Combinatorics and graph theory II.

4th practice 5th of October, 2022

Planar graphs, Duality

Good to know: 4-color theorem: If G is planar, then $\chi(G) \leq 4$.

The *dual* of a planar graph G, which is denoted by G^*: Each face of G correspond to a vertex of G^* and we put as many edges between two vertices of G^* as many edges separate the two corresponding faces of G. We draw these edges in such a way that each of them intersect one edge of G and different edges of G^* intersect different edges of G.

H is an *algebric dual* of G if there is a bijection between $E(G)$ and $E(H)$ which maps a cut to a cycle and a cycle to a cut.

H and G are *weakly isomorphic*: if there is a bijection between $E(G)$ and $E(H)$ which maps a cut to a cut and a cycle to a cycle.

Whitney 1: G has an algebric dual if and only if G is planar.

Whitney 2: Let G be a planar graph and let H and G be weakly isomorphic. Then:
1. H is planar
2. G^* and H^* are weakly isomorphic,
3. G and G^{**} are weakly isomorphic.

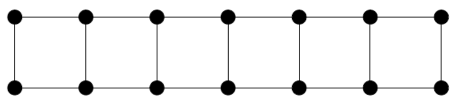
Whitney 3: Assume, that G and H are weakly isomorphic. Then H can be obtained from G by the iterative application of the following three operations:

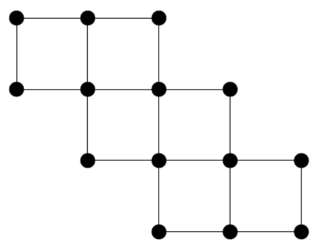
(a) If v is a cut vertex of the graph, then we cut the graph by the deletion of v and put two seperate copies of v back to the graph forming two connected components, one to each component.

(b) Two connected components are glued at a vertex.

(c) If the graph contains two vertices $\{u, v\}$ which form a cutset, then we cut the graph among $\{u, v\}$, invert one connected component and glue the components among $\{u, v\}$: Lets fix a connected component C obtained by the deletion of $\{u, v\}$, so it was not a connected component before. If $w \in V(C)$ and w was adjacent to u (v) but it was not adjacent to v (u), then now it is adjacent to v (u) but it is not adjacent to u (v).

1. Are these graphs weakly isomorphic?





2. Prove that two trees are weakly isomorphic if and only if they have the same amount of vertices.

3. Let $G(V, E)$ be a simple planar graph. Show that E can be partitioned to E_1 and E_2 such that (V, E_1) and (V, E_2) are bipartite graphs.

4. Let graph G and G^* be finite simple graphs. We know that G and G^* are duals of each other. Show that, $\min\{\delta(G), \delta(G^*)\} = 3$, where δ is the minimum degree.

5. Let G be an $n \geq 3$ vertex simple plane graph which has $3n - 6$ edges. What is the maximum degree of the dual of G?

6. Assume that G is a plane graph, each face of G is a triangle and each face of G^* is a quadrilateral. How many edges and how many vertices does G have?

7. How big can be the chromatic number of a perfect planar graph?

8. G is a connected plane graph which has 200 edges. The dual of G is a simple bipartite graph. Prove that G contains at most 100 vertices.

9. Let $F_n = K_{n,n} - nK_2$ be the bipartite graph which we can obtain from $K_{n,n}$ by deleting the edges of a perfect matching. For which n is F_n planar?

10. Prove that for all simple planar graphs, containing at least three vertices, have at least three vertices whose degree is less than 6.

11. G is a simple, connected plane graph. G has $n \geq 3$ vertices and G does not contain a cycle whose length is at most 5. Show that G^*, the dual of G, is not a simple graph.

12. Let G be a connected planar graph. Give a plane graph G' which is the dual of itself, so $G'^* \cong G'$ and G is a spanning subgraph of G'.

13. A connected graph G has 200 vertices and 300 edges. The dual of G is simple. Show that the maximum degree of G is 3.

14. Let G be a plane graph. Prove that the faces of G can be colored by 2 colors (s.t. adjacent faces have different colors) if and only if all degrees of G are even.

Homework

1. Let G be a bipartite plane graph. G' is created in the following way: Place a vertex at each face of G and put an edge between two newly added vertices if their faces are adjacent. Furthermore we put an edge between each newly added verex and each vertex of the face where the newly added vertex has been placed. Prove that $\chi(G') \leq 6$.

Give a bipartite plane multigraph G such that $\chi(G') = 5$ where G' is obtained from G by the construction which is given above.