Combinatorics and graph theory 2.

3rd practice, 28th of September, 2022.

Planar graphs

Good to know:

G is planar if it has an embedding to the plane such that two edges cross each other only at their common endpoint. Such an embedding of G is called as a plane graph.

Euler's theorem: If G is a plane graph, then $n + f = e + 1 + c$, where c is the number of components.

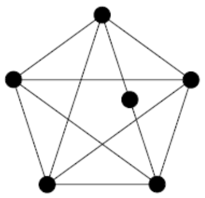
Claim, If G is a simple planar graph, then $e \leq 3n - 6$.

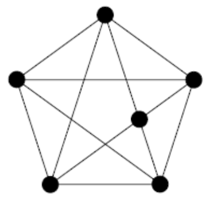
Claim, if G is planar and the shortest cycle in G has length at least 4, then $e \leq 2n - 4$.

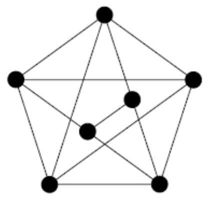
If we replace some edges of graph G by some paths then the obtained graph is called as a subdivision of G.

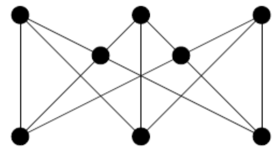
Kuratowki's theorem: G is planar iff K_5, $K_{3,3}$ and any subdivision of K_5 or $K_{3,3}$ are not subgraphs of G.

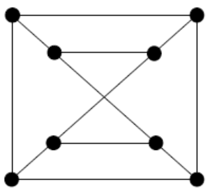
1. Decide whether the following graphs are planar or not:

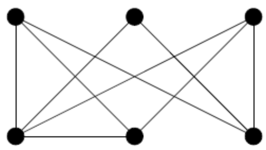
a) 

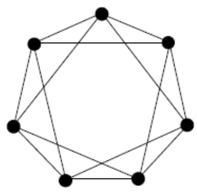
b) 

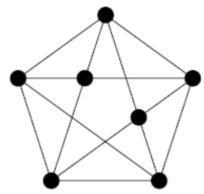
c) 

d) 

e) 

f) 

g) 

h) 

2. Is there a plane graph in which the number of vertices, edges and faces are all divisible by 3?

3. Is there a simple connected plane graph which has half as many vertices as faces?

4. In a simple, connected plane graph the degree of each vertex is 4, and the number of the edges is 16. Determine the number of faces.

5. A convex polyhedron has 20 vertices and 12 faces. Each face of the polyhedron is bounded by the same number of edges. What is this common number?

6. a) Show that in a simple planar graph the minimum degree is at most 5.
   b) In the simple planar graph G the minimum degree is 5. Show that in this case G contains at least 12 vertices of degree 5.
   c) Is the statement true with 13 instead of 12?

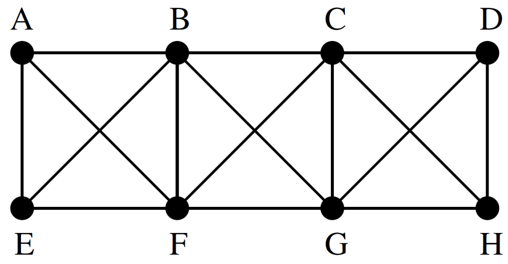
7. a) Pove that if G is a simple planar graph on at least 11 vertices then at least one of G and its complement $\overline{G}$ is not planar.
   b) Give a simple graph G on 8 vertices such that both G and $\overline{G}$ are planar.

8. a) Let G be a simple graph on at most 6 vertices. Pove that at least one of G and its complement $\overline{G}$ is planar.
   b) Give a simple graph G on 8 vertices such that neither G nor $\overline{G}$ is planar.

9. The graph G doesn't contain a subgraph which is $K_{2,3}$ or a subdivision of $K_{2,3}$. (the complete bipartite graph on 2+3 vertices). Does it follow that G is planar?

10. Let G be a simple, connected plane graph on $n \geq 3$ vertices, all of whose faces are triangles. Show that G has exactly $3n - 6$ edges. Draw such a graph.

11. At most how many edges can be added to the graph below in such a way that we get a simple planar graph? (We add edges only between already existing vertices.)



12. Suppose that $G = (V, E)$ is a simple graph whose set of edges E is the union of the disjoint sets of edges E_1, E_2 and E_3, where all the three subgraphs (V, E_1), (V, E_2) and (V, E_3) are spanning trees of G. Show that in this case G is not planar.

13. (MT+'19) Let G be a simple 4-regular bipartite graph (i.e. in which the degree of each vertex is 4). Can G be a planar graph?

14. a) Determine the maximum of $n(G) + e(G) - 2f(G)$ over all connected simple plane graphs on 100 vertices.
b) Determine the maximum of $n(G) + 2e(G) - f(G)$ over all connected simple plane graphs on 100 vertices.

15. Each face of a convex polyhedron is a (not necessarily regular) quadrangle or an octogon. Furthermore we know that exactly 3 faces meet at each vertex of the polyhedron. What is the difference of the number of quadrangular and octogonal faces?

Homework

1. In the graph G the degree of each vertex is at most 3. Furthermore we know that each cycle of G contains at most 5 edges. Show that G is a planar graph.

2. (a) Show that in a drawing of K_8 in which three edges cannot go through a common point there are at least 10 edge-crossings.
   (b) At least how many edge-crossings are there in a drawing of $K_{4,4}$ if three edges cannot go through a common point?