Extremal set theory

Let A be a base set. Then 2^A is the power set of A, so it contains all subsets of A.
$\mathcal{F} \subseteq 2^A$ is called a set-system or a family of subsets.

We do not lost generality if we say that A={1,2,3,..,n}=[n].
A set-system can be interpreted as a hypergraph where the verticess are the elements of A, the hyperedges are the elements of $\mathcal{F}$, which are subsets of A.

A set-system/hypergraph is k-uniform if each element of $\mathcal{F}$/hyperedge is a k element set.
A 2-uniform set-system/hyperedge is a graph.

Extremal set theory: How big can $\mathcal{F}$ be if it satisfies some conditions?

Examples:

$\mathcal{F} \subseteq 2^{[n]}$, for any pair $A, B \in \mathcal{F}$: if $A \neq B \Rightarrow A \cap B = \emptyset \Rightarrow \max|\mathcal{F}| = n+1$

The maximum is attained when $\mathcal{F}$ contains each one element subset and the empty set.

$\mathcal{F} \subseteq 2^{[n]}$, for any pair $A, B \in \mathcal{F}$: if $A \neq B \Rightarrow A \cap B \neq \emptyset \Rightarrow \max|\mathcal{F}| = 2^{n-1}$

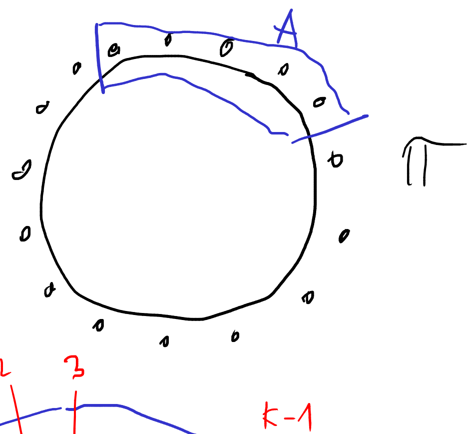
Proof: If $A \in \mathcal{F}$ then $\overline{A} \notin \mathcal{F}$, therefore at most half of the subsets of [n] can be included in $\mathcal{F}$. And this bound can be attained if we choose all subsets of [n] which contains 1.

Theorem: Erdős-Ko-Rado 1961: $\mathcal{F} \subseteq 2^{[n]}$ and $\mathcal{F}$ is k-uniform where $k \leq n/2$ and for any pair $A, B \in \mathcal{F}$: $A \cap B \neq \emptyset$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Note that if k>n/2, then any two k element subsets of [n] intersect each other, this is why the opposite case is interesting.

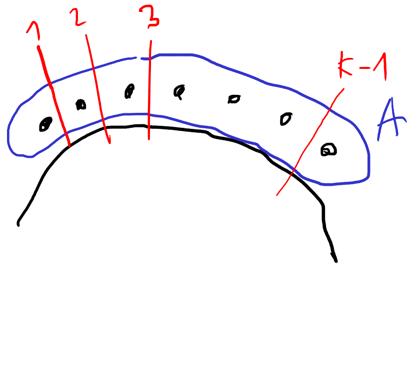
$\binom{n-1}{k-1}$ can be attained if we pick all the k element subsets of [n] which contain 1.

Proof by Gyula Katona OH '72:
Let Π be a circular permutation of [n]. So we imagine that the n elements are seated around a table.
A set A is an arc in Π if its elements are consecutive in Π:

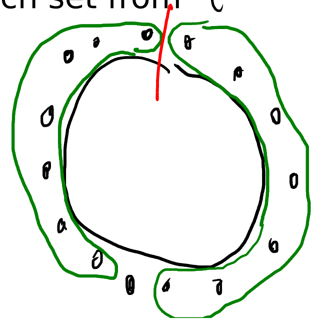


Fix a circular permutation Π.
There are at most k sets from $\mathcal{F}$ which are arcs in Π:

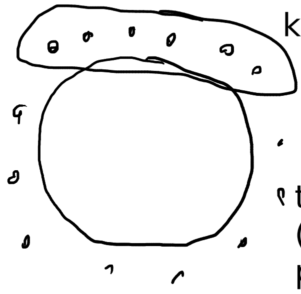
Pick a set A from $\mathcal{F}$ which is an arc in Π. An other set B from $\mathcal{F}$ which is also an arc in Π can ends at k-1 different positions, because B intersect A.



For each such position there is only one set from $\mathcal{F}$ which is an arc in Π and ends there: Each set from $\mathcal{F}$ has the same size k, therefore two arcs B and C of size k which end at the same place are either the same or disjoint because $k \leq n$.
So for Π there are at most k set from $\mathcal{F}$ which are arcs.



The total number of circular permutations is $(n-1)!$
So if we consider all of the circular permutations, we find at most $k(n-1)!$ arcs which correspond to sets from $\mathcal{F}'$.
A set from $\mathcal{F}'$ is an arc in $k!(n-k)!$ circular permutations.
Therefore:



$$|\mathcal{F}| \leq \frac{k(n-1)!}{k!\,(n-k)!} = \binom{n-1}{k-1}$$

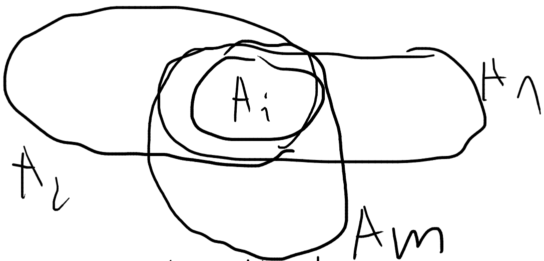
$\square$.

Theroem: Fischer's inequality:

$\mathcal{F} = \{A_1, A_2, \ldots A_m\} \subseteq 2^{[n]}$, $\lambda > 0$ and for $\forall i \neq j: |A_i \cap A_j| = \lambda \Rightarrow m \leq n$

Proof: For each i $|A_i| \geq \lambda$.

Case 1: There is an element A_i s.t $|A_i| = \lambda$. Then every other set A_j contains A_i as a subset. But then $\{A_1 \setminus A_i, A_2 \setminus A_i, \ldots A_{i-1} \setminus A_i, \emptyset, A_{i+1} \setminus A_i \ldots, A_m \setminus A_i\}$
is an m element set-system such that any two elements are disjoint, and its base set contains $n-\lambda$ elements, therefore this set system contains at most $n-\lambda+1 \leq n$ sets, so $m \leq n$.



Case 2: $\forall i: |A_i| > \lambda$:

Let $\bar{a}_i$ be the characteristic vector of A_i, so it is an n dimensional vector and $a_{i,j} = \begin{cases} 1 & \text{if } j \in A_i \\ 0 & \text{if } j \notin A_i \end{cases}$

Example: If n=7 and $A_i = \{1, 3, 5\}$ then $\bar{a}_i = [1, 0, 1, 0, 1, 0, 0]$

We are going to show that $\bar{a}_1, \bar{a}_2, \ldots \bar{a}_m$ are linearly independent.

$$\bar{a}_i \cdot \bar{a}_j = \begin{cases} \lambda & \text{if } i \neq j \\ |A_i| & \text{if } i = j \end{cases}$$

Consider a linear combination $c_1\bar{a}_1 + c_2\bar{a}_2 + c_3\bar{a}_3 + \ldots + c_m\bar{a}_m = \underline{0}$

Take the dot product of this vector and $\bar{a}_i$:

$$0 = \underline{0} \cdot \bar{a}_i = (c_1\bar{a}_1 + c_2\bar{a}_2 + \ldots + c_m\bar{a}_m)\bar{a}_i = (c_1 + c_2 + \ldots + c_m)\lambda + c_i\overbrace{(|A_i| - \lambda)}^{>0}$$

If $c_1 + c_2 + c_3 + \ldots + c_m = 0 \Rightarrow c_i = 0 \;\forall i$ $\Rightarrow$ $\bar{a}_1, \bar{a}_2, \ldots \bar{a}_m$ are linearly independent

If $c_1 + c_2 + c_3 + \ldots + c_m > 0 \Rightarrow c_i < 0 \;\forall i$ $\lightning$

If $c_1 + c_2 + c_3 + \ldots + c_m < 0 \Rightarrow c_i > 0 \;\forall i$ $\lightning$

If m vectrors which are n dimensional are linearly independent, then $m \leq n$.

$\square$.

Theorem: Ray-Chaudhuri-Wilson:
$L=\{l_1,\dots,l_s\}$, $\mathcal{F}\subseteq 2^{[n]}$, $\forall A,B\in\mathcal{F}$, $A\neq B: |A\cap B|\in L$ then: $|\mathcal{F}|\leq \sum_{i=0}^{s}\binom{n}{i}$

This bound can be attained, for example when L={0,1,2,...,s-1}, then $\mathcal{F}=\{A\subseteq[n] \mid |A|\leq s\}$ and $|\mathcal{F}|=\sum_{i=0}^{s}\binom{n}{i}$.

When s=1, Ray-Chaudhuri-Wilson gives that $\forall A,B\in\mathcal{F}$ $A\neq B: |A\cap B|=l \Rightarrow |\mathcal{F}|\leq \sum_{i=0}^{1}\binom{n}{i}=n+1$

Fischer's inequality gives: $\forall A,B\in\mathcal{F}$ $A\neq B: |A\cap B|=\lambda \Rightarrow |\mathcal{F}|\leq n$

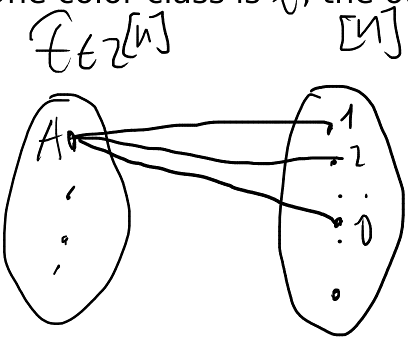
The difference comes from that $l\geq 0$ but $\lambda>0$!

Theorem: De Bruijn-Erdős: $\mathcal{G}\subseteq 2^{[m]}$, $\forall i,j\in[m]$, $i\neq j$: there is exactly one set $A\in\mathcal{G}$ such that $i,j\in A$. and if $B\in\mathcal{G} \Rightarrow |B|\geq 2$.
$\Rightarrow |\mathcal{G}|=1$ or $|\mathcal{G}|\geq m$.

This theorem can be interpreted in the following way: If m points in the projective plane are given then either they are on the same line or they determine at least m different lines.

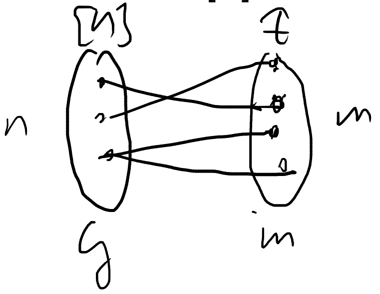
We prove this theorem by using hypergraph duality:

Any hypergraph $\mathcal{F}\subseteq 2^{[n]}$ can be represented by a bipartite graph in the folowing way: One color class is $\mathcal{F}$, the other one is [n], and a set A is incident to an element j if $j\in A$.

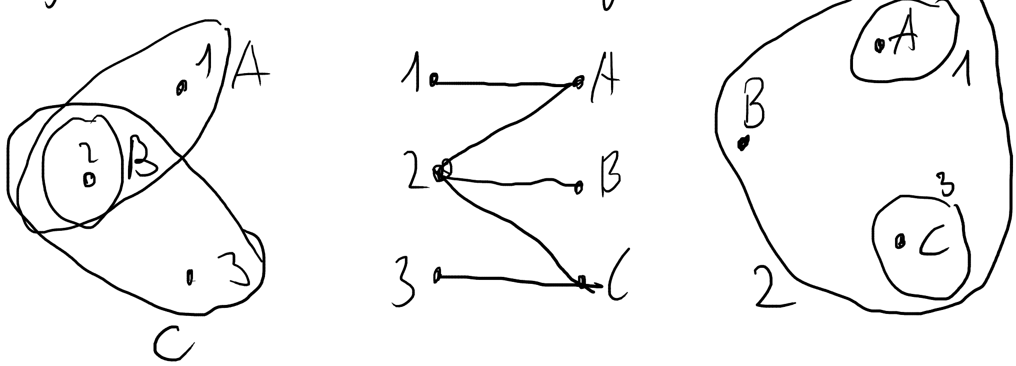


this can be a multiset of sets

If $\mathcal{F}\subseteq 2^{[n]}$, $|\mathcal{F}|=m$ then the dual hypergraph of $\mathcal{F}$ is a hypergraph $\mathcal{G}\subseteq 2^{[m]}$, s.t. $|\mathcal{G}|=n$ and the bipartite graph representation of $\mathcal{G}$ and $\mathcal{F}$ are the same, except that the role of the two color classes have been swaped. So the class which contained elements of $\mathcal{F}$ is containing the elements of [m] and the other class which contained the elements of [n] is containing the elements of $\mathcal{G}$.



Example:
$\mathcal{F}\subseteq 2^{\{1,2,3\}}$ $\mathcal{F}=\{A,B,C\}$ $\mathcal{G}\subseteq 2^{\{A,B,C\}}$, $\mathcal{G}=\{1,2,3\}$



Proof of De Bruijn-Erdős:

Let $\mathcal{G}$ be a set system which satisfies the conditions of the theorem and let $\mathcal{F}$ be its dual.

Case 1: We obtained the same set twice by the construction of $\mathcal{F}$, so there are A,B $\in \mathcal{F}$ s.t. A=B, so $\mathcal{F}$ is not a set but a multiset.
In this case A and B contain the same elements. What does it mean for $\mathcal{G}$?
A and B correspond to elements a and b, respectively and any set of $\mathcal{G}$ which contains a or b it contains the other. But there is exactly one such set. If c is another element from [m], then there must be a set which contains a and c, but then it also contains b. So c is also contained in that set. Therefore $\mathcal{G} = \{[m]\}$.

Case 2: Every set appears only once during the construction of $\mathcal{F}$, so it is a set of subsets.
Then $\forall a,b \in [m] \Rightarrow \exists!\, C \in \mathcal{G} : a,b \in C \Longrightarrow \forall A,B \in \mathcal{F} \Rightarrow \exists!\, c \in [n] : c \in A \cap B.$

So we can apply Fischer's inequality and we obtain that $m \le n = |\mathcal{G}|$. $\square$

m can be attained by the following example which is called as the "near pencil":