Combinatorics and Graph Theory II.

8th practice, 7th of November, 2022.

Turán

Good to know

For any graph H $ex(n, H)$ denotes the maximum number of edges of an n vertex simple graph which does not contain H as a subgraph. $Ex(n, H)$ denotes the set of n vertex simple graphs which does not contain H as a subgraph and the number of their edges is $ex(n, H)$.

Let $n, r \geq 1$. The n vertex r class Turán graph $T(n, r)$ has n vertices and r classes in such a way that if $n = ar + b$, $r > b \geq 0$, theb b classes contain $\lceil n/r \rceil$ vertices and $r - b$ classes contain $\lfloor n/r \rfloor$ vertices. If two vertices are contained in different classes, then they are adjacent, otherwise they are not adjacent.

Turán's theorem (1941). $ex(n, K_{r+1}) = |E(T(n, r))|$. If G is an n vertex graph which does not contain K_{r+1} as a subgraph and $|E(G)| = |E(T(n, r))|$, then G is isomorphich to the Turán graph $T(n, r)$, so $Ex(n, K_{r+1}) = T(n, r)$.

1. What is the maximum number of edges which an n-vertex simple graph can have if it does not contain
   - a cycle?
   - an odd cycle?
   - an even cycle?
   - a path of length 3 nor a cycle of length 3?
   - a spanning tree?
2. In a university class there are 90 students. Some students have private conversations in Teams. It does not matter how we choose 10 students, there are at least two of them which have a private conversation. Prove that the number of private conversations are at least 405.
3. Prove that the Turán graph $T_{n,m}$ does not contain a Hamiltonian cycle if and only if $m = 2$ and n is odd.
4. Let $v_1, v_2, \ldots, v_n$ be vectors from a plane, $|v_i| \geq 1$. What is the minimum number of pairs which satisfy $|v_i + v_j| \geq 1$?
5. Let G be a simple graph which does not contain a triangle and $|E(G)| \geq 2|E(K_k)|$. What is the minimum of $|V(G)|$?
6. What is the maximum number of edges that an n vertex graph can have if its edges can be colored by two colors such that there is no monochromatic triangle.
7. There are n people in a party where there are no two people who knows each other. What is the minimum number of introductions (of two people to each other) which are needed to obtain the following properties:
   1. In any group of three people there are two of them who have been introduced to each other.
   2. Anybody can send a message to anybody else in such a way that the message is handed between people who know each other.
8. A graph has 49 vertices and 1030 edges. Show that the chromatic number of this graph is at least 8 and it can be exactly 8.
9. In a party there are n people, in any group of k (k is a fixed number) people there are two who have shaken hands with each other. What is the minimum number of handshakings which have happened?
10. Let H be the 5 vertex graph which is the disjoint union of an edge and a triangle. Determine the value of $ex(n, H)$. (Let $n \geq 100$.)

11. n not necesearily different points are given on the plane. What is the maximum number of vertices which can be choosen in such a way that the distance between any two points is exactly one?

Homework

1. a. G is an n vertex graph and the degree of each vertex of G is at least 100. Prove that G contains a path of length 99 (a path which contains 100 vertices).

   b. G is an n vertex graph having e edges, $e > 100n$. Prove that G contains a path of length 99 (a path which contains 100 vertices).

2. Give an n vertex graph for each n such that $e > 40n - 100000$ and n does not contain a a path of length 99 (a path which contains 100 vertices).