Combinatorics and graph theory II.

11th practice, 7th of December, 2022.

Homogenous linear recurrences, Catalan numbers, generating functions.

Good to know.

Fibonacci numbers: $F_0 = 0$, $F_1 = 1$ and for all $n > 1$: $F_{n+1} = F_n + F_{n-1}$. Then

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Catalan numbers: $C_0 = 1$, $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. Then $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Some definitions for the Catalan numbers:

1. The number of ways how n open and n close parentheses can be placed such that it is a good parenthesization, so in each prefix the number of close parentheses does not exceed the number of open parentheses.

2. The number of total paranthesizations of a product containing $n+1$ terms.

3. Number of paths moving from $(0,0)$ to (n,n) such that at each step they go right or up and and they never cross the line $y = x$.

4. Number of different triangulations of a convex $n+2$-gon if the vertices are labelled.

5. Number of nonintersecting perfect matchings of the vertices of a convex $2n$-gon where the vertices are labeled.

6. The number of permutations of the elements $1, 2, \ldots, n$ which avoid a given 3 long pattern. Pattern of length 3: the permutaion of $1, 2, 3$. For example a $(2,1,3)$ pattern in a permutation: $\ldots a \ldots b \ldots c \ldots$, $b < a < c$.

1. Find the closed-form expression of the following sequence: $a_0 = 1, a_1 = 0 \ \ a_n = 5a_{n-1} - 6a_{n-2}$.
2. Find the closed-form expression of the following sequence: $a_0 = 3, a_1 = -3 \ \ a_n = -6a_{n-1} - 9a_{n-2}$.
3. Find the closed-form expression of the following sequence: $a_0 = 3, a_1 = 6, a_2 = 0 \ \ a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$.
4. Find the closed-form expression of $a_0 = 0, a_1 = 0 \ \ a_n = a_{n-1} + a_{n-2} + 1$, which is a non homogenous linear recurrence.
5. Let $g_0 = 1$ and $g_n = g_{n-1} + 2g_{n-2} + \ldots + (n-1)g_1 + ng_0$. Find the closed-form expression of g_n.
6. Let $g(n)$ denote the number of n long not self intersecting walks which starts from the origin and each step goes to west, north or east. Find the closed-form expression of $g(n)$.
7. How many ways can we reach the top of a ladder containing n rungs if we can move 1 or 2 rungs up each step?
8. How many ways can we cover the $2 \times n$ table by 1×2 and 2×2 dominoes?
9. Let K be a real number and let $a_n = 2Ka_{n-1} - K^2 a_{n-2}$.

   a. $a_0 = 1$, $a_1 = K$. Prove that $a_n = K^n$.

   b. $a_0 = 0$, $a_1 = K$. Prove that $a_n = nK^n$.
10. Give a linear recurrence relation for c_n if $c_n = \frac{1}{2}\left(\frac{\sqrt{17}-3}{2}\right)^n + \frac{1}{3}\left(\frac{-\sqrt{17}-3}{2}\right)^n$.
11. $2n$ children are waiting at the cashier of the cinema. They would like to buy a 10EUR ticket. n of them pay by 10EUR notes and the other n pay by a 20EUR notes. In the begining the cashier does not have any money. How many ways can the children buy their tickets if the cashier always can pay the change.

12. How many ways can we triangulate a convex n-gon if the vertices are labeled?
13. Determine the number of permutations of $1, 2, \ldots, n$ which avoid the pattern (132), so there are no three elements a_i, a_j, a_k, $i < j < k$ in the permutation such that $a_j > a_k > a_i$.
14. How many ways can 15 boys and 12 girls enter a room if it is not allowed at any step that there are more girls than boys in the room?
15. There are 25 red, 25 white and 50 green ball in an urn. How many ways can we remove all the 100 balls from the urn if at each step the number of red balls is at least as much as the number of white ones and the green balls are never the majority in the urn?
16. What is the generating function of the sequence
    (a) $1, 1, 1, \ldots,$
    (b) $1, 2, 4, 8, \ldots,$
    (c) $1, 2, 3, 4, \ldots,$
    (d) $1, 0, 1, 0, 1, \ldots$?
17. Find the closed-form expression of the following sequence: $a_1 = 0$ and $a_{n+1} = \frac{n+1}{n} a_n + n^2 - 1$ if $n \geq 1$.
18. Find the closed-form expression of the following sequence: $a_0 = 1, \;\; a_n = 8a_{n-1} + 10^{n-1}$