1. Let $a_1, a_2, \ldots, a_n$ be a number sequence. We create graph G over the vertex set $v_1, v_2, \ldots, v_n$ by the following rules: for all $n \geq i > j \geq 1$ pair v_i and v_j are adjacent if and only if $a_i > a_j$.

   Show, that for any number sequence $a_1, a_2, \ldots, a_n$, the obtained graph is perfect.

Let A={a1,a2,...an} and we define a partial order $\leq_A$ over A:

$a_i \leq_A a_j \iff a_i < a_j$ and $i<j$ OR $i=j$

This relation is a partial order, because it is
-reflexive: by def
-antisymmetric: $a_i \leq_A a_j$ and $a_j \leq_A a_i$
i<j and j<i cannot hold at the same time so i=j and ai=aj
-transitive: $a_i \leq_A a_j$ and $a_j \leq_A a_k$
case 1: i=j and j=k then i=k and $a_i \leq_A a_k$
case 2: i=j and aj<ak, j<k, then i<k and ai<ak so $a_i \leq_A a_k$
case 3: j=k and ai<aj, j<k, then i<k and ai<ak so $a_i \leq_A a_k$
case 4: i<j and ai<aj, j<k and aj<ak, then i<k and ai<ak, so $a_i \leq_A a_k$

The comparability graph of the poset (A,$\leq$) is the given graph and since all comparability graphs are perfect, it is perfect.

2. (Erdős-Szekeres theorem) Let $A = a_1, a_2, \ldots, a_m$ be a number sequence which does not contain the same number twice. Let $m = kl + 1$, $k, l > 0$.

   a. Prove that A contains a $k+1$ long increasing or an $l+1$ long decreasing subsequence.

   b. Prove that this statement is not necessarily true when $m = kl$. 2413 k=2, l=2,

Let A={a1,a2,...am}. We create a partial order over $\leq_A$ this set:

ai$\leq_A$aj iff ai<aj and i<j or i=j. In the previous excersise we have shown that this is partial order.

Consider the poset (A,$\leq$). In this poset an increasing subsequence is a chain, a decreasing subsequence is an antichain. If the poset contains an at least k+1 long chain then we are done. Otherwise the size of a maximum chain is at most k. In this case by Mirsky's theorem (A,$\leq$) can be partitioned into at most k antichains. The total number of elements in A is kl+1, so at least one of the obtained antichains must contain at least l+1 elements. We are done.

3. We have a lot of boxes which correspond to the vertices of graph G. Two vertices of G are adjacent if and only if neither of the corresponding two boxes can be put in the other one. Show, that G is perfect.

Let B={b1,b2...bn} be the set of boxes. We define a partial order $\leq_B$ over B:
bi$\leq_B$bj if and only if th ith box fit in the jth box or i=j.
This is reflexive: by def
antisymmetric: if $b_i \leq_B b_j$ and $b_j \leq_B b_i$, then clearly cannot happen that the ith box fits in the jth and the jth fits in the ith at the same time. Therefore i=j.
transitivity: easy to see by case analysis

The comparability graph of $(B, \leq_B)$ is $\overline{G}$, So $\overline{G}$ is perfect. By the weak perfect graph theorem G is perfect as well.

4. Several discs [redacted] are given on the plane. We create the following graph G: The vertices of G are correspond to the discs. Two vertices are adjacent if and only if one of the two corresponding discs contains the other one. Prove that G is perfect.

We define a partial order over the discs: $d_i \le_D d_j$ if and only if disc j contains disc i or i=j.
G is the comparability graph of the defined poset, so it is perfect.

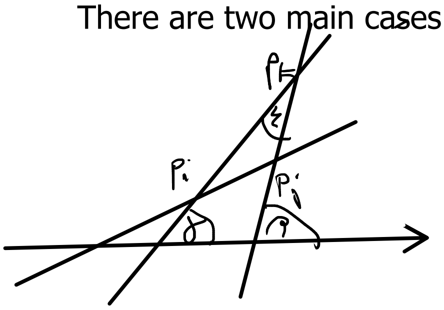
5. n points are given on the plane. Prove that, either we can choose $\lfloor\sqrt{n}\rfloor$ points from the given ones such that the angle between each line which is incident with two choosen points and the x axis is at least 30 degrees or we can choose $\lfloor\sqrt{n}\rfloor$ points from the given ones such that the angle between each line which is incident with two choosen points and the x axis is at most 30 degrees .

Let P be the set of points. We define a partial order $\le$ over these:

pi $\le_P$ pj <-->i=j or the angle between the line determined by pi and pj and the x axis is at most 30 degree and the x coordinate of pj is bigger than the x coordinate of pi.

reflexivity and antisimmetry is trivial, but transitivity requires a reasoning:
pi $\le_P$ pj and pj $\le_P$ pk: the cases when i=j or j=k are trivial so assume that i $\neq$ j $\neq$ k:

There are two main cases: 1st:

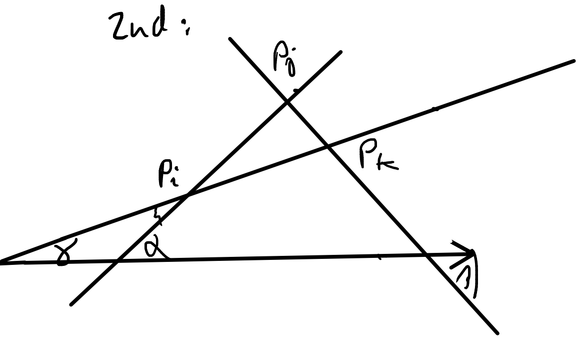


In $CP_kB\triangle$:

$\gamma+\xi+180^\circ-\beta=180^\circ \Rightarrow \gamma+\xi=\beta\le 30$

$\Rightarrow \gamma \le 30^\circ$ ✓

2nd:



$\gamma+\xi+180^\circ-\alpha=180^\circ \Rightarrow \gamma+\xi=\alpha\le 30^\circ$

$\Rightarrow \gamma \le 30^\circ$

So (P,$\le$) is a poset. If there are n points such that the slope of any line which is determined by any two of the n points is in [-30°,30°], then we are done. Note that such points form a chain in (P, $\le$).
If there are no such n points, then the size of a maximum chain is <$\sqrt{n}$, therefore by Mirsky's thm there is a chain which contains at least n/$\sqrt{n}$ =$\sqrt{n}$ elements. An antichain here is a set of points such that no two determine a line whose slope is in [-30°,30°], so we are done.

6. 50 different intervals of the same size are given on the line. Prove that
   (a) either there is a vertex which is contained in 8 intervals or there are 8 pairwise[redacted] disjoint intervals.
   (b) either there is a vertex which is contained in 7 intervals or there are 9 pairwise[redacted] disjoint intervals.

We define a partial order $\le$:
I1$\le$I2 if I1 and I2 are disjoint and the right endpoint of I1 is smaller than the left endpoint of I2 or I1=I2.

Then we can use dilworth theorem for the poset. If there are no 8 pairwise disjoint intervals (an antichain of size 8) then the poset can be partitioned into at most 7 chains and one of them must have size at least $\lceil 50/7 \rceil$=8.

b) If there are no 9 pairwise disjoint intervals (an antichain of size 9), then the poset can be partitioned into at most 8 chains and one of them must have size at least $\lceil 50/8 \rceil$=7.

7. Let $(H, \prec)$ be a partially ordered set. An element x is *maximal* (*minimmal*), if there is no element $y \in H$ which satisfies $x \prec y$ ($y \prec x$).

a. Prove that the set of maximal (minimal) elements of H forms an antichain.

b. Assume that the set of maximal and minimal elements together form an antichain. Prove that in this case all elements of H form an antichain as well.

If x and y are maximal elements then they cannot be comparable, otherwise one of them is not maximal. So maximal elements form an antichain and similarly minimal elements form a antichain as well.
b) If there are two comparable elements, then the lenght of a longest chain is more than one. The maximum and minimum elements of that chain are maximal and minimal in the poset and they are comparable, so all the maximal and minimal elements cannot form an antichain.
Therefore if all minimal and maximal elements form an antichan there cannot be two comparable elements in the poset, so the whole poset is an antichain.

8. Let G be an n vertex perfect graph. Prove that $\omega(\overline{G})\omega(G) \geq n$.

By Lovász' 2nd perfect graph thm: $n \leq \omega(G)\alpha(G) = \omega(G)\omega(\overline{G})$ ✓