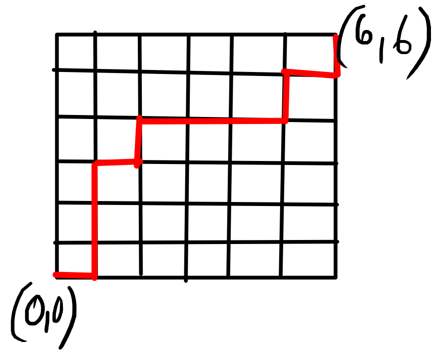
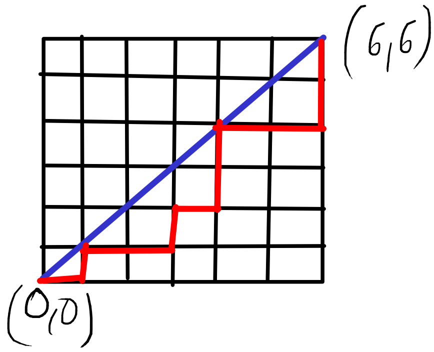
Catalan numbers

There are many different definitions of the catalan numbers. Our first definition is the following:

Consider an n by n square grid and consider the paths using the edges of the grid. First, lets count the number of paths which start at the bottom left point, so at (0,0) end at the top right point, so at (n,n) and at each step they either move up or right.
Each such path correspond to a 2n long sequence which contains n $\uparrow$ and n $\rightarrow$. Each such sequence encode a path from (0,0) to (n,n), so there is a bijection between these paths and the sequences.
The number of such sequences, and therefore the number of such paths, is $\binom{2n}{n}$.



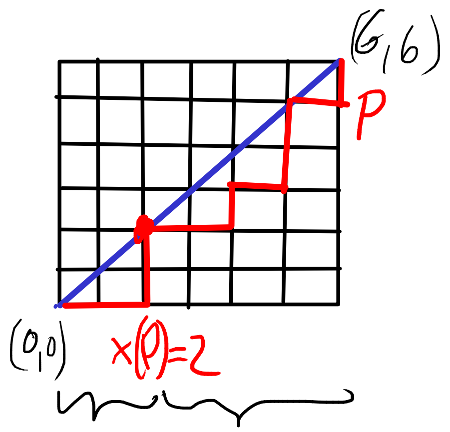
Now lets consider the paths in the n by n square grid which have these properties, so they start at (0,0), end at (n,n) and at each step they either move up or right, but furthermore they do not move above the main diagonal between (0,0) and (n,n). So they can touch the main diagonal, but cannot cross it.
The number of such paths is C_n, and it is called the nth Catalan number. $C_n = \frac{1}{n+1}\binom{2n}{n}$.



First, let's prove a recursion for the Catalan numbers.
Claim: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$

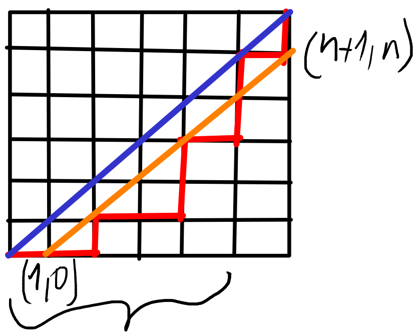
Proof: Let's say that a path P is good, if it start at (0,0), ends at (n+1,n+1), at each step it goes up or right and it never crosses the main diagonal. Let P be a good path and let x(P) denote the x coordinate of the first point (after (0,0)) where P touches the main diagonal. Clearly: 1<=x(P)<=n+1

$C_{n+1} = \#$good paths $= \sum_{i=1}^{n+1} \#$ good paths where x(P)=i.



first, second part

So how many good paths are there which touches the main diagonal at i=x(P) firstly?
Such good path have two parts. The second part goes from (i,i) to (n+1,n+1) and it never goes above the main diagonal, so the number of possible second parts is C_{n+1-i}
The first part is a little bit more complicated. It moves from (0,0) to (i,i) but it cannot touch the main diagonal except its endpoints. Therefore the number of possible first parts is not C_i. However the first part must start with a right and must end with an up, and it cannot cross the diagonal between (1,0) and (i,i-1). Therefore the number of possible first parts is C_{i-1}. So the total number of good paths which has x(P)=i is $C_{i-1}\cdot C_{n+1-i}$. Therefore:



first part

$$C_n = \sum_{i=1}^{n+1} c_{i-1}\cdot C_{n+1-i} = \sum_{i=0}^{n} c_i \cdot C_{n-i} \qquad \square$$

Let's prove the closed-form expression of the Catalan numbers by the generating function method. Note that $C_0=C_1=1$. Let C(x) be the generating function of he Catalan numbers.

$$C(x)=\sum_{i=0}^{\infty} C_i x^i \qquad \left(\text{Since } C_n<\binom{2n}{n}<4^n \text{ it is convergent when } |x|<\frac{1}{4}\right)$$

Def: generalised binomial coefficient: Let c be any real number and k be a nonnegative integer, then: $\binom{c}{k}=\frac{c(c-1)\cdot\ldots\cdot(c-k+1)}{k!}$, $\binom{c}{0}=1.$

Newton's generalisation of the Binomial theorem: If a,b and c are real numbers, then:

$$(a+b)^c=\sum_{i=0}^{\infty}\binom{c}{i}a^i b^{c-i}$$

Let's use these.

$$C(x)^2=\underbrace{C_0^2}_{C_1}+(\underbrace{C_1C_0+C_0C_1}_{C_2})x+(\underbrace{C_2C_0+C_1C_1+C_0C_2}_{C_3})x^2+\ldots$$

$$xC(x)^2=C_1x+C_2x^2+C_3x^3+\ldots=C(x)-C_0=C(x)-1$$

$$1+xC(x)^2=C(x)$$

$$xC(x)^2-C(x)+1=0 \Rightarrow C(x)_{1,2}=\frac{1\pm\sqrt{1-4x}}{2x}=\frac{2}{1\mp\sqrt{1-4x}}$$

Which one is the solution? When C(0)=C_0=1, so $C(x)=\frac{2}{1+\sqrt{1-4x}}=\frac{1-\sqrt{1-4x}}{2x}$

Note that when we divide by 0 we take the limit of the function instead.

$$xC(x)=\frac{1-\sqrt{1-4x}}{2}=\frac12-\frac12\sum_{i=0}^{\infty}\binom{1/2}{i}(-4x)^i$$ binomial series expansion

$$\binom{1/2}{i}=\frac{\frac12\left(-\frac12\right)\left(-\frac32\right)\cdot\ldots\cdot\left(-\frac{2i-3}{2}\right)}{i!}=\frac{1}{2^i}(-1)^{i-1}\frac{1\cdot3\cdot5\cdot\ldots\cdot(2i-3)}{i!}=$$

$$=(-1)^{i-1}\cdot\frac{1}{2^i}\cdot\frac{1\cdot3\cdot5\cdot\ldots\cdot(2i-3)}{i!}\cdot\underbrace{\frac{2\cdot4\cdot6\cdot\ldots\cdot(2i-2)}{2^{i-1}\cdot(i-1)!}}_{=1}=$$

$$=(-1)^{i-1}\frac{1}{i\cdot2^{2i-1}}\cdot\frac{(2i-2)!}{(i-1)!(i-1)!}=(-1)^{i-1}\cdot\frac{1}{i\,2^{2i-1}}\binom{2i-2}{i-1}=-\frac{2}{i}\left(-\frac14\right)^i\binom{2i-2}{i-1}$$

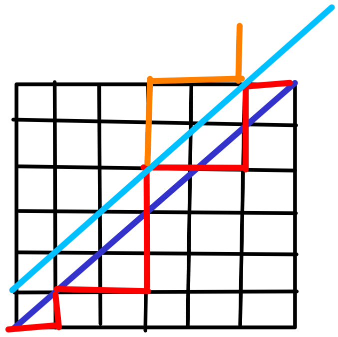
$$xC(x)=\frac12-\frac12\sum_{i=0}^{\infty}-\frac{2}{i}\left(-\frac14\right)^i\binom{2i-2}{i-1}(-4x)^i=\sum_{i=1}^{\infty}x^i\cdot\frac1i\binom{2i-2}{i-1}$$

$$C(x)=\sum_{i=1}^{\infty}x^{i-1}\frac1i\binom{2i-2}{i-1}=\sum_{i=0}^{\infty}x^i\underbrace{\frac{1}{i+1}\binom{2i}{i}}_{=C_i}$$

$\square$.

A much easier proof of the closed-form expression:

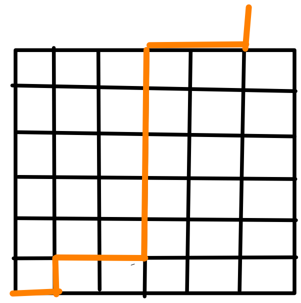
Lets say that a path is good if it never crosses the main diagonal in the nxn grid, starts from (0,0), ends at (n,n) and goes up or right at each step. Lets say that a path is bad if it crosses the main diagonal in the nxn grid, starts from (0,0), ends at (n,n) and goes up or right at each step.



bad path

Clearly $C_n = \#\text{good paths} = \binom{2n}{n} - \#\text{bad paths}$

Let's calculate the number of bad paths! Let P be a bad path. Let (i,i+1) be the first point where P goes above the main diagonal. Reflect the part of P which start at (i,i+1) to the diagonal between (0,1) and (n,n+1), denote this new path by P'. P' start at (0,0) and the endpoint of P' is (n-1,n+1) and at each step P' either goes up or right. So the number of such paths is $\binom{2n}{n-1}$
If we take any such path and use the same reflecting trick we get a bad path. So there is a bijection between these paths and bad paths. So the number of bad paths is also $\binom{2n}{n-1}$. So:



reflected path

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

beacuse: $\binom{2n}{n-1} = \frac{2n(2n-1)\cdot\ldots\cdot(n+2)}{(n-1)!} = \frac{n}{n+1}\,\frac{2n(2n-1)\cdot\ldots\cdot(n+1)}{n!} = \frac{n}{n+1}\binom{2n}{n}$

$\square$.

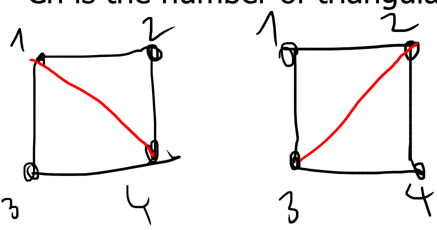
Another definitions for the catalan numbers:

1.
Cn is the number of 2n long sequences which contain n open and n close parentheses which are correctly matched. So each open paranthesis can be matched to a later close one.
Example: (), (()), ()(()()),

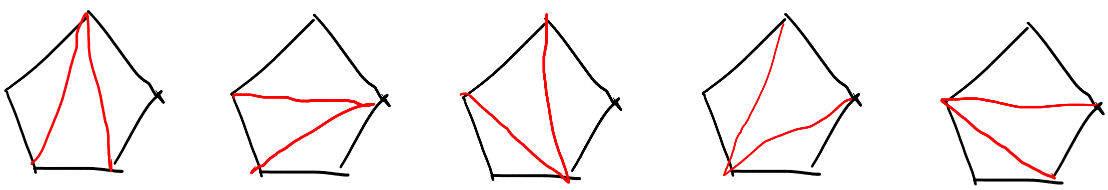
Note that n open and n close parentheses are correctly matched if and only if in any prefix there are no more close parantheses than open parentheses.

2.
Cn is the number of ways n+1 factors can be completly parenthesized (when the order of multiplications are completely detemined by parentheses and no precedence is need to be knows for evaluation).
Example: ((ab)c)d (a(bc))d (ab)(cd) a((bc)d) a(b(cd)) $C_3 = 5$

3.
Cn is the number of triangulations of an n+2 vertex convex polygon with labeled vertices.

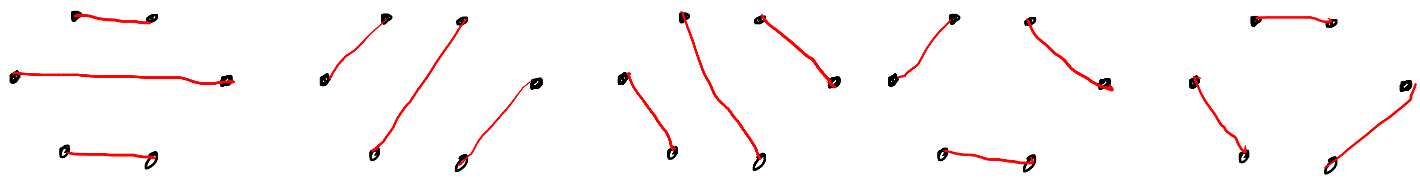


$C_2 = 2$



$C_3 = 5$

4. Cn is the number of nonintersecting perfect matchings of the vertices of a 2n convex polygon.



5. Let π be a permutation of [n].

We say that π contains a pattern 123 if there are i<j<k such that $\pi(i)<\pi(j)<\pi(k)$.

We say that π contains a pattern 132 if there are i<j<k such that $\pi(i)<\pi(k)<\pi(j)$

We say that π contains a pattern 231 if there are i<j<k such that $\pi(j)<\pi(k)<\pi(i)$

Let P be an arbitrary pattern of length three.

Cn is the number of permutations of [n] which do not contain pattern P.