Combinatorics and graph theory II.

6th practice, 19th of October, 2022.

Ramsey

Good to know:

Ramsey's theorem (for graphs with two colors): For all integers $k, l > 0$ exists a (smallest possible) number $R = R(k,l)$ which satisfies that it does not matter how we color the edges of the R vertex complete graph by red and blue, there is either a blue clique of size k or a red clique of size l. Erdős–Szekeres: $R(k,l) \leq \binom{k+l-2}{k-1}$.

$2^{\frac{k}{2}} \leq R(k,k) \leq 4^{k-1}$

Ramsey's theorem (for graphs with r colors): For all $r > 0$, $k_1, k_2, \ldots, k_r > 0$ numbers there is a (smallest possible) $R = R(k_1, k_2, \ldots, k_r)$ number such that it does not matter how we color the edges of the R vertex complete graph by colors $1, 2, \ldots, r$ there is a color i, such that there is a clique of size k_i whose edges have color i. ra

Ramsey's theorem (for k-uniform hipergraphs with r colors): Fopr all $r > 0$, $k \geq 2$, $k_1, k_2, \ldots, k_r > 0$ numbers there is a (smallest possible) number $R = R_k(k_1, k_2, \ldots, k_r)$ such that it does not matter how we color the edges of complete k-uniform hypergraph by colors $1, 2, \ldots, r$ there is a color i such that there are k_i vertices which induce a subhipergraph whose all edges have color i.

1. Let G be any graph which has at least 10 vertices. Show that $\omega(G) \geq 4$ or $\alpha(G) \geq 3$.
2. Prove that $R(3,3,3) \leq 17$ and prove that $R(3,4) = 9$.
3. a. We have colored the edges of a complete graph by red and blue. Prove that it has a monochromatic spanning tree.

   b. Is it true that a monochromatic Hamiltonian-path also exists?
4. Prove, that $R_3(4,4) \leq 21$.
5. Prove the following inequality: $R_3(k,l) \leq R_2(R_3(k-1,l), R_3(k,l-1)) + 1$. What kind of upper bound (in magnitude) on $R_3(k,k)$ comes from this inequality?
6. a) Prove, that if $c \geq 3$, then $R(n_1, n_2, \ldots, n_c) \leq R(n_1, n_2, \ldots, n_{c-2}, R(n_{c-1}, n_c))$.

   b) Prove, that if $c \geq 3$, then $R_t(n_1, n_2, \ldots, n_c) \leq R_t(n_1, n_2, \ldots, n_{c-2}, R_t(n_{c-1}, n_c))$.
7. Show that for each positive integer k there is a thresshold $N(k)$, such that if $n > N(k)$ and we color the subsets of the set $[n] := \{1, 2, \ldots, n\}$ by k colors, then there are disjoint subsets X_1 and X_2 of $[n]$ such that the color of X_1, X_2 and $X_1 \cup X_2$ is are the same. Is this statement true for 3 disjoint subsets?
8. Let $H(V,E)$ be a k uniform hypergaph which has less than 2^{k-1} edges. Show that the vertices of H can be colored by red and blue in such a way that no edge is monochromatic. (Each edge have blue and red vertices.)
9. Show that if G is an n vertex graph, then $\max(\alpha(G), \omega(G)) \geq 1 + \log_4 n$.
10. (Infinite Ramsey's theorem) We color the edges of a complete graph which has infinite number of vertices by red and blue. Show that there is a monochromatic infinite clique in the graph.

Homework:
Show that the Ramsey number $R(16;17) \geq 241$.