Combinatorics and graph theory II.

7th practice, 26th of October, 2022.

Ramsey type problems II.

Good to know:

Schur's theorem: For all $t > 0$ there is a (smallest possible) number $N = N(t)$ such that it does not matter how we color the integers $1, 2, \ldots, N$ by t colors, there are always exist x, y, z which have received the same color and $x + y = z$.

Van der Waerden's theorem: For all $t, k > 0$ there is a (smallest possible) number $N = N(t, k)$ such that it does not matter how we color the integers $1, 2, \ldots, N$ by t colors, there is always a monochromatic arithmetic progression of length k.

1. Prove, that if $c \geq 3$, then $R(n_1, n_2, \ldots, n_c) \leq R(n_1, n_2, \ldots, n_{c-2}, R(n_{c-1}, n_c))$.
2. Assume that we know that the Van der Waerden's theorem is true when we color by two colors. By using this, show that it is true when we color by three or more colors.
3. Prove that for all k positive integers there is a number $N(k)$ such that it does not matter how we color the numbers $1, 2, 3, \ldots, N(k)$ by k colors, then there are three different numbers x, y, z which have the same color and $x + y = 2z$.
4. Prove that for all $t, k > 0$ there is a number $M = M(t, k)$ such that it does not matter how we color the numbers $1, 2, \ldots, M$ by t colors there is always a monochromatic geometric progression whose length is k.
5. Let a_n be an infinite monotone increasing sequence of natural numbers. Show that there is either an arbitrary long subsequence of a_n such that any two elements of it are coprimes or there is an arbitrary long subsequence of a_n such that no two elements of it are coprimes.
6. Prove that for all $t > 0$ there is a number $M = M(t)$ such that it does not matter how we color the numbers $1, 2, \ldots, M$ by t colors there are numbers x, y, z which have the same color and $x + y = z$ and $x \neq y$.
7. Can we color the integers by two colors such that there is no monochromatic infinite arithmetic progression? What about infinite monochromatic geometric progressions?
8. Show a $(k-1)^2$ vertex graph which neither contains a clique of size k nor contains an independent set of size k.
9. Prove that for all n there is a number $K(n)$, such that any $K(n)$ distinct points on the plane determines at least n different distances.
10. The points of the plane are colored by red, white and green. Show that there are two points which have the same color and their distance is 1.
11. Show that the points of the plane can be colored by 9 colors in such a way that there are no two points which have the same color and their distance is 1. Show this with 7 colors as well.
12. Assume that the points of the plane are colored by red and green in such a way that both colors have been used. Show that there are two points such that their distance is one and they have received different colors. Is it true that there is a monochromatic equilateral triangle whose edge length is one?

Homework

1. Show that for each positive integer k there is a number $N(k)$ such that if we color the numbers $2, 3, \ldots, N(k)$ by k colors, then there are numbers x, y, z such that they have received the same color and $x + y = z + 1$.