Combinatorics and graph theory 2.

2nd practice, 21st of September, 2022.

Posets, Comparability graphs, Perfect graphs

Good to know:
Let H be a set and $\preceq$ be a relation over its elements. Relation $\preceq$ satisfies:
reflexivity:, for all $a \in H$: $a \preceq a$;
antisymmetry, if $a \preceq b, b \preceq a \longrightarrow a = b$;
transitivity, if $a \preceq b, b \preceq c \longrightarrow a \preceq c$.

A reflexive, antisimmetric and transitive relation is called a **partial order**, and the pair $(H, \preceq)$ is called as a **partially ordered set**.

If $a \preceq b$ and $a \neq b$, then we denote this by $a \prec b$.

If $a \preceq b$ or $b \preceq a$, then we say that a and b are **comparable.** If neither $a \preceq b$ nor $b \preceq a$, then we say that a and b are **incomparable.**

From this point, Let $(H, \preceq)$ be a partially ordered set. The subset $\{a_1, a_2, \ldots, a_k\} \subset H$ is called as a chain if $a_1 \prec a_2 \prec \cdots \prec a_k$, and it is called as an **antichain**, if for any i, j where $1 \leq i < j \leq k$ a_i and a_j are incomparable.

Dilworth's theorem: Let a be the size of the maximum antichain in the partially ordered set $(H, \preceq)$. Then H can be partitioned into a chains but it cannot be partitioned into $a - 1$ chains.

Dual of Dilworth's theorem: Let l be the size of the maximum chain in the partially ordered set $(H, \preceq)$. Then H can be partitioned into l antichains but it cannot be partitioned into $l - 1$ antichains.

Denote the comparability graph which corresponds to the partially ordered set $(H, \preceq)$ by $G(H)$. The vertices of $G(H)$ are the elements of H and two vertices are adjacent if and only if the corresponding elements are comparable.

Lemma. Comparability graphs are perfect.

1. Let $a_1, a_2, \ldots, a_n$ be a number sequence. We create graph G over the vertex set $v_1, v_2, \ldots, v_n$ by the following rules: for all $n \geq i > j \geq 1$ pair v_i and v_j are adjacent if and only if $a_i > a_j$.

   Show, that for any number sequence $a_1, a_2, \ldots, a_n$, the obtained graph is perfect.

2. (Erdős-Szekeres theorem) Let $A = a_1, a_2, \ldots, a_m$ be a number sequence which does not contain the same number twice. Let $m = kl + 1$, $k, l > 0$.

   a. Prove that A contains a $k + 1$ long increasing or an $l + 1$ long decreasing subsequence.

   b. Prove that this statement is not necessarily true when $m = kl$.

3. We have a lot of boxes which correspond to the vertices of graph G. Two vertices of G are adjacent if and only if neither of the corresponding two boxes can be put in the other one. Show, that G is perfect.

4. Several discs are given on the plane. We create the following graph G: The vertices of G are correspond to the discs. Two vertices are adjacent if and only if one of the two corresponding discs contains the other one. Prove that G is perfect.

5. n points are given on the plane. Prove that, either we can choose $\lfloor \sqrt{n} \rfloor$ points from the given ones such that the angle between each line which is incident with two choosen points and the x axis is at least 30 degrees or we can choose $\lfloor \sqrt{n} \rfloor$ points from the given ones such that the angle between each line which is incident with two choosen points and the x axis is at most 30 degrees .

6. 50 different intervals of the same size are given on the line. Prove that

   (a) either there is a vertex which is contained in 8 intervals or there are 8 pairwise disjoint intervals.

   (b) either there is a vertex which is contained in 7 intervals or there are 9 pairwise disjoint intervals.

7. Let $(H, \prec)$ be a partially ordered set. An element x is *maximal* (*minimmal*), if there is no element $y \in H$ which satisfies $x \prec y$ $(y \prec x)$.

   a. Prove that the set of maximal (minimal) elements of H forms an antichain.

   b. Assume that the set of maximal and minimal elements together form an antichain. Prove that in this case all elements of H form an antichain as well.

8. Let G be an n vertex perfect graph. Prove that $\omega(\overline{G})\omega(G) \geq n$.

9. Let $G(V, E)$ be a graph such that $E = E_1 \cup E_2$, $G_1(V, E_1)$ and $G_2(V, E_2)$ are perfect graphs and $|V| = 65$. Prove that $\max(\alpha(G), \omega(G)) \geq 5$.

10. A triange ABC and n points inside of ABC are given. Prove that we can choose $\lfloor \sqrt[3]{n} \rfloor$ points from the given ones such that each line which is incident with two choosen points intersects the same two edges of ABC.

11. Let $(H, \prec)$ be a partially ordered set. Let L be a maximal chain whose maximal and minimal elements are x and y, respectively. Let $A = \{z_1, \ldots, z_a\}$ be a maximal antichain in H.

    Finally let
    $$H^+ = \{\ h \in H \mid \exists z \in A : z \preceq h\},$$
    and
    $$H^- = \{\ h \in H \mid \exists z \in A : h \preceq z\}.$$

    a. Prove that $H^+ \cap H^- = A$.

    b. Prove that $H^+ \cup H^- = H$.

    c. Prove that $x \in H^+$, $y \in H^-$.

Homework

1. Give a partially ordered set $(H, \prec)$, a maximal chain C and a maximal antichan A in $(H, \prec)$ such that C and A are disjoint.

2. $G(V, E)$ be a graph such that $E = E_1 \cup E_2$, $G_1(V, E_1)$ is perfect, $G_2(V, E_2)$ is bipartite and $|V| = 163$. Prove that $\max(\alpha(G), \omega(G)) \geq 10$.

Hint: Start with a perfect graph theorem.