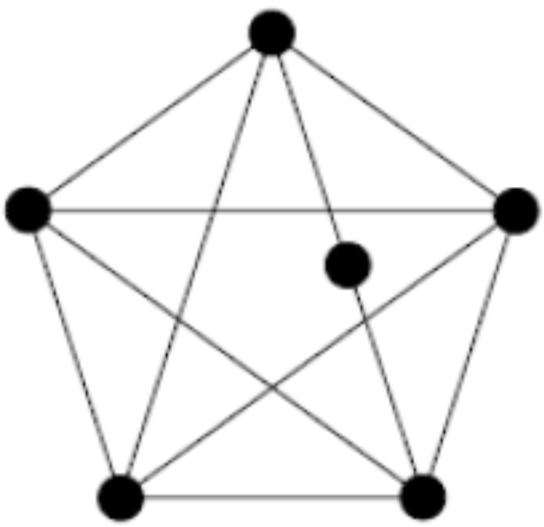
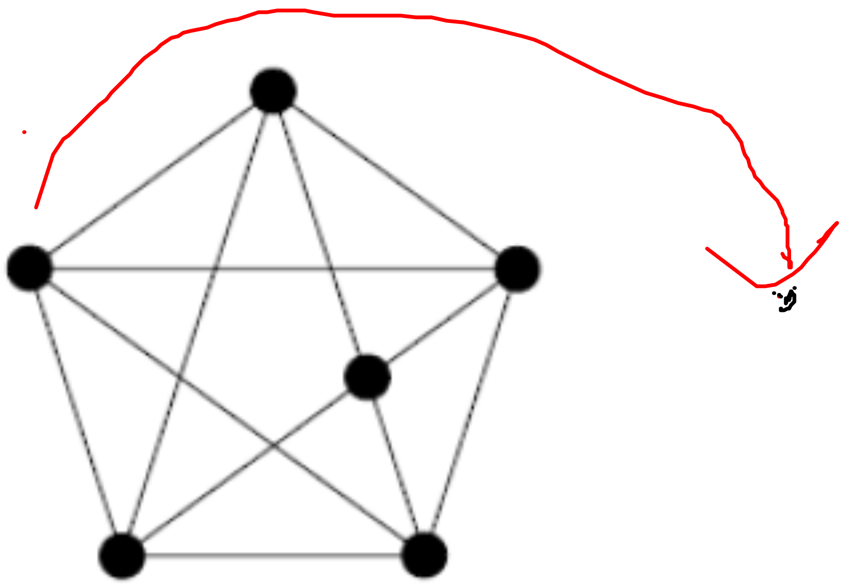
1. Decide whether the following graphs are planar or not:

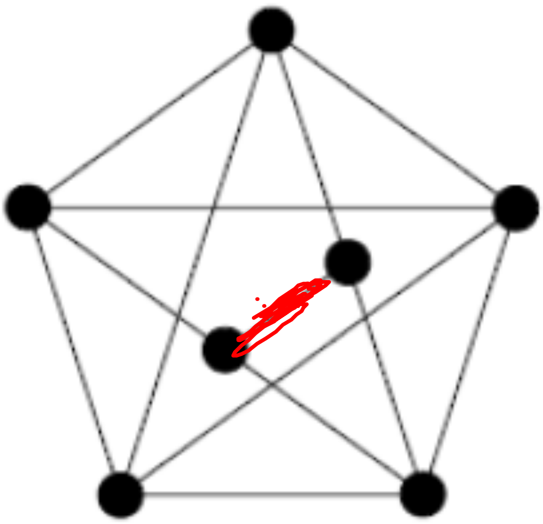
a) 

This is a subdivision of K5 therefore by Kuratowskys thm this is not planar.

b) 

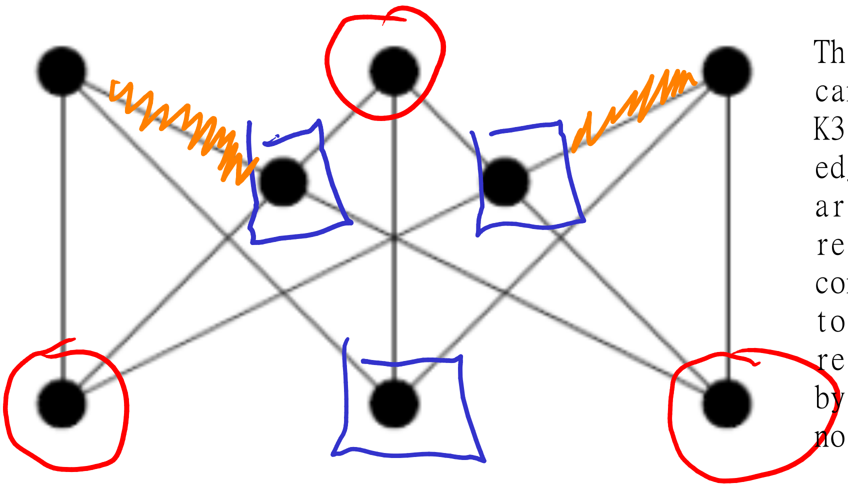
This is not a subdivision of K5 and if you delete some edge then you do not have enough degree 5 vertices to contain a subdivision of K5 as a subgraph. It also does not contain a K3,3 but it a little bit harder. An esasy reasoning:
It can be drawn without edge crossings as you can see above.

c)



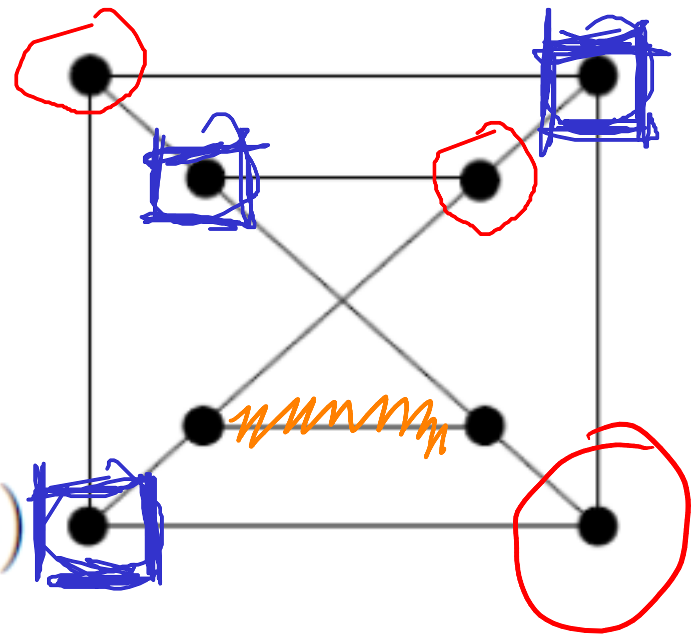
This is not planar by Kuratowskys theorem. If you delete the red edge then you obtain a subdivision of K5.

d)



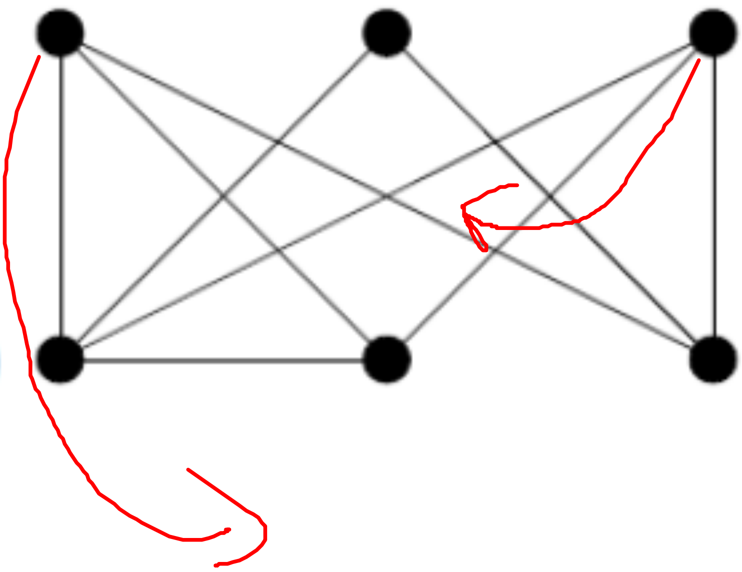
This is not planar because you can find the subdivision of a K3,3 after deleting the orange edges. The red and blue vertices are the two classes of K3,3, respectively. Each red cycle is connected by a private edge or path to all 3 red cycles. So it is really a subdivision of K3,3 and by Kuratowsky's thm the graph is not planar.

e)



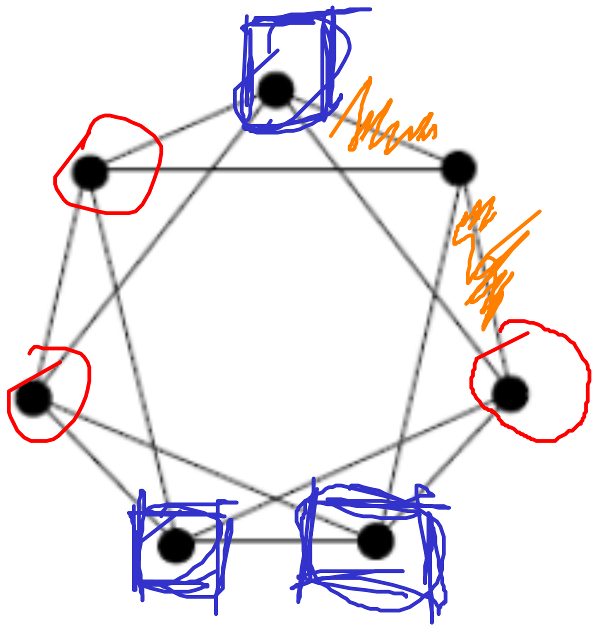
THis is not planar because it contains a subdivision of K3,3 as a subgraph.

f)



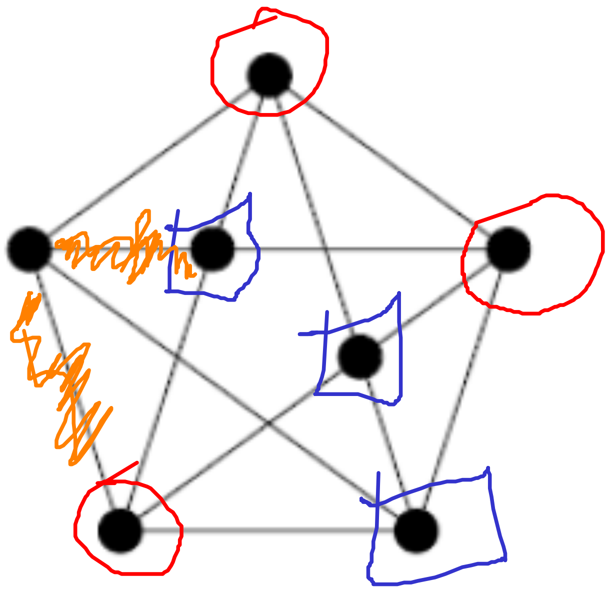
This is planar because we can draw it without a crossing.

g)



Not planar because it contains a subdivision of K3,3.

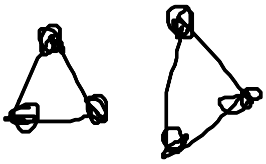
h)



Not planar because it contains a subdivision of K3,3

2. Is there a plane graph in which the number of vertices, edges and faces are all divisible by 3?

Yes. n+f=e+1+c by Euler's formula, therefore it must have two connected components or more.



3. Is there a simple connected plane graph which has half as many vertices as faces?

No because n+f=e+2 and f=2n would mean that 3n=e+2 but we know that e<=3n-6.

4. In a simple, connected plane graph the degree of each vertex is 4, and the number of the edges is 16. Determine the number of faces.

The number of edges is 4n/2=2n=16 therefore n=8. By Euler's formula f=e+2-n=16+2-8=10.

5. A convex polyhedron has 20 vertices and 12 faces. Each face of the polyhedron is bounded by the same number of edges. What is this common number?

Let x denote the number of edges bound a face. e=12x/2
The graph of the polyhedron is connected planar graph therefore by Euler's Formula:
n+f=e+2 so 6x=20+12-2=30 and x=5.

6. a) Show that in a simple planar graph the minimum degree is at most 5.
b) In the simple planar graph G the minimum degree is 5. Show that in this case G contains at least 12 vertices of degree 5.
c) Is the statement true with 13 instead of 12?

a) Assume the contrary, so each verte has degree at least six. Since the graph is planar and simple we know that e<=3n-6. But:

$$6 \cdot n \le \sum_{i=1}^{n} d(v_i) = 2e \le 2(3n-6) = 6n-12$$

$$6n \le 6n-12 \quad ↯$$

b) Let x denote the number of degree 5 vertices.

$$5x + 6(n-x) \le \sum_{i=1}^{n} d(v_i) = 2e = 6n-12$$

$$6n - x \le 6n - 12$$

$$-x \le -12 \Rightarrow \underline{x \ge 12}$$

c) No it is not true because th icosahedron has 12 vertices and each of them has degree 5. The graph of the icosahedron is planar.

7. a) Pove that if G is a simple graph on at least 11 vertices then at least one of G and its complement $\overline{G}$ is not planar.
b) Give a simple graph G on 8 vertices such that both G and $\overline{G}$ are planar.

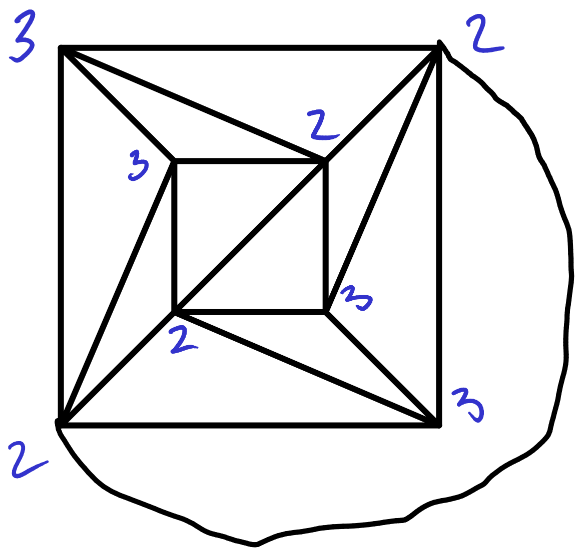
Assume that G has 11 vertices. We know that $e_G + e_{\overline{G}} = e_{K_{11}}$

$$e_{K_{11}} = \frac{11 \cdot 10}{2} = 55$$

If an 11 vertex graph is simple and planar then it has at most $3 \cdot 11 - 6 = 27$ edges. So if G is planar, then $\overline{G}$ has at least 28 edges, so it is not planar.

If G has more than 11 vertices, then take an 11 vertex subgraph H. If H is not planar then G is also not planar. If H is planar then $\overline{H}$ is not planar which is a subgraph of $\overline{G}$, therefore $\overline{G}$ is not planar.

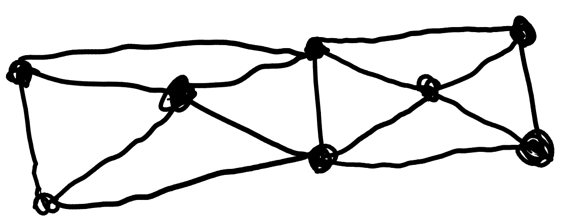
b) Take the graph of the cube and triangulate it. The degree sequence of the complement graph is 3,3,3,3,2,2,2,2. So it neither contains a subdivision of K3,3 which would require 6 degree 3 vertices nor K5 which require 5 degree 4 vertices. So by Kuratowski's theorem the complement of this planar graph is also planar.



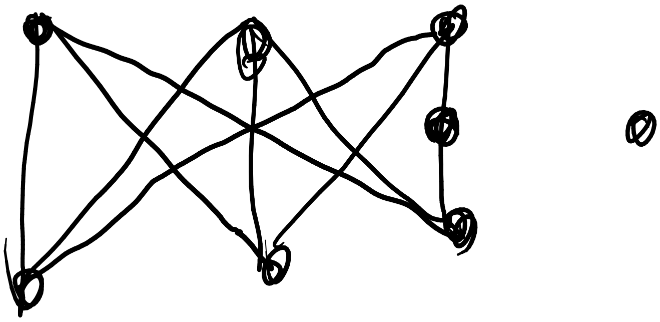
8. a) Let G be a simple graph on at most 6 vertices. Pove that at least one of G and its complement $\overline{G}$ is planar.
b) Give a simple graph G on 8 vertices such that neither G nor $\overline{G}$ is planar.

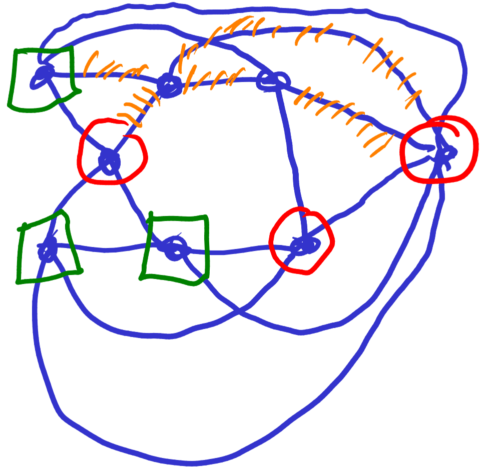
Assume that G has 6 vertices. If G is planar then we are done. Assume that G is not planar. We utilise Kuratowski's thm:
Case 1: G contains a K5 or its subdivison: Then in the complement 5 vertices has degree 0 or 1 and one vertex can have higher degree. So the complement graph is a star and some isolated vertices which is planar.
Case 2: G contains a K3,3 or its subdivison: Then in the complement each verex has degree 0,1 or 2. So The complamant is a union of cycles, paths, and isolated vertices which is planar.

b) Let G be K3,3 and two isolated vertices. Its complement looks like this:



This does not work. It is easy to see that the complement of K5 plus something will be planar. So instead of these pick a subdivision of K3,3 plus an isolated vertex:





This looks nice, lets haunt a K3,3 in this,

So this graph an also its complement is not planar.