Combinatorics and Graph Theory II.

9th practice, 16th of November, 2022.

Erdős-Ko-Rado, Fischer, De Bruijn-Erdős

Good to know

Erdős-Ko-Rado theorem. (1961) If $\mathcal{F} \subseteq 2^{[n]}$ is a k-uniform set-system ($k \leq n/2$) with the property that for any $A, B \in \mathcal{F} : A \cap B \neq \emptyset$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and this can be attained.

Fischer's inequality (1940) $\mathcal{F} = \{A_1, A_2, \dots, A_m\} \subseteq 2^{[n]}$ is a set system such that for any $i, j \in [n]$ when $i \neq j$: $|A_i \cap A_j| = \lambda > 0$. Then $m \leq n$.

Ray-Chaudhuri-Wilson theorem (1975) Let $L = \{l_1, l_2, \dots l_s\}$ and let $\mathcal{F} = \{A_1, A_2, \dots, A_m\} \subseteq 2^{[n]}$ be a set system such that for any $i, j \in [n]$ when $i \neq j : |A_i \cap A_j| \in L$. Then $m \leq \sum_{i=0}^{s} \binom{n}{i}$.

De Bruijn − Erdős theorem (1948) $\mathcal{F} \subset 2^{[n]}$, for all $A \in \mathcal{F}$ $|A| \geq 2$, and for arbitrary elements $1 \leq i < j \leq n$ there is exactly one $A \in \mathcal{F}$ such that $i, j \in A$. Then $|\mathcal{F}| = 1$ or $|\mathcal{F}| \geq n$.

1. Let $n = p_1 p_2 \cdots p_k$, where every $p_i > 1$ and p_i is a prime, furthermore $p_i \neq p_j$ if $i \neq j$. At most how many divisor of n can be choosen in such a way that no two of them are coprimes?
2. King Arthur sends out a reconing team every day. Each team must consist exactly k knights and two teams cannot be the same. However every two team must share a common member. How many days can Arthur sends out a reconing team following these rules?
3. m lines are given on the plane. Assume that all lines are not incident to the same point and no two lines are paralell to each other. Prove that these lines determine at least m intersection points.
4. A chemical company produces pesticides. They are testing the effect of their new product on m different plants at n different fields. They plant k different plants at each field and they plant each plant at r different fields. For each pair of plants there are exactly $l \geq 1$ fields where both of them have been planted. We know that $l \neq r$. Show that $n \geq m$.
5. There are some k element sets and any two of them intersect at l elements. Prove that there is an element which is contained in at most k sets.
6. There is a 100 element set. We choose some of its 20 and 80 element subsets in such a way that any two choosen subsets intersect each other. At most how many such subsets can be choosen?
7. Show that, if the family $\mathcal{F} \subseteq 2^{[n]}$ contains 2^{n-1} sets and any two of them intersect each other, then there are sets $F_1, F_2 \in \mathcal{F}$ such that $|F_1 \cap F_2| = 1$.
8. Assume, that $k < n/2$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is a Family such that any two sets from $\mathcal{F}$ intersect each other and $|\mathcal{F}| = \binom{n-1}{k-1}$. Show that $[n]$ has an element i such that all sets of $\mathcal{F}$ contain i. (The equality case of Erdős-Ko-Rado theorem)

Homework:

Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting family ($A, B \in \mathcal{F} \implies A \cap B \neq \emptyset$). Show that there is an intersecting family $\mathcal{G} \subseteq 2^{[n]}$ which satisfy $\mathcal{F} \subseteq \mathcal{G}$ and $|\mathcal{G}| = 2^{n-1}$.