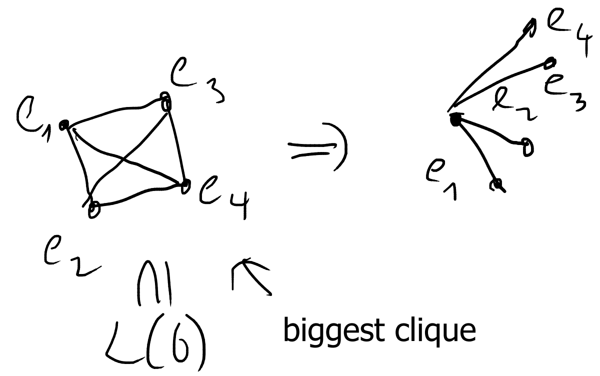
1 G is bipartite.

$\omega(L(G)) = \Delta(G)$, because: $\Delta(G) \geq \omega(L(G))$:



e1 and e2 are adjacent in L(G) means that they are incident to a vertex v. e3 is adjacent to both of them, therefore either e3 is adjacent to v or it is adjacent to the other endpoints of e1 and e2. If that is the same vertex then let v be that vertex. Otherwise e1, e2 and e3 form a triangle in G which is impossible since G is bipartite. We keep adding the edges contained in a biggest clique of L(G) and all of them are adjacent to v in G, so we get that $\Delta(G) \geq \omega(L(G))$.

$\Delta(G) \leq \omega(L(G))$:

Let v be a vertex with highest degree in G. Let e1, e2,.. ek be the edges which are incident to it. Note that there are no loops in bipartite graphs. In L(G) e1, e2,.. ek form a clique of size $\Delta(G)$.

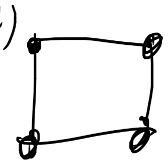
A proper coloring of L(G)'s vertices correspond to a proper coloring of G's edges. So $\chi(L(G)) = \chi_e(G)$ where $\chi_e(G)$ is the edge coloring number (chromatic index) of G.

By Kőnig's line coloring theorem if G is bipartite, then $\chi_e(G) = \Delta(G)$.

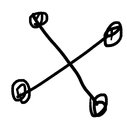
$$\chi(L(G)) = \chi_e(G) = \Delta(G) = \omega(L(G)).$$

Furthermore an induced subgraph of L(G) is a line graph of a bipartite graph, since the deletion of vertices of L(G) corresponds to the deletion of edges of G.
So any induced subgraph H of L(G) satisfies $\chi(H) = \omega(H)$. Therefore L(G) is perfect.

2 a)

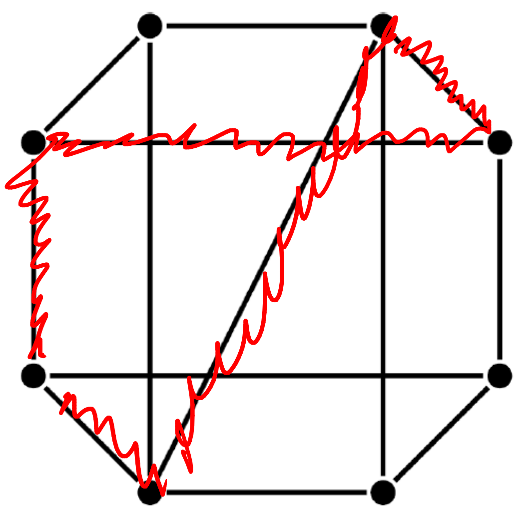


b)



3 a) It is bipartite, therefore it is perfect.

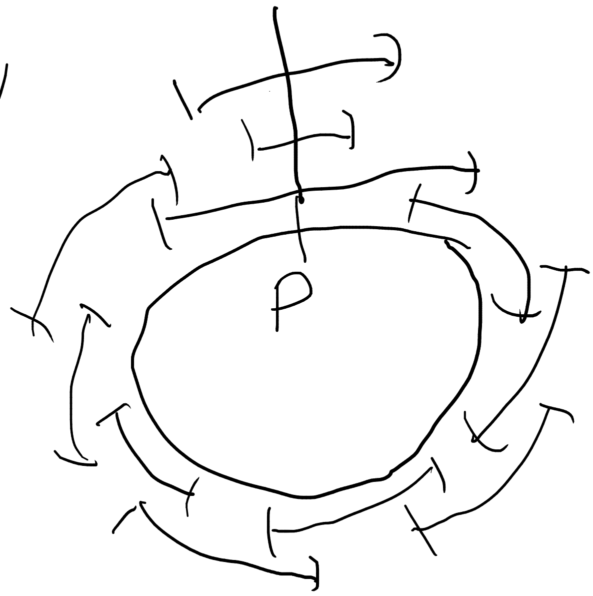
b)



Here is an induced C5 whose chromatic number is 3 and its clique number is 2, so this is not a perfect graph.

4 a) C5 is a circular-arc graph. Here is its representation:

b)

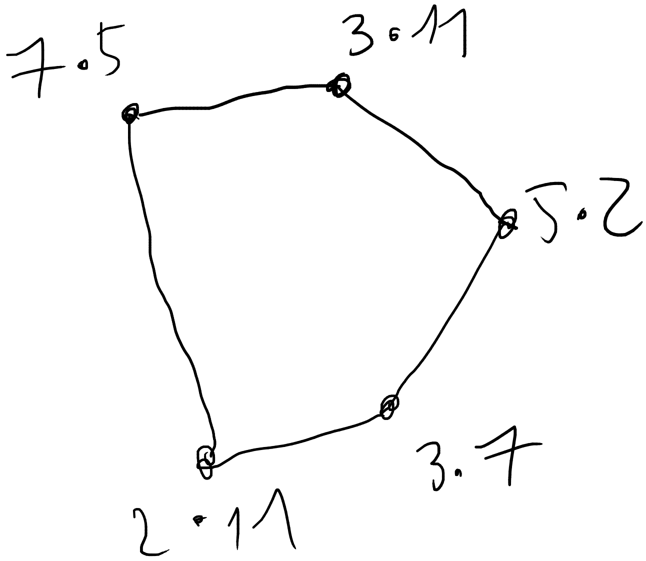


Fix a point P on the circle and throw out all the intervals which contain P. These intervals form a clique in the graph, so we have thrown out at most $\omega(G)$ intervals. Since no remaining interval contains P, they represent an interval graph whose clique number is at most $\omega(G)$ (clique number of a subgraph cannot be bigger) and it is perfect so it has a proper coloring with $\omega(G)$ colors. We can use a new color for each deleted interval and we obatin a proper coloring of G which uses at most $2\omega(G)$ colors.

5 $\omega(G_n) = \pi(n)+1$ so the number of primes until n +1. Because 1 is contained in all cliques and if there is a composite in the clique, then we can replace it with its prime factors and we obtain a bigger clique.

$\chi(G_n) = \pi(n)+1$ 1 receive color 1, 2 and its multiples receive color 2, 3 and its uncolored multiples receive color 3, 5 and its uncolored multiples receive color 4, etc. In this way we create a proper coloring with $\omega(G_n)$ colors, so it is an optimal coloring.

It is not perceft if n>32 because then it contains an induced C5:



6. Show that the complement of bipartite graphs are perfect graphs. Do not use the two perfect graph theorems.

Let G be a bipartite graph, so we want to show that $\overline{G}$ is perfect.
The complement of an induced subgraph of $\overline{G}$ is a bipartite graph, therefore it is enough to show that $\chi(\overline{G})=\omega(\overline{G})$.

For any graph $\omega(\overline{G})=\alpha(G)$ and therefore $\chi(\overline{G})\geq\alpha(G)$.

$\chi(\overline{G})$ can be thinked as a vertex coloring of G with the fewest possible colors such that any two not adjacent vertices have different colors. Assume for a while that G does not contain an isolated vertex. Pick a smallest edge cover set, whose size is $\rho(G)$, and go along this set and color any uncolored vertices of the ith edge with color i. This is a proper coloring of $\overline{G}$ with $\rho(G)$ colors.
By Gallai's 1st theorem if G is isolated vertex free, then $\alpha(G)+\tau(G)=|V(G)|$.

By Kőnig's theorem if G is bipartite, then $\tau(G)=\nu(G)$.
By Gallai's 2nd theorem if G is loop free, then $\nu(G)+\rho(G)=|V(G)|$

G is a bipartite isolated vertex and loop free graph, so by combining these we get:

$$\rho(G)=|V(G)|-\nu(G)=|V(G)|-\tau(G)=\alpha(G).$$

So we have a proper coloring of $\overline{G}$ whose size equals its clique number so it is an optimal coloring and we are done.

If G contains some isolated vertices then they are connected to all other verties in $\overline{G}$ so each of them raise $\chi(\overline{G})$ and $\omega(\overline{G})$ by one. So if we remove them, then in G' we can repeat the previous reasoning an the puting back the isolated vertices we get the desired result.

7. We create graph G' from a perfect graph G in the following way: We add a new vertex v and join v to all vertices of a clique of G. Show that G' is a perfect graph.

Let H be an induced subgraph of G'. If H does not contain v, then it is also an induced subgraph of G and we are done, the equation holds for it.
So assume that H contains v. H\v is an induced subgraph of G, so since G is perfect $\chi(H\setminus v)=\omega(H\setminus v)$. If d(v)< $\omega(H\setminus v)$ then $\omega(H)=\omega(H\setminus v)$ and the neighbours of v do not use all colors in an optimal coloring of $H\setminus v$ therefore there is a free color for v. In this case $\chi(H)=\chi(H\setminus v)=\omega(H\setminus v)=\omega(H)$.

Otherwise d(v)= $\omega(H\setminus v)$. In this case lets use an optimal coloring of H\v and use a new color for v. We get a proper coloring which uses $\chi(H\setminus v)+1=\omega(H\setminus v)+1=\omega(H)$ colors, therefore it is an optimal coloring, and the required equation is satisfied.

8. The vertices of graph G is the tiles of an 8x8 chessboard. Two vertices are adjacent in G if the knight can move from one to the other in one move. (The knight moves in a 3x2 L shape.) Show that G is perfect.

This graph is bipartite, therefore it is perfect.

9. Show that a graph G is perfect if and only if each induced subgraph G' of G contains an independent set A, such that A intersects each maximum clique of G'.

First we show that if this property does not hold then G is not perfect. Let G' be an induced subgraph which does not have such an independent set. It means that if we have a proper coloring and pick the vertices colored by red it forms an independent set and there is a maximum clique which does not contain red vertices. Therefore $\chi(G') > \omega(G')$. and G is not perfect.

Now lets assume that the property holds and let G' be any induced subgraph of G. We show that we can color G' with $\omega(G')$ colors. Pick an independent set A' which intersects alllargest cliques of G'. Color it with 1. The remaining vertices induce an induced subgraph G' with one smaller clique number. It contains an independent set A'' which intersect a largest clique of G''. Color it with 2. Repeat this procedure. After $\omega(G')$ steps no more vertex remains and G' is colored with $\omega(G')$ colors properly.