Combinatorics and graph theory II.

5th practice, 15th of October, 2022.

List coloring

Good to know:

G is a graph, and for each vertex v of G a list of colors $L(v)$ is assigned. G is L-colorable if G has a proper coloring such that for each vertex v the color of v is an element of $L(v)$.

The list coloring number of G, denoted by $ch(G)$, is the least k, such that if for each vertex v $|L(v)| = k$, then G is L-colorable.

For all G: $\chi(G) \leq ch(G)$, and for any k there is a graph G, such that $\chi(G) = 2$ but $ch(G) > k$.

For all G $ch(G) \leq \Delta(G) + 1$.

List coloring conjecture: If G is a line graph, then $\chi(G) = ch(G)$.
Galvin's theorem: If G is the line graph of a bipartite graph, then $\chi(G) = ch(G)$.
Thomassen 94: If G is planar, then $ch(G) \leq 5$.
Voigt 93: There is a planar graph G such that $ch(G) = 5$.

1. Determine the value of $ch(K_{2,4})$. ($K_{2,4}$ is the complete bipartite graph, where the two color classes contain 2 and 4 vertices.)

2. Let $K_{2,2,\ldots,2}$ be the graph whose complement is n disjoint edges. Determine $ch\left(K_{2,2,\ldots,2}\right)$.

3. Is it true that if $\chi(G) = ch(G)$, then $\chi(\overline{G}) = ch(\overline{G})$?

4. Let G be a graph. Is it true, that if for each vertex of G an at least $ch(G)$ long list is given, then there is an order of the vertices such that we can color the vertices greedily according to that order by selecting the smallest color which has not been used at the neighbors at each vertex?

5. Let G be a finite simple graph. Prove that if for each vertex v of G $|L(v)| > d(v)$ is satisfied, then G is L-colorable. ($d(v)$ denotes the degree of v)

6. Show that if T is a tree which has at least two vertices, then $ch(T) = 2$.

7. Prove that if C is an odd cycle, then $ch(C) = 3$.

8. Let G be an arbitrary graph. We construct the graph $3G$ in the following way: We pick 3 disjoint copies of G and for each vertex v of G we connect the three copies of v to each other by 3 edges.

   Prove that if G is planar, then the list chromatic number of $3G$ is at most 7. (So $ch(3G) \leq 7$.)

9. We were able to draw G into the plane in such a way that there is only one edge crossing (a point which is not a vertex and two edge cross each other at that point). Prove that:

   a. $ch(G) \leq 6$,

   b. $ch(G) \leq 5$.

10. Show a graph which is not a line graph of a graph.

11. Show that if G is a line graph of a graph, then $ch(G) \leq 2\chi(G) - 1$.

12. Let G be a simple graph. A proper total coloring of G is a coloring of the edges and the vertices in such a way that adjacent vertices, adjacent edges and incident elements (edges and vertices) receive different colors. The total chromatic number of G, denoted by $\chi''(G)$, is the least k such that G has a total coloring with k colors. Prove that if the list coloring conjecture is true, then $\chi''(G) \leq \Delta(G) + 3$ for all G. (List coloring conjecture: If G is a line graph, then $\chi(G) = ch(G)$.)

13. Let G be a plane graph which has at least 4 vertices. 7 different colors are given: $\{1, 2, \ldots, 7\}$. We would like to color the vertices of G by these 7 colors, but somebody has already colored 4 vertices with colors 1,2,3 and 4 and these 4 vertices induce a K_4. Show that we can finish the coloring by the given 7 colors to obtain a proper coloring.

14. Let G be a graph. It does not matter how we remove four edges from G the obtained graph is always planar. Prove that $ch(G) \leq 6$.

15. Let G be a planar graph. All but one vertices of G have a color list of length 5, the exceptional vertex has a color list of size one.

    Prove that G is colorable by these lists.

Homework

1. Prove that for all positive integer n, $ch(K_{n,n^n}) = n + 1$. (K_{n,n^n} is the complete bipartite graph where the two color classes contain n and n^n vertices.)

2. Prove that if C is an even cycle, then $ch(C) = 2$.