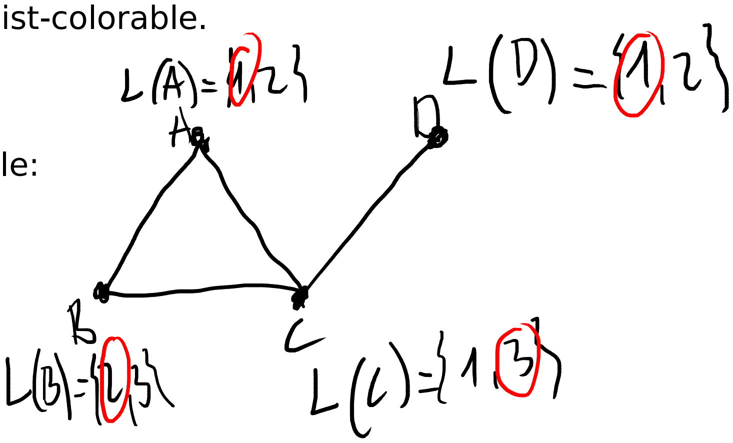
List coloring

definition: Let G be a graph, and for each vertex v of G a set (called list) L(v) is given. Each L(v) contains colors which can be used to color that vertex. Graph G can be colored from the lists L(v) if it has a proper vertex coloring c such that c(v) is an element of L(v) for each vertex v.
G is called k-list-colorable or k-choosable if G can be colored from any set of lists L(v) which satisfy |L(v)|=k, so the length of each list L(v) is k.
The list-chromatic number (or choice number) of G, denoted by ch(G) is the least k, such that G is k-list-colorable.

Example:



This graph can be colored from these lists, however its list-chromatic number is 3.

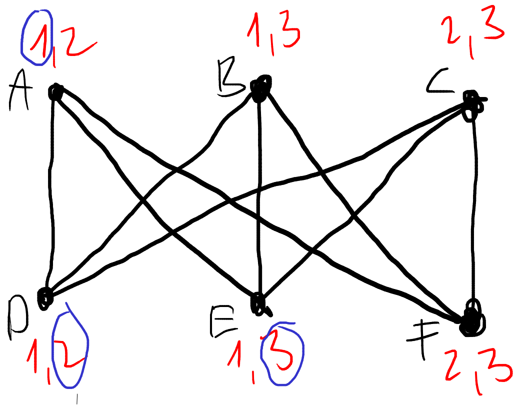
List coloring is invented and investigated by Vizing, Erdős, Rubin and Taylor.

Claim: For each graph G $\chi(G) \le ch(G)$.

Why? Because if each list is (1,2,3...$\chi(G)-1$), then G cannot be colored from those lists. These are lists of length $\chi(G)$-1, therefore ch(G)$\geq \chi(G)$.

What if the lists are different? The first idea is that if the lists are different, then we can use shorter lists and we can color the graph from these shorter different lists, but this is not true.

Example: $K_{3,3}$



Let's try to color this graph from these lists. Without loss of generality we can assume that, c(A)=1, and c(D)=2. Then c(E)=3 and c(C)=3, but E and C are adjacent, so this is not a proper coloring.

$\Rightarrow ch(K_{3,3}) \geq 3$ $\qquad \chi(K_{3,3})=2$

This is a bipartite graph, so its chromatic number is 2. To tell the truth ch(K_3,3)=3.

Claim: For each integer k, there is a graph G such that: $\chi(G)=2$ but $ch(G)>k$.

Proof: Let G be $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$, so the complete bipartite graph whose color classes separately contains $\binom{2k-1}{k}$ vertices.

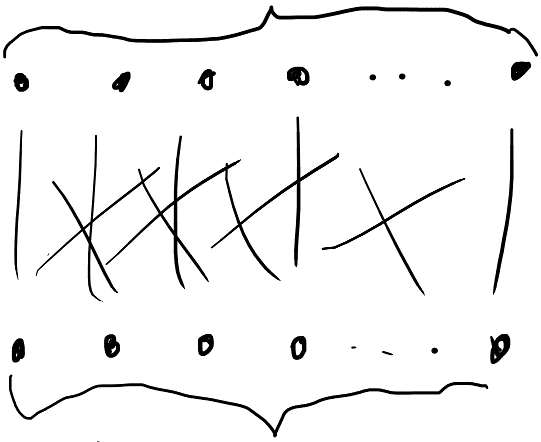
$$ch\left(K_{\binom{2k-1}{k},\binom{2k-1}{k}}\right) > k:$$

We give a set of lists whose length is k and G cannot be colored from these lists.

From 2k-1 colors we can choose $\binom{2k-1}{k}$ different lists of length k.

Assign list of length k containg colors from 1,2,3..2k-1 to each vertex of G in such a way that any two vertex from the same color class recieve different lists.
So we use each possible list once in each color class. Note, this is what we did for $K_{3,3}$.

$\binom{2k-1}{k}$ vertices, any two of them have different lists



$\binom{2k-1}{k}$ vertices, any two of them have different lists

We cannot color G from these lists, because:
Indirectly assume that we colored G from these lists.
If we use k-1 or less colors at the top vertices, then we do not use at least k colors, but there is a vertex whose list contains exactly those k colors, so we cannot pick a color for that vertex. Therefore at least k colors must be used at the top vertices and similarly at least k colors must be used at the bottom vertices. Since we have 2k-1 colors in total, there is a color which is used at a bottom and a top vertex as well. But these two vertices are adjacent, so this is not a proper coloring.

So we have shown a set of lists whose length is k, but G cannot be colored from these lists, therefore ch(G)>k.

Claim: For any graph G: $ch(G) \leq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G.

Proof: We can greedy color G if each vertex has $\Delta(G)$+1 available colors.

Theorem (Generalization of Brook's Thm): If G is connected, G is neither a complete graph nor an odd cycle, then $ch(G) \leq \Delta$.

Claim:
If T is a tree, then ch(T)=2.
If G is an even cyclen, then ch(G)=2.
If G is an odd cycle, then ch(G)=3.

We can define list coloring for edges. Instead of writting such a definition we can talk about the list coloring number of line graphs. The list coloring of edges is equivalent to the list coloring of the line graph.
We have seen that the chromatic and the list-chromatic number can be very different. On the other hand, if we list color the edges, so when we consider the list coloring number of line graphs, it looks like that the situation is the opposite.

List coloring conjecture: If G is a line graph, so there is a graph H such that G=L(H), then ch(G)= $\chi(G)$.

This is an open question. What we know, is the following theorem:

Galvin's Theorem: If H is a bipartite graph, then: $ch(L(H)) = \chi(L(H))$

remainder: Kőnig's theorem: If H is biparite, then $\chi'(H) = \chi(L(H)) = \Delta(H) = ch(L(H))$

by Galvin's Thm

List colroing of planar graphs

Remainder: 4-color theorem: If G is planar, then $\chi(G)\leq 4$.

Theorem (Thomassen '94): If G is planar, then $\chi(G)\leq 5$.

Voigt '93: There is a planar G such that ch(G)=5.
The example of Voigt contains 130 vertices.

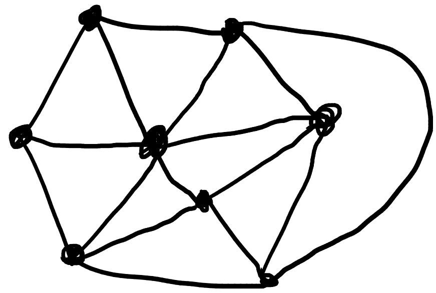
Mirzakhani '96: She have constructed a planar G whose vertex number is 63, ch(G)=5 and $\chi(G)=3$.

Proof of Thomassen's theorem:

We are going to proof by induction on the number of vertices, but the inductional hypothesis will be a strengthened version of the statement.

Notation: Let G be a plane graph and call it as an almost triangulated graph if the outer face is bounded by a cycle, and the other faces are triangles.
Let B denote the boundary of the outer face.

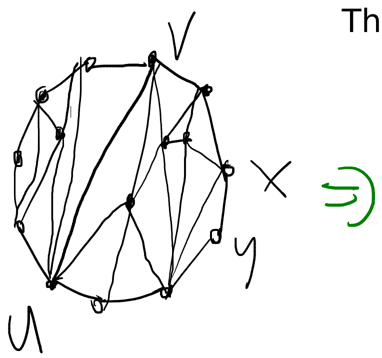
Example:



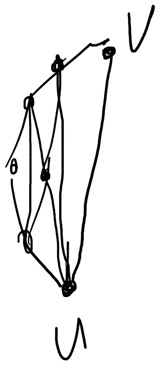
Lemma: Let G be an almost triangulated graph
Let x and y be two adjacent vertices at B, and
let L(x)=α and L(y)=β
If v is a vertex at B but v is neither x nor y, then
let L(v) be a list of length 3. Let the lenght of
the list of other vertices be 5.
Then G is colorable from lists L(v).

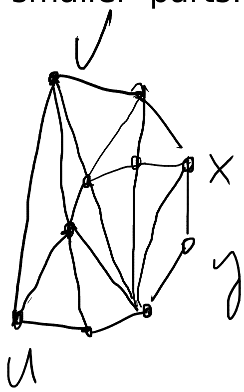
Clearly, if the lemma is true, then it implies Thomassen's theorem.

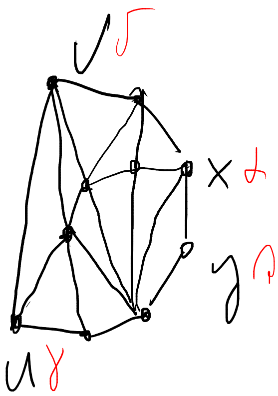
Proof of the lemma:
Induction on the number of vertices denoted by n. Clearly the lemma is true when n=3.
So assume that the statement is true for graphs having less vertices.
Case I: There are two vertices u and v at B such that (u,v) is a diagonal edge:



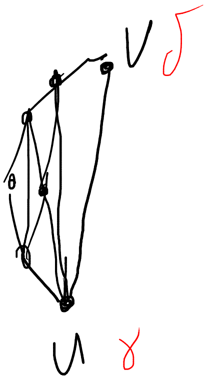
Then we can break G to two smaller parts:



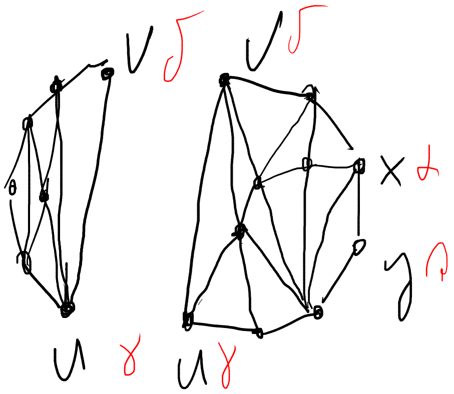




According to the inductional hypothesis, we can color this graph, from the lists. Choose such a coloring. c(x)=α and c(y)=β c(u)=γ c(v)=δ.
Note that it is not a problem if x or y is v or u. Also α and γ can be the same color, and etc, what we require that $\alpha\neq\beta$ and $\gamma\neq\delta$.



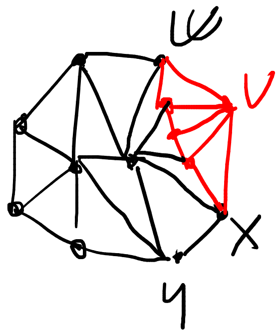
In this part let erase everything from L(u) except γ and similarly let erase everything from L(v) except δ, Then by the inductional hypothesis, this part can be colored from these lists.



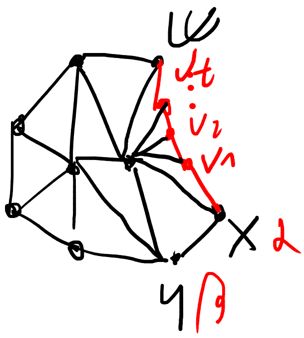
We have a coloring of both parts and we can join them, to obtain a coloring of G from lists L(v), because the color of u is the same in both parts and the same thing can be told about v.

So we have handled this case.

Case 2: There are no u and v at B such that (u,v) is a diagonal edge.



Lets denote the other neighbor of x which is at B by v. Lets denote the other neighbor of v which is at B by w. Since G is almost triangulated, the neighbors of v induce a path between v and w. Therefore if we delete v, we obtain a smaller almost triangulated graph.

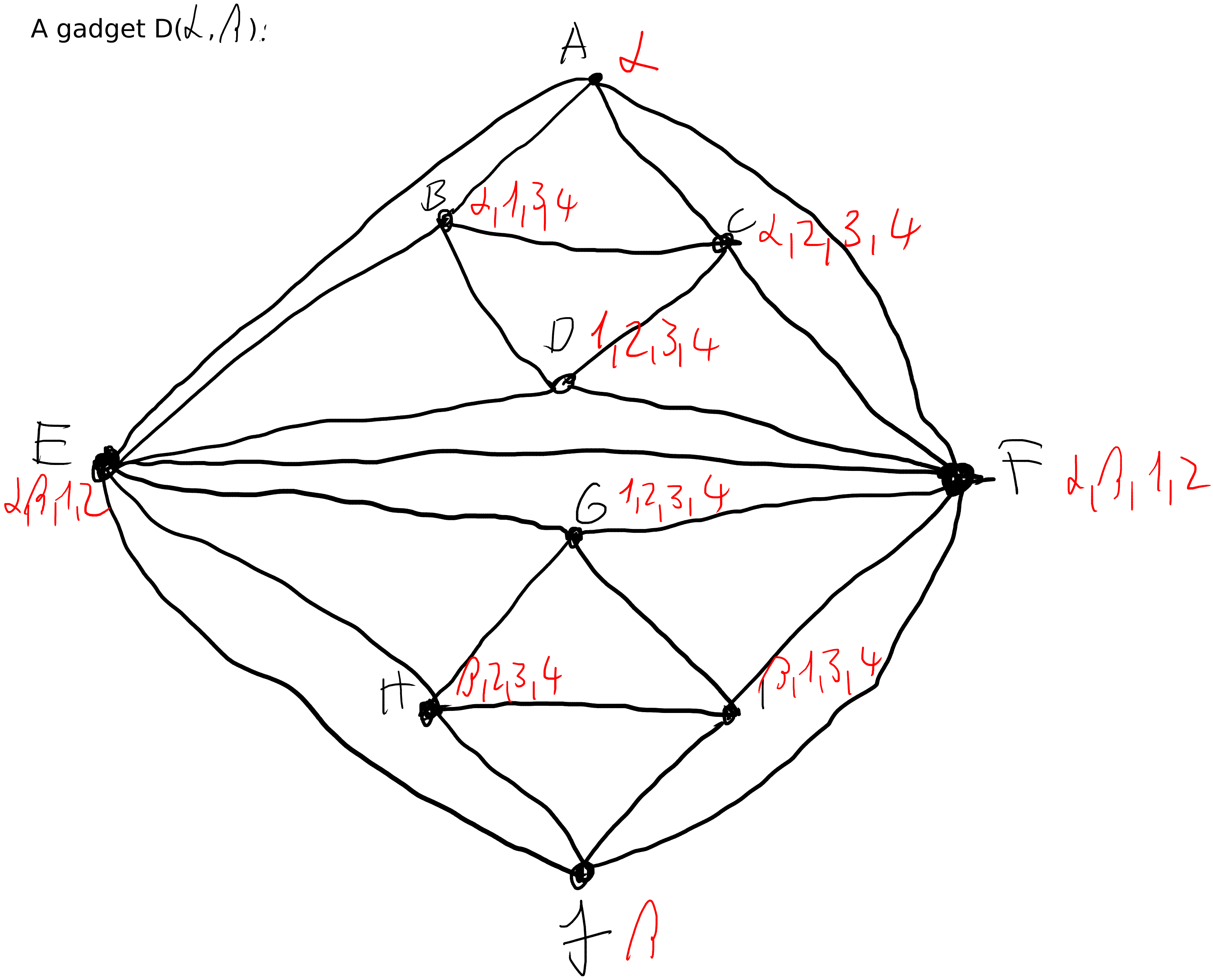


This new triangulated graph have bigger outer face, let's denote its boundary by B'. Lets denote the vertices of the path section of B' between x and w by $v_1, v_2, \dots, v_t$. These with x and w were the neighbours of v.
The list L(v) contains 3 colors, at least two of them are not α.
Lets denote two of these colors by γ and δ.
Delete these two colors, γ and δ from the lists of $v_1, v_2, \dots, v_t$.
If the length of $L(v_i)$ is bigger than 3, then delete one or two arbitrary colors from it to make its length 3.
By the inductional hypothesis we can color this smaller graph from these lists.
Now we just need a color for v which does not conflict with the color of its neighbours. γ and δ are included in L(v) and x, $v_1, v_2, \dots, v_t$ have not received these two colors.
If c(w) uses one of these two colors then color v with the other one, otherwise we can color v with any of them.

So we have proved the lemma and therefore the theorem as well.

Example of Voigt: A planar graph G having 130 vertices, lists of length 4 such that G is not colorable from these list.

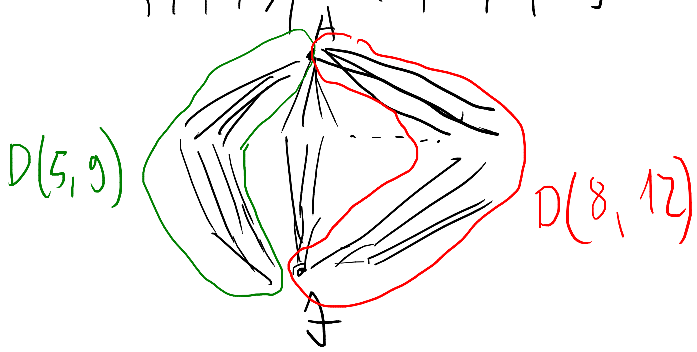
A gadget D(α, β):



Claim: This gadget is not colorable from the given lists.

Proof:
Let's try to color it from the given lists.
$c(A)=\alpha$, $c(J)=\beta$ $\Rightarrow$ due to symmetry, without loss of generality we can assume that c(E)=1, c(F)=2. We need color 3 and 4 to color B and C but, then no color remains for D.

The graph of Voidt contains 16 gadgets D(α, β), where α and β taking all possible combinations when $\alpha \in \{5,6,7,8\}$, $\beta \in \{9,10,11,12\}$ and these 16 gadgets are glued together among A and J.



This graph is planar and each list has length 4.
This is not colorable from these lists because if we try to color it and color A and B first, then the gadget D(c(A), c(B)) cannot be colored from the lists.
The number of vertices are 16*8+2=130.