Perfect graphs

Motivation:

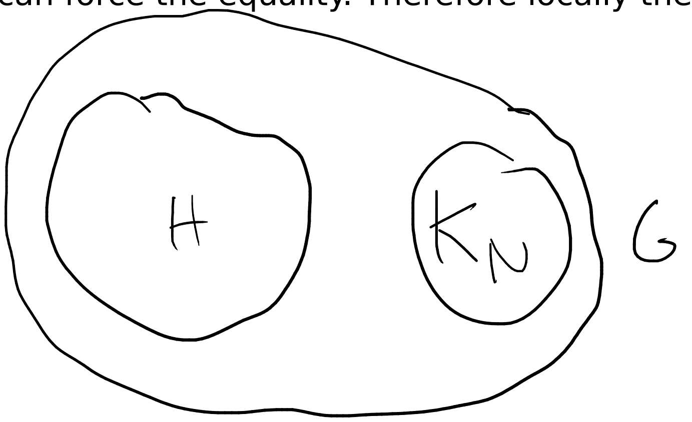
$\omega(G)$: clique number of G: Size of the biggest complete subgraph of G.

$\chi(G)$: the chromatic number of G: Least k such that G has a proper vertex coloring with k colors. A vertex coloring is proper if adjacent vertices receive different colors.

Remainder: $\omega(G) \leq \chi(G)$ (We need different colors for each vertex of a clique)

Question: What graph G satisfies $\chi(G) = \omega(G)$?

Answer: G can be any graph which contains a huge a cliqe. If we have a graph H whose clique and chromatic number are far away then by puting a huge clique in the graph we can force the equality. Therefore locally the graph can be anything.



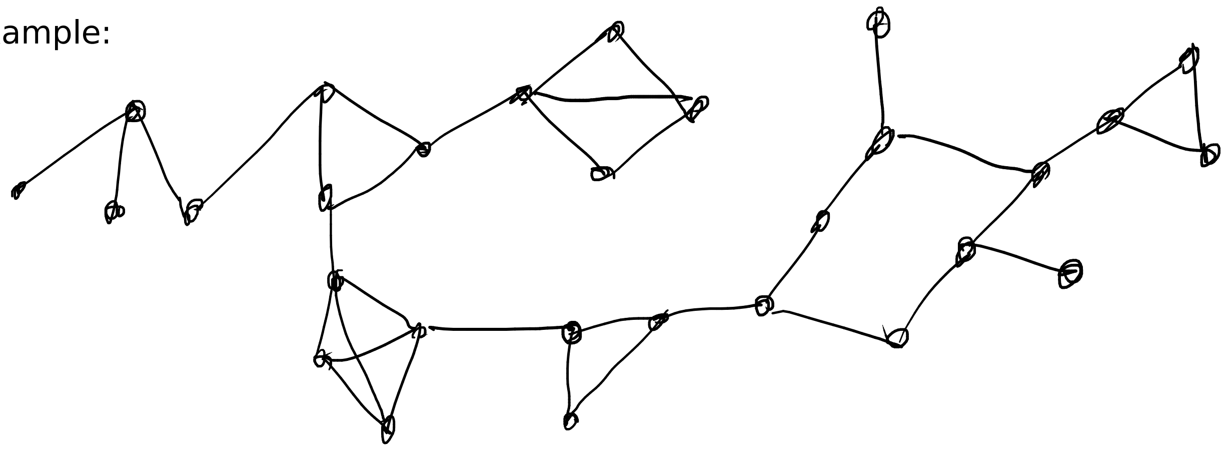
$$\chi(H) = N > \omega(H)$$
$$\chi(G) = N = \omega(G)$$

So this graph class is not so interesting.

Question II: What graph G satisfies that any subgraph H of G satisfies $\chi(H) = \omega(H)$?

Answer: $\omega(C_{2k+1}) = 2$ and $\chi(C_{2k+1}) = 3$ if k>1, therefore such a graph cannot contain an odd cycle except triangles. And it can be proven that these graph have the required property. These graphs are not so interesting.

Example:



definition: Induced subgraph:
Let G=(V(G),E(G)) be a graph. H is an induced subgraph of G if:
1) V(H) $\subseteq$ V(G) and
2) h1, h2 $\in$ V(H) and (h1,h2) $\in$ E(H) then (h1,h2) $\in$ E(G)

Alternative definition: H is an induces subgraph of G if H can be obtained from G by deleting some of its vertices and the edges which are incident to the deleted vertices.

definition: (Berge '63) A graph G is called perfect if each induced subgraph H of G satisfies that $\chi(H)=\omega(H)$.

It turns out that the interesting question is that what are the perfect graphs. We will see later that perfect graphs have many interesting applications.

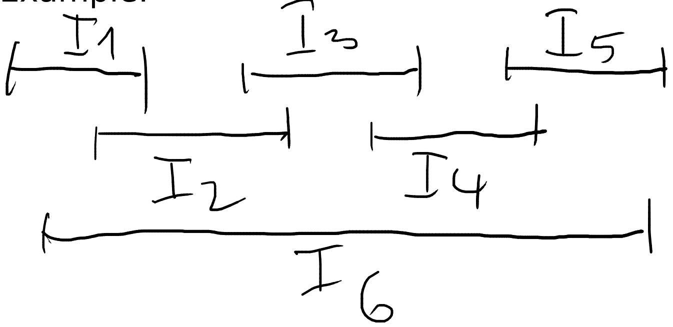
Examples: Complete graphs are perfect, empty graphs are perfect
If a graph contains an induced odd cycle which is not a triangle, then it is not perfect.

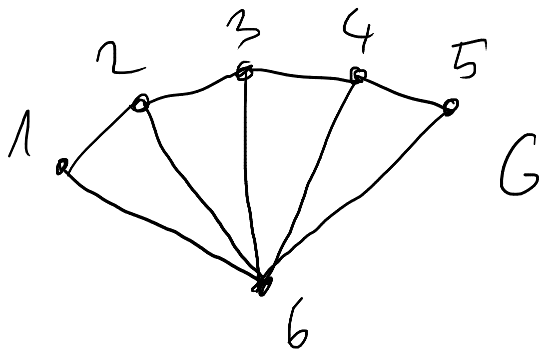
Claim: Bipartite graphs are perfect

Proof: Let G be a bipartite graph. We need to show for all induced subgraph H of G that $\chi(H)=\omega(H)$. Fortunately an induced subgraph of a bipartite is also a bipartite graph: Remember that G is bipartite <--> G does not contain an odd cycle. By deleting some vertices and edges to obtain an induced subgraph we cannot create odd cycles.
Therefore it is enough to show that if G is bipartite, then $\chi(G)=\omega(G)$.
This is true. We know that either $\chi(G)=1=\omega(G)$ or $\chi(G)=2=\omega(G)$ when G is bipartite. So we are done. $\square$

Interval graphs
definition: G is an interval graphs if it can be represented by several intervals on a line in the following way:
1) There is a bijection between the vertices of G and the intervals
2) Two vertices of G are adjacent if and only if the corresponding intervals intersect.

Example:





Note: Not all graphs are interval graphs. For example C4 is not an interval graph.

Claim: Interval graphs are perfect.

Proof: An induced subgraph of an interval graph is an interval graph: If you delete a vertex from a intervalgraph then you delete the corresponding interval and obtain the interval representation of the obtained graph.
Therefore it is enough to show that if G is an interval graph, then $\chi(G)=\omega(G)$.

The following greedy algorithm colors an interval graph G with $\omega(G)$ colors:
Order the intervals according to their left endpoints, so the first interval in the ordering has the leftmost left endpoint. Color the first interval with color 1. If the 2nd and the 1st interval intersect then color the second interval with color 2, otherwise color it with color 1. The general step: color the next interval with the smallest available color, so with a color which is not used by already colored intervals which intersect the current interval. Clearly this algorithm creates a proper coloring of the intervals and the corresponding interval graph.

Claim: This algorithm uses at most $\omega(G)$ colors.
Proof: Indirectky assume that for some interval Ik it uses color $\omega(G)+1$.
Then it means that all the smaller colors were not available, so Ik intersects with $\geq\omega(G)$ intervals which have been colored so their leftmost endpoint is not to the right.
In this case all of these intervals contain the leftmost endpoint of Ik, so they form a clique of size $\omega(G)+1$ in G, which is a contradiction. $\square$

So G has a proper coloring with $\omega(G)$ colors and since $\chi(G) \geq \omega(G)$ it is an optimal coloring. Therefore G is perfect. $\square$.

Some theorems about perfect graphs:

Weak perfect graph theorem (Lovász '72): G is perfect <--> $\bar{G}$ is perfect.

2nd perfect graph theorem of Lovász: G is perfect $\Leftrightarrow$ $\forall H$ induced subgraph of G: $|V(H)| \leq \alpha(H) \cdot \omega(H)$

Strong perfect graph theorem (Chudnovsky, Robertson, Seymour, Rhomas '02):
G is perfect if and only if G does not contain an induced subgraph H which is an odd cycle of length at least five or H is a complement of an odd cycle of length at least five.