Combinatorics and graph theory 2.

1st practice, 7th of September, 2022.

Perfect graphs

Good to know

$\omega(G)$: clique number, size of the biggest complete subgraph.

$\chi(G)$: chromatic number, the least number of colors needed to color G such that adjacent vertices receive different colors.

$\chi(G) \leq \omega(G)$.

G graph is perfect if for each G' induced subgraph of G (including G as well): $\chi(G') = \omega(G')$.

Interval graphs: Each vertex correspond to an interval on the line. Two vertex is adjacent if and only if the corresponding intervals intersect each other.

Bipartite graphs, interval graphs, line graph of bipartite graphs and the complementers of these are perfect graphs

Weak perfect graph theorem: (Lovász 72): G is perfect if and only if $\overline{G}$ is perfect.

Strong perfect graph theorem (Chudnovsky, Roberts, Seymour, Thomas, 2002): G is perfect if and only if G does not contain an odd cycle of length at least five or its complement as an induced subgraph.

1. The line graph of G, denoted by $L(G)$, is the following:

   (a) $V(L(G)) = E(G)$

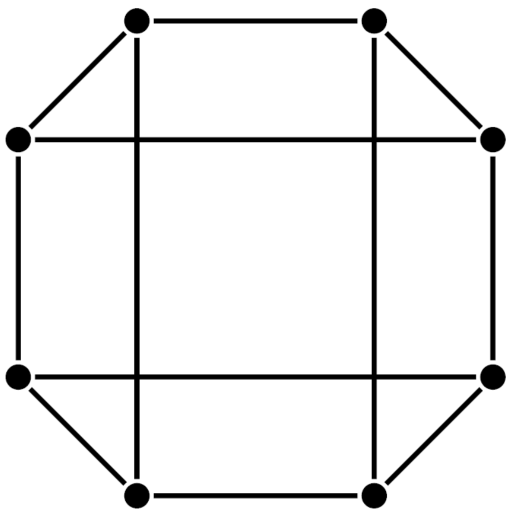
   (b) $\{e_1, e_2\} \in E(G) \iff e_1$ and e_2 share a common endpoint in G.

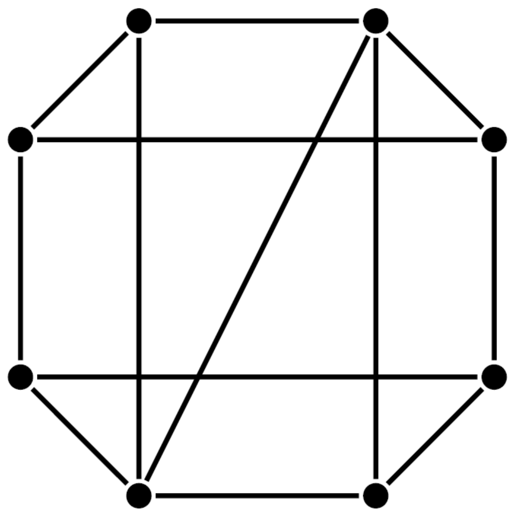
   Show that the line graph of a bipartite graph is perfect.

2. a. Show a perfect graph which is not an interval graph.

   b. Show a perfect graph which is not a complement of an interval graph.

3. Are these graphs perfect?





4. G is called as a Circular-arc graph if its vertices correspond to intervals on a circle and two vertrices are adjacent if and only if the corresponding intervals intersect each other.

   a. Show a Circular-arc graph which is not perfect.

   b. Prove that if G is a Circular-arc graph, then $\chi(G) \leq 2\omega(G)$.

5. Let G_n be a graph whose vertices are $1, 2, 3, \ldots, n$, and ij is an edge of G_n if and only if, i and j are coprimes. Determine $\chi(G_n)$ and $\omega(G_n)$. Is G_n perfect when n is big?

6. Show that the complement of bipartite graphs are perfect graphs. Do not use the two perfect graph theorems.

7. We create graph G' from a perfect graph G in the following way: We add a new vertex v and join v to all vertices of a clique of G. Show that G' is a perfect graph.

8. The vertices of graph G is the tiles of an 8x8 chessboard. Two vertices are adjacent in G if the knight can move from one to the other in one move. (The knight moves in a 3x2 L shape.) Show that G is perfect.

9. Show that a graph G is perfect if and only if each induced subgraph G' of G contains an independent set A, such that A intersects each maximum clique of G'.

10. a) Is there an interval graph whose complement is not an interval graph?

    b) Is there a graph which is a complement of an interval graph but it is not an interval graph?

11. G is a split graph if its vertex set is a union of a clique and an independent set. Show that split graphs are perfect graphs.

12. (Shift graph) Let $m > 1$. Let the vertices of the shift graph S_m be (i, j) number pairs, where $1 \le i < j \le m$. (We can imagine (i, j) as an interval.) Two vertices, i.e. (i, j) and (i', j') are adjacent if and only if either $i = j'$ or $j = i'$. In other words one of the two intervals starts where the other ends.

    a. Show that S_m does not contain a triangle.

    b. Show that $\chi(S_m) \ge \log_2 m$.

13. Show that the complement of interval graphs are perfect graphs. Do not use the two perfect graph theorems.

14. A tree T is given and $F_1, \ldots, F_n$ are subtrees of T. We give a graph G over the set $\{F_1, \ldots, F_n\}$: F_i and F_j $(i \ne j)$ are adjacent if and only if they share a common vertex. Show that G is perfect.

Homework

Remainder: $L(G)$ is the line graph of G.

1. Is $L(K_n)$ perfect?

2. We know that G is a perfect graph. X is a subset of $V(G)$ which satisfies that each element of X is adjacent to all elements of $V(G) \setminus X$. Let G' be the graph which we obtain from G by deleting all the edges running between X and $V(G) \setminus X$. Show that G' is perfect.