Linear recurrence

def: The sequence $a_0, a_1, a_2, \dots$ is given by a linear recurrence relation with constant coefficients if $a_0, a_1, \dots, a_{p-1}$ and p+1 numbers $c_1, c_2, \dots c_p, d$ given which satisfies the following equality: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_p a_{n-p} + d$. when $n \geq p$

We call this equality as a linear recurrence relation. If d=0, then we say that it is a homogenous linear recurrence relation with constant coefficients.

Example: The Fibonacci sequence is given by a homogenous linear recurrence relation:

$$F_0 = 0;\ F_1 = 1;\ F_n = F_{n-1} + F_{n-2}$$

The first 9 elements of the series are: 0,1,1,2,3,5,8,13,21,

Its closed-form expression is: $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

Today we are going to see two methods how we can obtain the closed-form expression of any sequence given by a homogenous linear recurrence relation.

1st method: generating function

def: Let $a_0, a_1, a_2, a_3, \dots$ be a number sequence, then the power series $\sum_{i=0}^{\infty} a_i x^i$ is called as the generating function of the sequence and usually denoted by $F(x)$

We can simply use formal power series for calculation, but remember, the power series $F(x) = \sum_{i=0}^{\infty} a_i x^i$ is convergent on the open disc (-r,r) where r= $\frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$

it is divergent outside of the closed disc (-r,r) and sometimes it is convergent, sometimes it is divergent at points -r and r. Also F(x) is differentiable in (-r,r) infinitely many times.

Some operations:

$$a_0, a_1, a_2, a_3, a_4, \dots \Leftrightarrow F(x)$$
$$a_1, a_2, a_3, a_4, a_5 \Leftrightarrow \frac{F(x) - a_0}{x}$$
$$0, a_0, a_1, a_2, a_3 \Leftrightarrow F(x) \cdot x$$
$$a_1, 2a_2, 3a_3, 4a_4 \dots \Leftrightarrow F'(x)$$

How can we use the generating function to find a closed-form expression of a sequence?

Let's see it for the Fibonacci numbers! So let F(x) be the generating function of the Fibonacci numbers.

$$F(x) = \sum_{i=0}^{\infty} F_i x^i = F_0 + F_1 x + \sum_{i=2}^{\infty} F_i x^i = 0 + x + \sum_{i=2}^{\infty} (F_{i-1} + F_{i-2}) x^i =$$

We use the recurrence relation here

$$= x + x\sum_{i=2}^{\infty} F_{i-1}x^{i-1} + x^2\sum_{i=2}^{\infty} F_{i-2}x^{i-2} = x + x\left(\sum_{i=1}^{\infty} F_{i-1}x^{i-1} - F_0\right) + x^2\sum_{i=2}^{\infty} F_{i-2}x^{i-2} =$$

$$= x + x\sum_{j=0}^{\infty} F_j x^j + x^2\sum_{j=0}^{\infty} F_j x^j = x + xF(x) + x^2(F) = F(x)$$

We solve this function equality:

$$F(x)\left(1-x-x^2\right) = x$$

$$F(x) = \frac{x}{1-x-x^2}$$

We calculate the partial fraction decomposition of the generating function.

$$F(x) = \frac{x}{1-x-x^2} = \frac{x}{\left(1-\frac{1+\sqrt{5}}{2}x\right)\left(1-\frac{1-\sqrt{5}}{2}x\right)} = \frac{1}{\sqrt{5}}\underbrace{\frac{1}{1-\frac{1+\sqrt{5}}{2}x}} - \frac{1}{\sqrt{5}}\underbrace{\frac{1}{1-\frac{1-\sqrt{5}}{2}x}} =$$

each of them is a sum of a geometric series

$$= \frac{1}{\sqrt{5}}\sum_{i=0}^{\infty}\left(\frac{1+\sqrt{5}}{2}x\right)^i - \frac{1}{\sqrt{5}}\sum_{i=0}^{\infty}\left(\frac{1-\sqrt{5}}{2}x\right)^i = \sum_{i=0}^{\infty}\underbrace{\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^i\right]}_{F_i}x^i$$

We have not discussed why this power series is convergent. We can verify the convergence criteria for the obtained Fi-s and we see that every operation was feasible.
On the other hand, it is not necceseary to think about convergece. We can always use formal power series to obtain a closed-form expression and we can use induction to show the correctness of the obtained result.

So the generating function method in general is the following:
Let say that the initial values $a_0, a_1, \dots a_{p-1}$ and a homogenous linear reccurrence relation with constant coefficients are given: $a_n = \sum_{i=1}^{p} c_i a_{n-i}$ when $n \geq p$

Then we calculate the generating function:

$$R(x) = a_0 + a_1x + a_2x^2 + \dots + a_{p-1}x^{p-1} + \sum_{i=p}^{\infty} a_i x^i =$$

We use the recurrence relation:

$$= a_0 + a_1x + a_2x^2 + \dots + a_{p-1}x^{p-1} + \sum_{i=p}^{\infty}\sum_{j=1}^{p} c_j a_{i-j}x^i =$$

and then we reorganize the equation to obtain $R(x) = \dfrac{q(x)}{1-\sum_{j=1}^{p} c_j x^j}$

Finally we use partial fraction decomposition to determine the coefficients in the power series of R(x).

This method requiers a lot of calculation, let's see an easier one.

def: $x^p = c_1 x^{p-1} + c_2 x^{p-2} + \dots c_{p-1} x + c_p$ is called the characteristic equation of the homogenous linear recurrence $a_n = \sum_{i=1}^{p} c_i a_{n-i}$.

Example: Consider the Fibonacci sequence again. Its characteristic equation is $x^2 = x + 1$

The roots of this equation are: $x_1 = \frac{1+\sqrt{5}}{2}, x_2 = \frac{1-\sqrt{5}}{2}$

Claim: The geometric sequences created from the powers of the roots of the characteristic equation are solutions of the reccurrence relation:

$a_i^1 = x_1^i$ and $a_i^2 = x_2^i$

Why? Because: $x_1^2 = x_1 + 1 \Leftrightarrow x_1^n = x_1^{n-1} + x_1^{n-2} \Leftrightarrow a_n^1 = a_{n-1}^1 + a_{n-2}^1$

$x_2^2 = x_2 + 1 \Leftrightarrow x_2^n = x_2^{n-1} + x_2^{n-2} \Leftrightarrow a_n^2 = a_{n-1}^2 + a_{n-2}^2$

Then any linear combination of these two geometric sequences is a solution of the reccurence relation. We pick a linear combination such that it also satisfies the inintial conditions:

$$F_n = \lambda_1 x_1^n + \lambda_2 x_2^n$$

$$\left.\begin{aligned} F_0 &= \lambda_1 x_1^0 + \lambda_2 x_2^0 = \lambda_1 + \lambda_2 = 0 \\ F_1 &= \lambda_1 x_1 + \lambda_2 x_2 = \lambda_1 \frac{1+\sqrt{5}}{2} + \lambda_2 \frac{1-\sqrt{5}}{2} = 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda_1 &= \frac{1}{\sqrt{5}} \\ \lambda_2 &= -\frac{1}{\sqrt{5}} \end{aligned}$$

So we have find the closed-form expression: $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

The characteristic equation method in general is the following:

We find the roots of the characteristic equation $x^p = \sum_{i=1}^{p} c_i x^{p-i}$

Lets say the roots are $x_1, x_1, \dots x_p$.

Case I: We have obtained p different roots so each of them is a simple root. Then we solve the following linear system:

$$\begin{aligned} a_0 &= \lambda_1 + \lambda_2 + \dots \lambda_p \\ a_1 &= \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_p x_p \\ &\vdots \\ a_i &= \lambda_1 x_1^i + \lambda_2 x_2^i + \dots + \lambda_p x_p^i \\ &\vdots \\ a_{p-1} &= \lambda_1 x_1^{p-1} + \lambda_2 x_2^{p-1} + \dots + \lambda_p x_p^{p-1} \end{aligned}$$

This system always has a unique solution because the following Vandermonde determinant is not 0:

$$\begin{vmatrix} x_1^0 & x_2^0 & \dots & x_p^0 \\ x_1^1 & x_2^1 & \dots & x_p^1 \\ & & & \\ x_1^{p-1} & x_2^{p-1} & & x_p^{p-1} \end{vmatrix} \neq 0$$

And the solutions are: $a_n = \sum_{i=1}^{p} \lambda_i x_i^n$

Case II: The roots are $x_1, x_2, \ldots x_l$ and x_i is a roof of multiplicity k_i. So $k_1+k_2+\ldots+k_l=p$.

Then the following geometric sequences satify the recurrence relation:

$$a_n^1 = x_1^n \quad a_n^2 = n x_1^n \quad a_n^3 = n^2 x_1^n \ldots a_n^{k_1-1} = n^{k-1} x_1^n, \quad a_n^{k_1} = x_2^n, \ldots$$

Because if the multiplicity of x_1 is k_1, then it is a root of the k_1-1 th derivative of the the characteristic function

$$C(x) = x^p - \sum_{i=1}^{p} c_i x^{p-i};$$

Clearly x_1 is also a root of the k1-1th derivative of the function $\rightarrow x^n - \sum_{i=1}^{p} c_i x^{n-i}$ if $n \geq p$

$$\frac{d}{dx}\left(x^n - \sum_{i=1}^{p} c_i x^{n-i}\right) = n x^{n-1} - \sum_{i=1}^{p} c_i (n-i) x^{n-i} \; (*)$$

and $n x_1^{n-1} - \sum_{i=1} c_i x^{n-1} = 0 \iff n x_1^n = c_1(n-1)x_1 + c_2(n-2)x_1^{n-2} + \ldots + c_d(n-d)x_1^{n-d}$

x1 is the root of multiplicity k1-1 of this derivative function. Lets multiply * by x. x1 is the roof of multiplicity k1-1 of this new function as well. Lets differentiate it:

$$\frac{d}{dx}\left(n x^n - \sum_{i=1}^{p} c_i (n-i) x^{n-i}\right) = n^2 x^{n-1} - \sum_{i=1}^{p} c_i (n-i)^2 x^{n-i-1} \quad \text{if } n \geq p$$

and $n^2 x_1^{n-1} - \sum_{i=1}^{p} c_i (n-i)^2 x_i^{n-i-1} = 0 \iff n^2 x_1^n = c_1(n-1)^2 x_1^{n-1} + \ldots + c_d(n-d)^2 x^{n-1}$

x1 is the root of multiplicity k1-2 of the obtained function. If we multiply it by x, then x1 is the root of multiplicity k1-2 of the new function.

...

We can repeat this procedure until x_1 is the root of the obtained derivative.

We do this for all the roots of the characteristic polynomial.

Then we can solve the following linear system:

$$a_0 = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_{k_1-1} + \lambda_{k_1} + \ldots$$

$$a_1 = \lambda_1 x_1 + \lambda_2 x_1 + \lambda_3 x_1 + \ldots + \lambda_{k_1-1} x_1 + \lambda_{k_1} x_2 + \ldots$$

$$\vdots$$

$$a_i = \lambda_1 x_1^i + \lambda_2 i x_1^i + \lambda_3 i^2 x_1^i + \ldots + \lambda_{k_1-1} i^{k_1-1} x_1^i + \lambda_{k_1} x_2^i + \ldots$$

$$\vdots$$

$$a_{p-1} = \lambda_1 x_1^{p-1} + \lambda_2 (p-1) x_1^{p-1} + \ldots + \lambda_{k_1-1} (p-1)^{k_1-1} x_1^{p-1} + \lambda_{k_1} x_2^{p-1} + \ldots$$

And the solutions are: $a_n = \lambda_1 x_1 + \lambda_2 n x_1 + \lambda_3 n^2 x_1 + \ldots + \lambda_{k_1-1} n^{k_1-1} x_1 + \lambda_{k_1} x_2 + \ldots$