Sperner's theorem, LYM inequality

definition: A family $\mathcal{F}$ is called a Sperner system/ Sperner family if for any A,B $\in \mathcal{F}$: $A \not\subset B$ and $B \not\subset A$.

Sperner's Theorem (1928):
If $\mathcal{F} \subseteq 2^n$ is a Sperner system, then $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and the only (even: one, odd: two) Sperner systems which reach this bound are $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ and $\binom{[n]}{\lceil \frac{n}{2} \rceil}$.

$\binom{[n]}{k}$: is the family which contains all the k element subsets of [n].

Of course a k element set cannot contain another k element set, therefore these are sperner systems and $\left|\binom{[n]}{k}\right| = \binom{n}{k}$. We are going to prove this theorem later.

LYM inequality (Lubell '66, Yamamoto '54, Meshalkin '63):

Let $\mathcal{F} \subseteq 2^n$ be a Sperner system and let f_k denote the number of k element sets contained in $\mathcal{F}$. Then: $\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}} \leq 1$. Equality is attained if and only if $\mathcal{F} = \binom{[n]}{k}$ for some k.

Proof of Sperner's theorem by LYM inequality:

$$1 \geq \sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}} \geq \sum_{i=0}^{n} \frac{f_i}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \frac{|\mathcal{F}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \Rightarrow \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq |\mathcal{F}|.$$

We have used here that $\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \binom{n}{i} \quad \forall i$

If $\binom{n}{\lfloor \frac{n}{2} \rfloor} = |\mathcal{F}|$, then every inequality is satisfied with equality, therefore by LYM inequality $\mathcal{F} = \binom{[n]}{i}$ for some i and $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{i}$ therefore either $\mathcal{F} = \binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ or $\mathcal{F} = \binom{[n]}{\lceil \frac{n}{2} \rceil}$. $\square$

We are going to give 2 different proofs for LYM inequality. The statement about the equality case will be obtained only from the second one.

LYM ineq 1st proof:
$A_0 \subset A_1 \subset A_2 \subset A_3 \subset \ldots \subset A_i \subset \ldots A_n$ where $\forall i \; |A_i| = i$ is called as an ascending chain. There is a bijection between ascending chains of [n] and the permutations of [n].
Example: $\emptyset \subset \{3\} \subset \{2,3\} \subset \{2,3,4\} \subset \{1,2,3,4\} \Leftrightarrow 3241$

So $\pi(j) = i \Leftrightarrow j \notin A_{i-1}$ but $j \in A_i$

Clearly a Sperner system can contain at most one set from an ascending chain.
A k element set from [n] is contained in exactly k!(n-k)! ascending chains:
We need to include its k elements one by one in the first k set of the ascending chain. It can be done in k! different ways. Then we can finnish the ascending chain by adding the remaining n-k elements one by one and that can be done (n-k)! different ways.

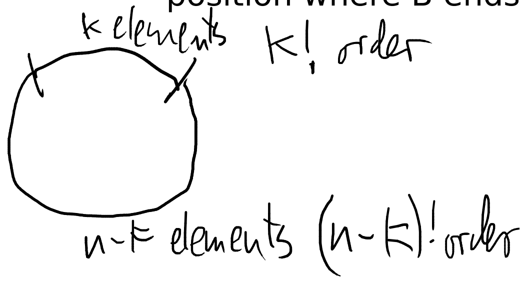
Lets calculate the number of ascending chains which contain a set from $\mathcal{F}$, An ascending chain can contain at most one set from $\mathcal{F}$, therefore:

$$\underbrace{n!}_{\#\text{ascending chains}} \geq \underbrace{\sum_{k=0}^{n} f_k\, k!(n-k)!}_{\#\text{ascending chains which contain an element from } \mathcal{F}}$$

$$\Rightarrow \quad 1 \geq \sum_{k=0}^{n} f_k \frac{k!(n-k)!}{n!} = \sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \qquad \square .$$

LYM inequality 2nd proof:

Let π be a circular permutation of [n]. A set A is an arc in π if its elements are consecutive in π. Fix π.
At most n sets from $\mathcal{F}$ can be an arc in π:
At a position at most one arc set from $\mathcal{F}$ can start,
because if A and B start at the same position, then one of them ends later and it contains the other one, but $\mathcal{F}$ is a Sperner system. There are n positions, at each of them at most one arc set from $\mathcal{F}$ can start, therefore at most n sets from $\mathcal{F}$ is an arc in π.
The number of circular permutations of [n] is (n-1)!.
A k element subset of [n] is an arc in exactly k!(n-k)! circular permutations.



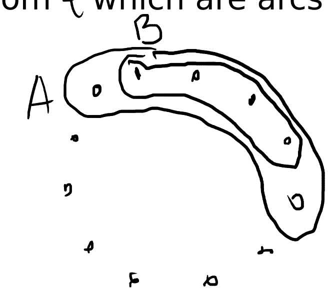
Lets calculate the total number of arcs in all circular permutations which are contained in $\mathcal{F}$:

$$\overbrace{\sum_{k=0}^{n} f_k\, k!(n-k)!} \leq n\underbrace{(n-1)!} \quad \Rightarrow \quad \sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1$$

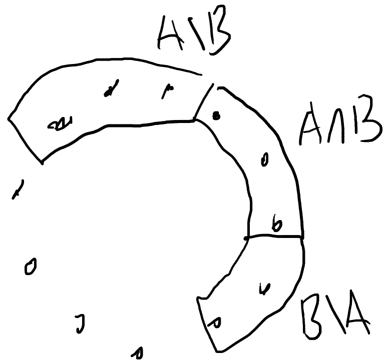
an upper bound on total number of arcs in all circular permutations which are contained in $\mathcal{F}$

What if $\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} = 1$?

Then our upper bound on the total number of arcs in all circular permutations is sharp, therefore for each circular permutation there π are exactly n sets from $\mathcal{F}$ which are arcs in π.
In this case all of these n sets must have the same cardinality:
At any position an arc from $\mathcal{F}$ starts, and if their size is not the same, then there are arcs A,B $\in \mathcal{F}$ such that A and B starts at adjacent positions, B starts later but |B|<|A|. It means $B \subset A$, but $\mathcal{F}$ is a Sperner system, so this cannot happen.
Therefore in π all arcs which are contained in $\mathcal{F}$ have the same size.



For any two sets $A,B \subseteq [n]$, there is a circular permutation such that A and B are arcs in it:



Therefore if $\mathcal{F}$ satisfies LYM with equality, then in any circular permutation the size of any arc which is contained in $\mathcal{F}$, has the same size. Thus $\mathcal{F} = \binom{[n]}{k}$ for some k.