Partially ordered sets (posets)

A relation $\leq$ is called a partial order over a set S, if it satisfies the following three properties:

-reflexivity: $\forall a \in S:\ a \leq a$

-antisymmetry: $\forall a,b \in S:$ if $a \leq b$ and $b \leq a \Rightarrow a=b$

-transitivity: $\forall a,b,c \in S:$ if $a \leq b$ and $b \leq c \Rightarrow a \leq c$

Definition: If we have a set S and a relation $\leq$ which is a partioal order over S, then we call the pair $(S,\leq)$ as a partially ordered set, or poset for short.

Definition: $a,b \in S$ are called comparable elements if $a \leq b$ or $b \leq a$. If two elements are not comparable, then we call them as incomparable elements.

Notation: If $a \leq b$ and $a \neq b$, then we simply write a<b.

Definition: {a1, a2, ..., an}$\subseteq$S is a chain in $(S,\leq)$ if there is a permutation $\pi \in S_n$ such that

$$a_{\pi(1)} < a_{\pi(2)} < a_{\pi(3)} < \dots < a_{\pi(n)}.$$

Definition: {a1, a2, ..., an}   S is an antichain in (S,   ) if there is no ai and aj $(i \neq j)$ which are comparable.

Example: S= the points of the plane
The definition of $\leq_S$: $(x_1,y_1) \leq_S (x_2,y_2) \Leftrightarrow x_1 \leq x_2$ and $y_1 \leq y_2$

This relation is a partial order over the plane:

reflexivity: $(x,y) \leq_S (x,y) \Leftrightarrow x \leq x$ and $y \leq y$ ✓

antisymmetry: $(x_1,y_1) \leq_S (x_2,y_2)$ and $(x_2,y_2) \leq_S (x_1,y_1)$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\underbrace{x_1 \leq x_2 \quad y_1 \leq y_2 \qquad x_2 \leq x_1,\ y_2 \leq y_1}$$

$$\Downarrow$$

$$x_1 = x_2,\ y_1 = y_2 \Rightarrow (x_1,y_1) = (x_2,y_2)$$

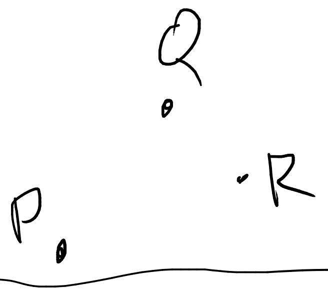
transitivity: $(x_1,y_1) \leq_S (x_2,y_2)$ and $(x_2,y_2) \leq_S (x_3,y_3)$

$$\Updownarrow \qquad\qquad \Updownarrow$$

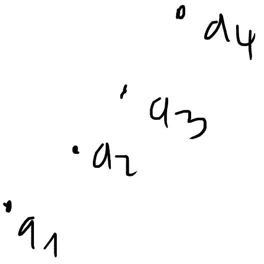
$$\underbrace{\begin{matrix} x_1 \leq x_2 \\ y_1 \leq x_2 \end{matrix} \qquad\qquad \begin{matrix} x_2 \leq x_3 \\ y_2 \leq y_3 \end{matrix}}$$

$$\Downarrow$$

$$\left.\begin{matrix} x_1 \leq x_3 \\ y_1 \leq y_3 \end{matrix}\right\} (x_1,y_1) \leq_S (x_3,y_3)$$

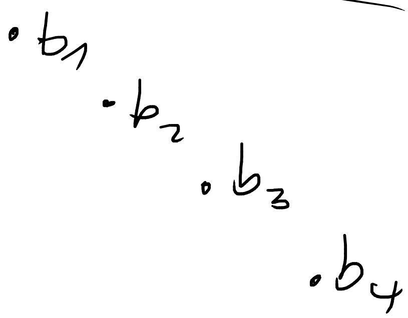


$P \leq Q$, $P \leq R$, R and Q are incomparable



{a1, a2, a3, a4} is a chain in this poset.

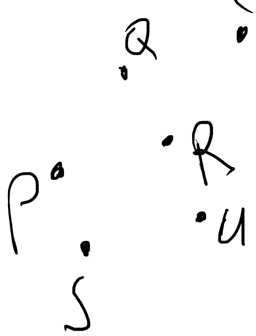
{b1, b2, b3, b4} is an antichain in this poset.



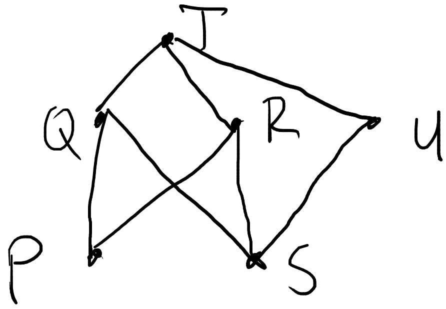
How to "draw" a poset?

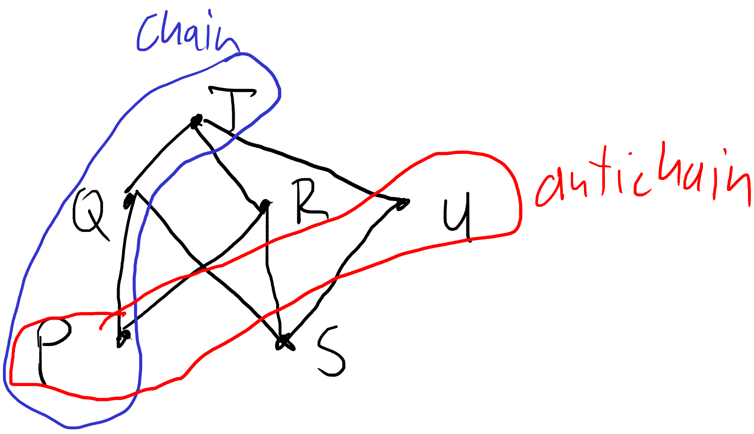
Hesse diagram of a poset:
Each element of $(S,\leq)$ is a vertex on the plane. If a<b, then the vertex of b appears above the vertex of a. Vertices a and b are adjacent if and only if a<b and there is no $c \in S$ s.t. a<c<b or b<a and there is no $c \in S$ s.t. b<c<a.

Example: Let S be the following points on the plane and let the partial order be the previous, so $(x_1,y_1) \leq (x_2,y_2) \iff x_1 \leq x_2$ and $y_1 \leq y_2$.



The Hasse diagram of $(S,\leq)$:





Dilworth's theorem ('50):
Let $(S,\leq)$ be a poset and let a be the size of a maximum antichain of $(S,\leq)$. Then $(S,\leq)$ can be partitioned into a chains but it cannot be partitioned into a-1 chains.

Dual of Dilworth's theorem/Mirsky's theorem:
Let $(S,\leq)$ be a poset and let c be the size of a maximum chain of $(S,\leq)$. Then $(S,\leq)$ can be partitioned into c antichains but it cannot be partitioned into c-1 antichains.

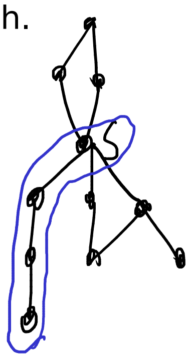
Note: If A is an antichain and C is a chain, then $|A \cap C| \le 1$.
Proof: If a,b are contained in a chain then they are comparable, so they cannot be in an antichain.

Corrollary: $(S,\le)$ cannot be partitioned into a-1 chains because to cover the maximum antichain, whose size is a, we need at least a chains.
Simillarly $(S,\le)$ cannot be partitioned into c-1 chains because to cover the maximum chain, whose size is c, we need at least c chains.

Proof of Dual Dilworth's thm:
We have seen that c-1 antichains are not enough to cover the poset. We give c antichains which cover it. Let $s \in S$. We define r(s), the rank of s, as the size of the largest chain whose maximum element is h.

Example:

largest chain whose maximum element is s



$r(s)=4$

Clearly for any $s \in S$: $1 \le r(s) \le c$.
Claim: If r(s1)=r(s2) and $s_1 \ne s_2$, then s1 and s2 are incomparable. $*$
To show that indirectly assume that r(s1)=r(s2), $s_1 \ne s_2$ and $s_1 \le s_2$.

Let C1 be the largest chain whose maximal element is s1. The size of C1 is r(s1).
Since s1<s2 and $\le$ is tansitive $C_1 \cup \{s_2\}$ is a chain whose maximum element is s2.
Then $r(s2) \ge r(s1)+1 = r(s2)+1$ which is a contradiction, so $*$ is true.

Now we give an antichain cover whose size is c:
Let $A_i = \{s \in S \mid r(s) = i\}$. $A_1, A_2, \dots, A_c$ are antichains and they cover S.

Proof of Dilworth: later

Comparability graphs

def: A graph G is a comparability graph if there is a poset $(V(G), \le)$ such that $\forall g, h \in V(G)$ are adjacent if and only if g and h are comparable and $g \ne h$.

Claim: Comparability graphs are perfect:
Proof: An induced subgraph of a comparability graph is a comparability graph, because if we restrict the relation $\le$ to a subset then it induces a subgraph in G. Therefore it is enough to show that if G is a comparability graph, then $\chi(G) = \omega(G)$.
Note that:

G		$(G, \le)$
clique	<-->	chain
maximum clique	<-->	maximum chain
$\omega(G)$	<-->	c
independent set	<-->	antichain
max independent set	<-->	maximum antichain
$\alpha(G)$	<-->	a

In an optimal coloring we divide V(G) to $\chi(G)$ disjoint inpedendent sets, so we divide the corresponding poset to the fewest number of antichains. By Mirsky's thm that number is c, the size of a maximum chain which is $\omega(G)$. So $\omega(G) = \chi(G)$. $\square$

Proof of Dilworth's theorem by the weak perfect graph theorem:

Let G be the comparability graph of the poset $(S,\preceq)$. Then $\overline{G}$ is the "incomparability" graph of $(S,\preceq)$ where two vertices are adjacent if and only if the two elements are incomparable. G is perfect, so we can use the weak perfect graph theorem and obtain that $\overline{G}$ is perfect as well. Thus $\chi(\overline{G})=\omega(\overline{G})$. Where $\omega(\overline{G})=\alpha(G)=$ size of the largest antichain in $(S,\preceq)$.

An independent set in $\overline{G}$ is a chain in $(S,\preceq)$, therefore $\chi(\overline{G})$ is the least number of chains which cover S. So $\chi(\overline{G})=\omega(\overline{G})$ is equivalent to the Dilworth's thm. $\square$.

A direct proof of Dilworth's theorem:

We use induction on |S|. |S|=1 is trivial. Let C be a maximum chain in $(S,\preceq)$.
Reminder: a denotes the size of a maximum antichain in $(S,\preceq)$.
Let A be a maximum antichain in $(S\setminus C,\preceq)$ The size of A is either a or a-1. If |A|=a-1, then by induction S\C can be covered by a-1 chains and by adding C we obtain a partition of $(S,\preceq)$ into a chains.
Otherwise |A|=a, so it is also a maximum antichain in $(S,\preceq)$.
Let A={s1,s2,..,sa}. Let x be the maximum element of C and let y be the minimum element of C.
Let $S^+=\{s\in S \mid \exists s_i\in A : s\succeq s_i\}$, $S^-=\{s\in S \mid \exists s_i\in A : s\preceq s_i\}$

Claim 1: $S^+\cap S^-=A$:

Clearly: $S^+\cap S^-\supseteq A$. Indirectly assume that $\exists s\notin A : s\in S^+\cap S^-$

By the def of S^+ $\exists s_i\in A: s_i\preceq s$

By the def of S^- $\exists s_j\in A: s_j\succeq s$

$\Big\}$ $s_i\preceq s\preceq s_j \Rightarrow s_i\preceq s_j$ ↯ A antichain

Claim 2: $S^+\cup S^-=S$:
Indirectly assume that $\exists s\in S\setminus\{S^+\cup S^-\}$: $\nexists s_i\in A: s_i\succeq s \Leftarrow s\notin S^-$, $\nexists s_i\in A\ s_i\preceq s \Leftarrow s\notin S^+$

$A\cup\{s\}$ is a bigger antichain of $(S,\preceq)$ and its size is a+1 ↯

a was the size of a maximum

Claim 3: $x\in S^+$, $y\in S^-$:

Indirectly assume that $x\notin S^+$. By Claim 2, then $x\in S^-$. $\Rightarrow \exists s\in S: x\preceq s$
$\Rightarrow C\cup\{x\}$ is a chain which is bigger than C, but C was a maximum chain. ↯.

By 3. $|S^-|<|S|$ and $|S^+|<|S|$ so we can use our inductional hypothesis for the posets $(S^-,\preceq)$ and $(S^+,\preceq)$. A is a maximum antichain in both. Therefore both one can be partitioned into a chains. Each chain of $(S^-,\preceq)$ contains one element of A and that is the maximum element of that chain. Similarly each chain of $(S^+,\preceq)$ contains one element of A and that is the minimum element of the chain. We can combine each chain of $(S^+,\preceq)$ to a chain of $(S^-,\preceq)$ to obtain a chain of $(S,\preceq)$.

By this construction we obtained a partition of $(S,\preceq)$ to a chains. $\square$.