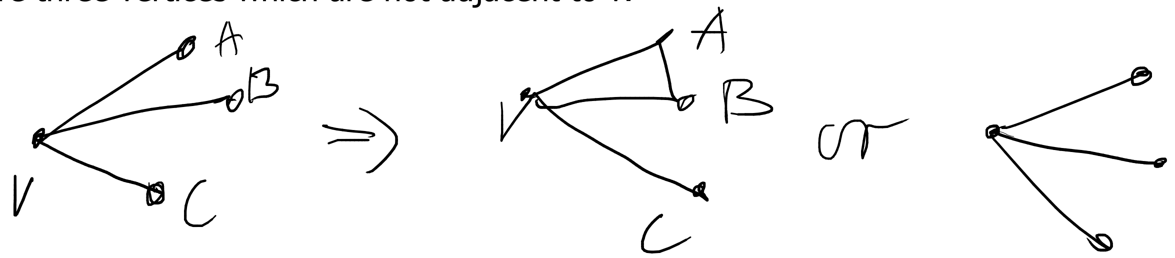
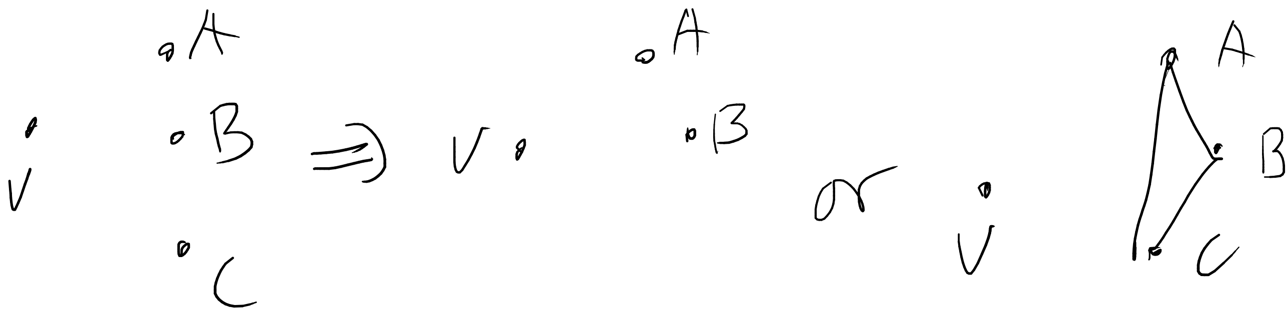
Ramsey Theory

Claim: In any six vertex graph either there are 3 pairwise adjacent vertices, a clique, or 3 pairwise non-adjacent vertices, an independent set.

Proof: Let G be a six vertex graph. A vertex v in G is either adjacent to three vertices or there are three vertices which are not adjacent to v.

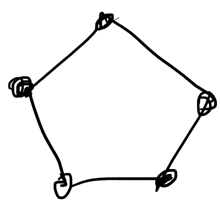


Assume that A, B and C are adjacent to v. If any two of these vertices are adjacent, then we have a clique of size 3, otherwise these three vertices are non-adjacent.



Now consider that A,B and C are not adjacent to v. If any two of these three vertices are not adjacent to each other, then they whit v form an independent set of size three. Otherwise A,B and C form a clique of size 3.

Note: This claim is not true when we have only 5 vertices. Example:



Ramey's Theorem: For any k and l, there is a number R(k,l), which is defined to be the smallest possible, such that if you take any graph containing at least R(k,l) vertices, then the graph contains a clique of size k or an independent set of size l.

Note: The previous claim shows that R(3,3)=6.

Proof of Ramsey's theorem by Erdős and Szekeres:

We are going to prove that for any l and k: $R(k,l) \le \binom{k+l-2}{k-1} = \binom{k+l-2}{l-1}$

We prove by induction on k and l.
R(2,l)=l because if a graph has l vertices, then either it contains an edge or all of its vertices are independent. Similarly R(k,2)=k. Furthermore:

$$R(2,l)=l=\binom{l}{1}=\binom{k+l-2}{k-1}\Big|_{k=2} \checkmark \quad \text{and} \quad R(k,2)=k=\binom{k}{1}=\binom{k+l-2}{k-1}\Big|_{l=2} \checkmark$$

So the base cases are OK, and we can start the induction.

We show that in the general case:
R(k,l)<=R(k-1,l)+R(k,l-1) and therefore R(k,l) exists.

Let G be a graph which contains at least R(k-1,l)+R(k,l-1) vertices and let v be a vertex of G. Either v is adjacent to R(k-1,l) vertices or it is not adjacent to R(k,l-1) vertices.

In the first case, by the inductional assumption either the neighbors of v induce a subgraph which contains a clique of size k-1, which can be extended by v to a clique of size k, or they contain an independent set of size l. In both cases we are done.

In the second case, by the inductional assumption either the neighbors of v induce a subgraph which contains an independent set of size l-1, which can be extended by v to an independent set of size l-1, or it contains a clique of size k. Again, in both cases we are done.

$$R(k-1,\ell)+R(k,\ell-1) \geq R(k)$$

$\wedge|$ by induction $\wedge|$

$$\binom{k+\ell-3}{k-2}+\binom{k+\ell-3}{k-1}=\binom{k+\ell-2}{k-1}$$

$\uparrow$

Remember that $\binom{N}{K}=\binom{N-1}{K}+\binom{N-1}{K-1}$ $\square$

Corollary: $R(k,k) \leq \underbrace{\binom{2k-2}{k-1}} < \underbrace{2^{2k-2}} = 4^{k-1}$

Number of k-1 element subsets of a 2k-2 element set

Number of subsets of a 2k-2 element set

Note: $\binom{2k-2}{k-1} \approx \frac{4^k}{\sqrt{k}}$

The best known upper bound for R(k,k) is $\approx \frac{4^k}{k}$

This problem can be told as an edge coloring problem in the following way:
We have a complete graph and color the edges by two colors, red and blue. If a an edge is blue, then it is an edge of G, otherwise it is not an edge of G.
Clearly if the complete graph has at least R(k,l) vertices, then it does not matter how we color its edges, there is either a blue clique of size k or a red clique of size l.
This coloring description allow us to generalise the problem:

Theorem: There is a number R(k1,k2,..km) such that if we color the edges of a complete graph which has at least R(k1,k2,..km) vertices by m colors, then there is a color i, such that there is a clique of size ki whose edges have color i.

Claim: $R(k_1,\dots,k_m) \le R(k_1-1,k_2,\dots,k_m)+R(k_1,k_2-1,\dots,k_m)+\dots+R(k_1,k_2,\dots,k_m-1)$

and $R(k_1,\dots,k_m) \le \dfrac{\left(\sum_{i=1}^{m}(k_i-1)\right)!}{\prod_{i=1}^{m}(k_i-1)!}$

The previous proof can be generalised to prove this claim.

A lower bound on R(k,k):

Claim: $R(k,k) \ge 2^{k/2}$ if $k \ge 3$

We are going to prove this result by the probabilistic method.

Proof: Let n $< 2^{k/2}$. Let g_n be the number of n vertex simple graphs over n labeled vertices.

So for example we distinguish the following two isomorphich graphs:

Let $g_{n,k}$ be the number of n vertex simple graphs over n labeled vertices which contain a clique of size k.

$g_n = 2^{\binom{n}{2}}$ Because from 1,2..n we can create $\binom{n}{2}$ pairs and for each of them we can either place or do not place an edge.

$g_{n,k} \le \binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}}$ ← The number of possible edges which are not determined by the choosen clique.

That many ways can we choose k vertices to be a clique

Clearly a labeled graph which contains a clique of size k can be counted many times at the right hand side. For example K_n is counted $\binom{n}{k}$ many times.

$$\frac{g_{n,k}}{g_n} \le \binom{n}{k}\cdot 2^{-\binom{k}{2}} < \frac{n^k}{k!\,2^{\binom{k}{2}}} < \frac{\left(2^{\frac{k}{2}}\right)^k}{k!\,2^{\binom{k}{2}}} = \frac{2^{\frac{k^2}{2}-\left(\frac{k(k-1)}{2}\right)}}{k!} = \frac{2^{\frac{k}{2}}}{k!} < \frac{1}{2}$$

the number of numerators is less than the number of denominators and each numerator is not bigger than a corresponding denominator

So less than half of the n vertex graphs contain a clique of size k if $n < 2^{k/2}$.

In the exact same way we can count the number of n vertex graphs which contain an independent set of size k and we obtain that less than half of the n vertex graphs contain an independent set of size k if $n < 2^{k/2}$

Therefore if $n < 2^{k/2}$, then there is an n vertex graph which neither contains a clique of size k nor contains an independent set of size k. Thus R(k,k) $\ge 2^{k/2}$.

$\square$

Note that the probabilistic method has not given us such an example. We only proved its existance. People are trying to construct such an example, but they are not even close.

The best known lower bound on R(k,k) is approxametly $k \cdot 2^{k/2}$.

Other generalization:

Assume that we have a set and we are coloring the p element subsets of the set. Note that p=2 means that we are coloring the edges of a graph. When p>2 we are coloring the edges of a hypergaph.

Theorem: There is a (smallest possible) number $R_p(k_1, k_2, \ldots k_t)$ such that if a set contains at least $R_p(k_1, k_2, \ldots k_t)$ elements, then it does not matter how we color the p element subsets of the set by t colors there is always a color i, such that there is a ki element subset whose all p element subsets have color i.

Theorem:

$$\exists c \in \mathbb{R}: \quad R_p(\underbrace{k, k, k, \ldots k}_{t}) \leq \left.2^{2^{2^{2^{\cdot^{\cdot^{2^{c \cdot t}}}}}}}\right\} p$$

Theorem (Schur 1916): For each t, there is a number N(t), such that if we color the numbers 1,2,3..N(t) by t colors, then there are numbers x,y,z which have obtained the same color and x+y=z.

Proof: Let N=$R(\underbrace{3, 3, \ldots 3}_{t})$. We color 1,2,3...N by t colors, and for such a coloring we create a coloring of the edges of the N vertex complete graph:

The edge (vi,vj) receives the color of the number |i-j|. Since N=R(3,3...3), there is a monchromatic triangle in that graph. Let a,b and c be its vertices such that a<b<c. Then: x:=c-b, y:=b-a, z:=c-a. x,y,z are the numbers whose colors are given to the edge of the monochromatic triangle, so they received the same color and x+y=z. $\square$

Theorem (Van der Waerden 1927): For each t,n there is a number N(t,n) such that if we color the numbers 1,2,3...N(t,n) by t colors then there is a monochromatic aritmetic progression of length n.

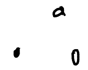
Corrolary: If we color all the natural numbers by t colors, then there is an arbitrary long finite monochromatic arithmetic progression.
On the other hand an infinite monochromatic arithmetic progression not necessarily exists.

Theorem (Szemerédi 1975): Let A be a subset of the natural numbers.

If $\liminf_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} > 0$, then A contains an arbitrary long finite arithmetic progression.

Theorem (Erdős-Szekers 1935 also known as the Happy ending problem):

For each n, there is a (smallest possible) F(n) such that if F(n) points in general position (no three points are incident to the same line) are given, then there are n of them which form the vertices of a convex polygon.

$F(3) = 3$ 

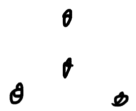
$F(4) = 5$ For example:  these does not form a convex polygon.

$F(5) = 9$ and $F(n) \simeq 2^n$

Proof (By Johnson)
We are going to show the following weak upper bound: $F(n) \leq R_3(n,n)$

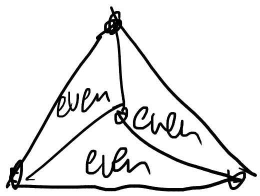
Assume that we have $R_3(n,n)$ points in general position.
Any three given points form a triangle and we color such a triangle by blue if its interior contains even number of given points and we color it red if its interior contains odd number of given points.

Since we have $R_3(n,n)$ points, there are n points such that any triangle which they span have the same color. In this case they form a vertex set of a convex polygon because:
Indirectly assume that they not. In these case there are four of them in the following position:



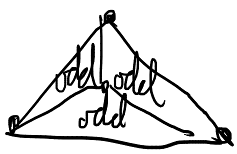
which means the following by the definition of the coloring:

Either their color is blue:



But in this case the outer triangle contains odd number of points, so it is red, which is a contradiction.

Or its color is red:



But in this case the outer triangle contains even number of points, so it is blue, which is a contradiction.

$\square$.