Combinatorics and graph theory II.

10th practice, 23rd of November, 2022.

Erdős-Ko-Rado, Sperner, LYM-inequality

Good to know

Erdős-Ko-Rado theorem. (1961) If $\mathcal{F} \subseteq 2^{[n]}$ is a k-uniform set-system ($k \leq n/2$) with the property that for any $A, B \in \mathcal{F} : A \cap B \neq \emptyset$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and this can be attained.

De Bruijn – Erdős theorem (1948) $\mathcal{F} \subset 2^{[n]}$, for all $A \in \mathcal{F}$ $|A| \geq 2$, and for arbitrary elements $1 \leq i < j \leq n$ there is exactly one $A \in \mathcal{F}$ such that $i, j \in A$. Then $|\mathcal{F}| = 1$ or $|\mathcal{F}| \geq n$.

Sperner's theorem (1928) $\mathcal{F} \subset 2^{[n]}$, for all $A, B \in \mathcal{F}$: $A \not\subset B$ and $B \not\subset A$. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. In case of equality: $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$.

LYM inequality $\mathcal{F} \subset 2^{[n]}$, for all $A, B \in \mathcal{F}$: $A \not\subset B$ and $B \not\subset A$. Let f_k denote the number of k element sets contained in $\mathcal{F}$. Then $\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1$. In case of equality: $\mathcal{F} = \binom{[n]}{k}$ for some k.

1. There is a 100 element set. We choose some of its 20 and 80 element subsets in such a way that any two choosen subsets intersect each other. At most how many such subsets can be choosen?

2. Let T be an n vertex tree.

   (a) At most how many *connected subgraphs* of T can be choosen if none of them is a subgraph of another one?

   (b) At most how many *induced subgraphs* of T can be choosen if none of them is a subgraph of the another one?

3. Show a set system $\mathcal{F} \subseteq 2^{[n]}$, such that the intersection of any two sets contained in $\mathcal{F}$ contains at least two elements and $|\mathcal{F}| = 2^{n-2}$. Is there a bigger $\mathcal{F} \subseteq 2^{[n]}$ which satisfies the first property? Useful: $\binom{2k}{k} \leq 2^{2k-1}$. (Furthermore, $\leq 2^{2k}/\sqrt{k}$ if k is big enough.)

4. Let $\mathcal{F} \subseteq 2^{[n]}$ be a set system which does not contain a chain of size $s+1$ (so there are no sets $A_1, A_2, \ldots A_{s+1}$ in $\mathcal{F}$ such that $A_1 \subset A_2 \subset \cdots \subset A_{s+1}$). Prove that $\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq s$, where f_k is the number of k element sets contained in $\mathcal{F}$..

5. Let $\mathcal{F} \subseteq 2^{[2n]}$ be a set system such that the cardinality of each set contained in $\mathcal{F}$ is even and any two set contained in $\mathcal{F}$ intersect each other. Show that if n is even, then $\mathcal{F}$ contains at most 2^{2n-2} sets.

6. Let $\mathcal{F} \subseteq 2^{[n]}$ be a set system such that the cardinality of each set contained in $\mathcal{F}$ is even but the cardinality of the intersection of any two sets contained in $\mathcal{F}$ is odd. Show that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

7. n people are living in a village. How many clubs can be established in the village such that if A_i denotes the members of club i, then $|A_i| \not\equiv 0 \pmod 3$ and for all $i \neq j$: $|A_i \cap A_j| \equiv 0 \pmod 3$.

8. Assume that any two edges of the hypergraph $\mathcal{H} = (V, \mathcal{E})$ are either disjoint or one of them contains the another. At most how many edges can $\mathcal{H}$ have if it does not have repeated edges?

9. Assume that $k < n/3$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is a k-uniform set-system which does not contain 3 pairwise disjoint sets. Show that $|\mathcal{F}| \leq 4\binom{n-1}{k-1}$.

10. Assume, that the hypergraph $\mathcal{H} = (V, \mathcal{E})$ does not contain a cycle, so there is no sequence containing pairwise different vertices and hyperedges such that $x_1, E_1, x_2, E_2, \ldots x_k, E_k, x_{k+1} = x_1$ where E_i contains vertices x_i and x_{i+1}. Show that if the $\emptyset$ is not an edge of $\mathcal{H}$ and $\mathcal{H}$ is connected (V is not a disjoint union of two nonempty sets V_1 and V_2 such that any hyperedge is a subset of V_1 or V_2), then $\sum\{|E| - 1 : E \in \mathcal{E}\} = |V| - 1$.

Homework

1. For each $k \geq 1$ show a k-uniform hypergraph which is isomorphic to its dual hypergraph.