Local topological toughness

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Abstract

We localize and strengthen Katona's idea of an edge-toughness to a local topological toughness. We disprove a conjecture of Katona concerning the conection between edge-toughness and factors. For the topological toughness we prove a theorem similar to Katona's $2k$-factor-conjecture, which turned out to be false for his edge-toughness. We prove, that besides this the topological toughness has nearly all known nice properties of Katona's edge-toughness and therefore is worth to be considered.

1 Preliminaries and Results

For notations not defined here we refer to [2]. Unless otherwise stated, t is an arbitrary non negative real number, k is an arbitrary integer, G is an arbitrary finite graph (loops and multiple edges allowed), U is an arbitrary subgraph of G, X and H are arbitrary disjoint subsets of $V(G)$ and Y is an arbitrary subset of $E(G-X-H)$. A cycle covering H is called *H-cycle*. The union of internally disjoint H-paths is called H-local k-factor, if all vertices of H have degree k in it. The maximum number of internally disjoint H-paths we denote by $p_G(H)$. With $G[X]$ we denote the subgraph of G induced by X, $[Y]$ denotes the graph with edge set Y whose vertex set is the set of all vertices incident with edges of Y. Instead of $G[V([Y])]$ we shortly write $G[Y]$. $E'(G)$ denotes the set of all edges in G except the loops. Let $\mathcal{C}(G)$ denote the set of components of G and $\partial_G(U)$ denote the set of vertices of U incident with edges of $G-E(U)$. For $V(U)-\partial_G(U)$ we will write shortly $in_G(U)$. According to [10] we define the permeability of a pair (X,Y) by:

$$\mathrm{perm}_G(X,Y) = |X| + \sum_{C\in\mathcal{C}([Y])} \left\lfloor \frac{|\partial_{G-X}(C)|}{2} \right\rfloor$$

In 1997 the second coauthor introduced the concept of edge-toughness (see [8]). It is strengthening the concept of toughness introduced by Chvátal in 1971. We will define these concepts.

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G is *t-tough* (in the sense of Chvátal, cf. [1]) if deleting of k vertices of G results in at most $\max\{1, \frac{k}{t}\}$ components. The *toughness of* G (denoted by $t(G)$) is the supremum over all reals t such that G is t-tough. It turns out that $t(G) = \infty$ if any two vertices of G are adjacent, and $t(G) = \min\left\{ \frac{|X|}{|\mathcal{C}(G-X)|} \;\middle|\; 2 \leq |\mathcal{C}(G-X)| \right\}$ otherwise.

G is *t-edge-tough* if for all (X, Y) with $[Y] = G[Y]$ we have:

$$|\mathcal{C}(G - X - Y - in_{G-X}([Y]))| \leq \max\{1, \frac{\mathrm{perm}_G(X, Y)}{t}\}$$

The *edge-toughness of* G (denoted by $te(G)$) is the supremum over all reals t such that G is t edge-tough.

The ideas are as follows: Every path passing a vertex of X needs another one of these vertices, every path not counted by this and passing a component C of [Y] needs two more vertices of the boundary of C in $G-X$. If G is hamiltonian, then deleting X, Y and $in_{G-X}(Y)$ therefore results in at most $\mathrm{perm}_G(X, Y)$ components incident with edges not contained in Y. This idea gives Chvatal's toughness if we restrict (X, Y) to that pairs with $Y = \emptyset$. Otherwise – as defined before – we get edge-toughness.

We are interested in a local toughness concept, since the topic of the existence of cycles through prescribed vertices of a graph seems to be of interest (cf. [6, 7, 5, 3, 10, 11]).

Local versions of Katona's edge toughness and Chvatal's toughness are naturally defined as follows:
H is called *k-edge-tough* (or *k-tough*) *in* G if for all (X, Y) with $[Y] = G[Y]$) (or (X, Y) with $Y = \emptyset$) the graph $G - X - Y$ has at most $\max\{1, \frac{\mathrm{perm}_G(X,Y)}{k}\}$ components containing a vertex of H. The local version of Chvatal's toughness occurs for instance in [5].

What would be useful properties we expect from a local version of toughness? First of all, we should be sure that in a graph G for a subset H of its vertices not being 1-tough in G the graph G contains no H-cycle. Second, if a set H is k-tough in G, then every at least two element subset of H should be k-tough, too. Third, the toughness of H in G should not depend on the length of paths in G, the inner vertices of which have degree 2 in G.

Obviously the local versions of the mentioned toughnesses fulfill the first and the second condition, but break the third. The latter is easy to see (e.g. intersect each edge of a complete graph).

The toughness concepts we have discussed by now deal with disconnecting graphs. Our idea is to complementary - it deals with connecting vertices.

Every cycle for every k element subset H of vertices has exactly k internally disjoint H-paths. This simple observation leads to the following definition: H is *topological t-tough in* G iff for all $H' \subseteq H$ with $|H'| \geq 2$ the graph G contains (at least) $t|H'|$ internally disjoint H'-paths. We chose this name because subdividing edges has obviously no effect on this value. Moreover, this definition ensures that the topological toughness fulfills all our three conditions.

If $V(G)$ itself is topological t-tough in G, we will say shorter that G is topological t-tough. The *topological toughness* of H in G is the maximal t such that H is t-tough in G, the topological toughness of G is the topological toughness of $V(G)$ in G.

We want to compare the ideas of edge toughness and topological toughness. For this we need Mader's theorem about the number of internally disjoint H-paths (cf. [12]) in G. We use it in the version of [2].

Theorem 1 (Mader, 1978) $p_G(H) = |E'(G[H])| + \min\{perm_G(X,Y) \mid \forall C \in \mathcal{C}(G - X - Y - E'(G[H])) : |V(C) \cap H| \leq 1\}$

Mader's theorem is often understood as a generalization of Menger's theorem (cf. [13]).

Theorem 2 (Menger, 1927) *Let a and b be nonadjacent vertices of G. The maximum number $p_G(\{a,b\})$ of internally disjoint ab-paths in G equals the minimum number of vertices of $G - \{a,b\}$ separating a from b in G.*

In [10] the concept of A-separators is introduced. In our notation we will replace A by H and call it small H-separator. For an independent set H a pair (X,Y) is called a *small H-separator* if $\mathrm{perm}_G(X,Y) < |H|$ and G has no H-path avoiding X and Y. Obviously, G can't have an H-cycle if it has a small H-separator. Our topological toughness by Theorem 1 generalises the idea of small H-separators: An independent set H is topological 1-tough in G if and only if G has no H-separator.

For a graph G being t-edge-tough means having a system of at least $t|H|$ H-paths for certain (but not all!) subsets H of the vertex set of G. Especially using Mader's theorem one can prove easily the following lemma which classifies the edge-toughness in a connector-language:

Lemma 3 *If for each independent set H there are at least $t|H|$ internally disjoint H-paths in G, then G is t-edge-tough. If G is t- edge-tough, then for each induced subgraph U there are at least $|\mathcal{C}(U)|t$ subgraphs of G being U-paths or cycles not disjoint to U which are disjoint out of U.*

Obviously Lemma 3 combined with the definition of topological toughness leads to

Corollary 4 *Every topological t-tough graph G is t-edge-tough and therefore t-tough in Chvátal's sense.*

All the toughnesses are constructed to detect non-hamiltonicity by a toughness value less than one (which one can prove by presenting a single separator). Let NC be the set of graphs not being 1-tough, NE be the set of graphs not being 1-edge-tough and NT be the set of graphs not being topological 1-tough. The following observation tells us, that beyond the mentioned versions of toughness, topological toughness detects non-hamiltonicity best:

Observation 1 *The following holds: $NC \subset NE \subset NT$*

Replacing edge-toughness by the topological toughness unfortunately doesn't preserve the strong (linear) connection to Chvátal's toughness:

Observation 2 *Let G be a complete graph on $2k^2+1$ vertices after deleting one of it's edges. Then the toughness of G is $k^2 - \frac{1}{2}$ and the topological toughness is $2k - \frac{3}{2}$.*

The toughness of G is $k^2 - \frac{1}{2}$ because by deleting vertices one can only separate the endvertices of the missing edge and has to delete all other vertices for this purpose. For an h-element subset H of the vertex set of G we find at least $(2k^2 + 1 - h) + \binom{h}{2} - 1$ internally disjoint H-paths and this bound is tight (if H contains the endvertices of the missing edge this becomes obvious). For the topological toughness of G we get therefore

$$\begin{aligned} t &= \min\left\{ \frac{\binom{h}{2} + (2k^2+1-h) - 1}{h} \;\middle|\; h = 2, \ldots, k^2+1 \right\} && (1) \\ &= -\frac{3}{2} + \min\left\{ \frac{h}{2} + \frac{2k^2}{h} \;\middle|\; h = 2, \ldots, k^2+1 \right\} && (2) \\ &= -\frac{3}{2} + \left[\frac{h}{2} + \frac{2k^2}{h} \right]_{h=2k} && (3) \\ &= 2k - \frac{3}{2} && (4) \end{aligned}$$

However, we prove the following:

Theorem 5 *If H is $(4t^2+2t)$-tough in G and $|H| \geq \frac{(t+\frac{3}{2})^2}{2}$, then H is topological t-tough in G.*

Here $4t^2 + 2t$ is not best possible, as it is seen in the next theorem for small values of t.

Theorem 6 *If $t \leq 1$, $|H| \geq 3$ and H is $2t$-tough in G, then H is topological t-tough in G.*

The connection between k-factors and toughness first was proved in [4]:

Theorem 7 (Enomoto, Jackson, Katerinis, Saito, 1985) *Let k be a positive integer and G be a k-tough graph such that $k|V(G)|$ is even. Then G has a k-factor.*

We know that a graph being 1-tough may have a hamiltonian cycle – which is a 2-factor (more precisely, we know that a graph not being 1-tough cannot be hamiltonian). Therefore the idea of toughness creates the conjecture that every k-?-tough graph has a $2k$-factor. This was conjectured by the second coauthor for the edge-toughness (see [9]) and proved for $k = 1$.

Unfortunately, this is not true in general.

Let C_q^p denote the p^{th} power of a cycle on q vertices. (In the p^{th} power of a cycle on vertices $v_1, \ldots, v_q$ the vertices v_i and v_j are connected iff $|i-j| \leq p$ or $|i-j| \geq q-p$.) Then take m disjoint copies of $C_{k^2+k-1}^{k-1}$ and denote these by $H_1, \ldots, H_m$. Moreover, take a complete graph K_x, where x is the largest integer satisfying $x < m\frac{k^2+k-1}{k}$ and connect each vertex of K_x to each vertex of each H_i. The resulting graph is denoted by $G_{k,m}$. It is worth of mentioning that the smallest such construction for $k = 2$ is obtained from a K_7 by deleting the edges of a cycle of length 5.

Theorem 8 *Then $G_{k,m}$ is k-edge-tough but has no $2k$-factor for all $m \geq 1$ and $k \geq 2$ integers.*

Even the localisation of Theorem 7 is not true.

Observation 3 *Let G be a copy of K_{24} after deleting an edge. Let H be a set of 6 vertices of G not inducing K_6. Then H is 11-tough in G but G has no X-local 11-factor because G has no 33 internally disjoint H-paths.*

However, the situation changes if we consider the topological toughness:

Theorem 9 *Every topological k-tough graph has a $2k$-factor.*

This is a consequence of our main result:

Theorem 10 *A set H^* of vertices of a graph G is topological k-tough in G if and only if for every $H \subseteq H^*$ with $|H| \geq 2$ the graph G has an H-local $2k$-factor.*

This theorem is a little surprising because it says that it is sufficient to have enough H-paths for each $H \subseteq H^*$, $|H| \geq 2$, to be able to arrange them in a regular way for each such H.

2 Proofs

We only need to prove Theorems 5, 6, 8 and 10.

Proof of Theorem 5. The proof is indirect. Suppose H is not topological t-tough in G. Then there is a set $H' \subseteq H$ with $|H'| \geq 2$ such the maximum number of internally disjoint H'-paths in G is smaller than $t|H'|$. By Mader's Theorem (Theorem 1) there is a total separator (X', Y') of H' in $G - E(G[H'])$ satisfying $\mathrm{perm}_G(X', Y') + |E'(H')| < t|H'|$. Let α be the independence number of $G[H']$ and set $x = \frac{|H'|}{\alpha}$. Let $\overline{G}$ be the simple graph on H' in which two vertices are adjacent exactly if they are not adjacent in G. Clearly $\overline{G}$ is $K_{\alpha+1}$-free. By Turán's Theorem (cf. [2]) it has at most as many edges as a complete r-partite graph with nearly equal partition classes, such that the sizes of classes differ by at most one. Thus $G[H']$ has at least $\alpha\frac{x(x-1)}{2}$ edges. Therefore we have $\alpha\frac{x(x-1)}{2} \leq |H'|t = \alpha xt$. This leads to $x \leq 2t + 1$.

Consider for the first case $\alpha \geq 2$. Let H'' be an independent subset of H' in G with $|H''| = \alpha$. Then in (X', Y'), we can replace each component C of $\mathcal{C}(Y')$ by all but one vertices of $\partial_{G-X}(C)$, and replace the edges induced by H' by the vertices of $H' \setminus H''$. This operation leads to a total separator $(X'', \emptyset)$ of H'' with $|X''| \leq 2\left(\mathrm{perm}_G(X', Y') + |E(G[H'])|\right)$. For the Chvatal-toughness t_c of G we get:

$$t_c \leq \frac{|X''|}{|H''|} = \frac{|X''|}{\alpha} = \frac{x|X''|}{|H'|} \leq \frac{2(\mathrm{perm}_G(X', Y') + |E(G[H'])|)x}{|H'|} < 2tx \leq 4t^2 + 2t$$

Consider now the other case $\alpha = 1$. If every vertex of $H \setminus H'$ is contained in X', we get $t|H'| > |H| - |H'| + \binom{|H'|}{2}$. This leads to:

$$|H| < \frac{1}{2}(2t + 3 - |H'|)|H'| \leq \frac{(t + \frac{3}{2})^2}{2}.$$

Therefore, such H is not considered in the theorem we have to prove. Hence we may and shall assume that H contains a vertex v not contained in $H' \cup X'$. Furthermore, we may assume that for every component C of $[Y']$ we have $\left\lfloor \frac{|\partial_{G-X'}(C)|}{2} \right\rfloor \geq 1$, that is, every component has a nonzero contribution to the permeability of (X', Y') in G. Let X^* contain all vertices of X' and for each component C of $[Y']$ all but one element of $\partial_{G-X'}(C)$, chosen such that X^* do not contain v. Clearly, $G - X^* - E(G[H'])$ has no component containing two vertices of H', but it may have a component containing v and a vertex w of H', but no other vertex of H'. If this is the case, set $X'' = X^* \cup \{w\}$, otherwise set $X'' = X^*$. In both cases let H'' consist of v and one element of $H' \setminus X''$. Then X'' separates H''. We get $|X''| \leq 2\mathrm{perm}_G(X', Y') + 1 <$

$2(t|H'| - |E'(G[H'])|) + 1 \le 2(t|H'| - \binom{|H'|}{2}) + 1 = (2t + 1 - |H'|)|H'| + 1 < (t + \frac{1}{2})^2 + 1$. Finally, the toughness of H' in G is at most $\frac{|X''|}{2} < 4t^2 + 2t$. □

Proof of Theorem 6. It suffices to prove the following for all H: If $|H| \ge 3$ and there is a set H' with $|H'| \ge 2$ such that G has no $t|H'|$ internally disjoint H'-paths, then H has an independent subset H'' such that there is a set $X'' \subseteq V(G - H'')$ being a total separator of H'' with $2t|H''| > X''$.

If $G[H']$ is connected, then it has at least $|H'| - 1$ edges. Its edges are internally disjoint H'-paths. Therefore it has at most $|H'| - 1$ edges. Thus it is a tree and $t > \frac{1}{2}$ holds. Furthermore, $G - E(G[H'])$ has no H'-path.

If, furthermore, $|H'| = 2$, then there is a vertex $h \in H \setminus H'$. In this case let $H' = \{h_1, h_2\}$. Clearly, either $G - h_1$ has no $\{h, h_2\}$-path or $G - h_2$ has no $\{h, h_1\}$-path. Suppose w.l.o.g. the latter is the case. Then with $H'' = \{h, h_1\}$ and $X'' = \{h_2\}$ we are done.

If, otherwise, $|H'| > 2$, then $G[H']$ has a cutvertex x. Let H'' consist of two vertices of different components of $G[H'] - x$. We are done with $X'' = \{x\}$.

So we may suppose that $G[H']$ contains at least 2 components. Choose for H'' one vertex of each of these components.

By Theorem 1 there is a set $X' \subseteq V(G - H')$ and a set $Y' \subseteq V(G - H' - X')$ such that $\mathrm{perm}_G(X', Y') + |E'(G[H'])| < t|H'|$ and $G - X' - Y' - E'(G[H'])$ has no H'-paths. Therefore, by the construction of H'', the graph $G - X' - Y'$ has no H''-paths. Furthermore, we get $\mathrm{perm}_G(X', Y') < t|H'| - |E'(G[H'])| < t|H''|$. Let X'' consist of all vertices of X' and all but one vertices of $\partial_{G-X'}(C)$ for all $C \in \mathcal{C}([Y])$. Then $|X''| \le \mathrm{perm}_G(X', Y') < t|H''|$ but $G - X''$ has no H''-path. □

Proof of Theorem 8. First we prove that $G_{k,m}$ has no $2k$-factor. Since each H_i is $2k-2$ regular, and they are disjoint from each other, each vertex of each H_i in a $2k$-factor must send at least 2 edges to K_x, so there must be at least $2m(k^2 + k - 1)$ edges ending in K_x. However, x is given such that this is not possible, because in a $2k$-factor each vertex of K_x is incident to at most $2k$ of these edges.

Now we prove that $G_{k,m}$ is k-edge-tough. By Lemma 3 it is enough to prove that for any independent vertex set $H \subset V(G_{k,m})$ there are at least $k|H|$ internally disjoint H-paths. It is clear that H cannot contain any vertex of K_x. Let $h_i = |H \cap V(B_i)|$. Since all B_i are copies of $C^{k-1}_{k^2+k-1}$, there are at most k independent vertices in it, so $h_i \le k$. On can easily see that if $i \le j - k$, then there are $k - 1$ disjoint paths from v_i to v_j using only vertices $v_{i'}$ with $i < i' < j$. This implies that in each B_i there are at least $(k - 1)h_i$ internally disjoint H-paths. Thus there we have $d := x + \sum_{i=1}^{m}(k - 1)h_i$ internally disjoint H-paths in $G_{k,m}$. Since $x > m\frac{k^2+k-1}{k} - 1$ and $\sum_{i=1}^{m} h_i \le km$ straightforward calculation gives that $d \ge k\sum_{i=1}^{m} h_i$ holds if $m \ge 2$, which proves our claim in this case. If $m = 1$, then $x = k$ and similar argument shows that our claim holds, so the proof is complete. □

Proof of Theorem 10. If all the local $2k$-factors exist, the assertion is trivial. Hence we only have to prove the other direction. Therefore H^* is assumed to be topological k-tough in G. Consequently, every subset of H^* is topological k-tough in G and thus it suffices to prove that there is an H^*-local $2k$-factor.

Let G' be the graph obtained from G by subdividing each edge once or twice depending on whether one or two of its ends are members of H^*. After this, replace each member v of H^* (together with the incident edges) by a star with $2k$ edges and connect each of its leafs to each neighbor (after the subdividing step) of v. The center of the star which was the replacement for v is called v', all these centers form the set H', the graph induced by all vertices having distance at most 2 from v' we call G'_v. For convenience let us call the edges incident with a $v' \in H'$ *yellow*, the edges incident to yellow edges are called *green*, and those edges which are incident to green ones but coloured to not yellow yet will be called *blue*.

If in G' we contract all graphs $G_{v'}$ to the corresponding vertex v, we get back the graph G. By this contraction an H'-local $2k$-factor of G' becomes an H^*-local $2k$-factor of G. Furthermore, it is easy to construct an H'-local $2k$-factor of G' from an H^*-local $2k$-factor of G. Therefore, G has an H^*-local $2k$-factor if and only if G' has an H'-local $2k$-factor.

The proof is indirect. Suppose G has no H^*-local $2k$-factor. We have to prove that there is a subset H of H^* such that G has no $k|H|$ internally disjoint H-paths. We may therefore assume, that G' has no H'-local $2k$-factor. Since every system of $k|H'|$ internally disjoint H'-paths of G' forms an H'-local $2k$-factor, we assume that G' has no such system.

We now make a choice for a set X' and a set Y': We choose X' and Y' such that $X' \subseteq V(G'-H')$, $Y' \subseteq E(G'-H'-X')$, (X',Y') is a total H'-separator of G', and $\mathrm{perm}_{G'}(X',Y') < k|H'|$. Such choice is possible by Theorem 1, since H' is indepenent in G'.

Now step by step we specify our selection:
Among all remaining possibilities for X' and Y' we choose X' and Y' such that the number of vertices in the neighborhood of H' is minimal.

If in the neighborhood of a vertex v' of H' there would be a vertex x' of X' and a vertex y' out of X' one could replace in (X',Y') the vertex x' by all edges from x' to the neighborhood of y' in $[Y']$. Replacing a vertex by edges here means taking the vertex out of X' and adding the edges to Y'. The permeability of the so changed pair would not increase: The border of one component of Y' increases by at most one while the number of vertices in X' is decreasing. Therefore the assumptions about x and y contradict our choice of (X',Y').

The following property follows immediately:

Observation 4 *If the neighborhood N of one vertex of H' in G' contains an element of X', then X' contains N.*

Among all remaining possibilities for X' and Y' we choose X' and Y' such that the second neighborhood of H' contains as few vertices of X' as possible.

Suppose there is an H'-path containing a vertex $x' \in X'$ in the second neighborhood of some $v' \in H'$ while avoiding all other elements of X' and Y' which does not contain the only blue edge incident with x'. This means that the path, which starts in v' with a yellow edge, continues with a few green edges to arrive to x', then continues with a few more green edges, and exiting the second neighborhood at x'' with a blue edge. It is clear that this part of the path can be replaced by a path from v' to x'' containing only one yellow and on green edge such that it avoids. Thus such path cannot exist.

Therefore, one could replace a vertex of the second neighborhood of H' by the only blue edge incident with it without breaking the former conditions. Consequently, we get

Observation 5 *X' is disjoint to the second neighborhood.*

Among all remaining possibilities we choose Y' such that the number of vertices of H' having a neighborhood not disjoint to $[Y']$ and not contained in the same component of $[Y']$ is minimal.

If a neighbor w of an element v' of H' is not incident with an edge in Y', then an H'-path can use w and any of the green edges incident to it. Therefore by removing all edges incident with the neighborhood of v' from Y' does not effect the number of disjoint H'-paths. This leads to

Observation 6 *For each $v \in H'$ the neighborhood of v is either contained in one component of $[Y']$ or disjoint to $[Y']$.*

To complete our choice we choose Y' among all remaining possibilities such that $|Y'|$ is maximal. This leeds to:

Observation 7 *If Y' contains a green edge incident to a vertex x in the second neighborhood of $v' \in H'$, then all green edges incident with x belong to Y'. Therefore each H'-path avoiding X' has a blue edge (which corresponds to an edge of G).*

Let x be the number of vertices of H' having a neighbor in X' and y be the number of vertices of H' having a neighbor incident with an edge in Y'. For the permeability of (X', Y'), by observations 4 and 6, we obtain the following.

Observation 8 *$k|H'| > perm_{G'}(X', Y') \geq 2kx + ky$ holds which implies $|H'| - x - y \geq |H'| - 2x - y > 0$.*

Suppose first, that $|H'| - x - y > 1$. Let H contain all vertices v of H^* for which the neighborhood of the corresponding vertex v' in H' is neither contained in X' nor contains a vertex incident with an element of Y'.

Let X contain all vertices v of H^* for which the neighborhood of the corresponding vertex v' in G' is contained in X' and all other vertices of G having corresponding vertices in X'. Let Y contain all edges of $G - H - X$ having corresponding edges in Y'. Then using Observation 7 we get that (X, Y) is a total H-separator of $G - E(G[H])$. Each edge of $G[H]$ has a corresponding edge in Y' being the only edge of one component of $[Y']$ the boundary of which in $G' - X'$ consists exactly of its 2 endpoints. Therefore we get $\mathrm{perm}_G(X, Y) + |E[H]| \leq \mathrm{perm}_{G'}(X', Y') - (2k-1)x - ky < k|H'| - kx - ky = |H|$. Applying Theorem 1 the proof is completed in this case.

In the remainder we may and shall assume that $|H'| - x - y = 1$. Obviously, Observation 8 then gives us $x = 0$. Let v' be the vertex of H' whose neighborhood has no vertex incident with an edge in Y' and w' be an arbitrary other vertex of H'. Call the vertices in the neighborhood of $H' \setminus \{v'\}$ *black* and all others *white*.

Observation 9 *All black vertices are contained in $[Y']$ and the number of black vertices in each boundary of a component of $[Y']$ is (by Observation 7) a multiple of $2k$.*

Let P' be a path from v' to w' in G'. Clearly, P' has vertices in $\partial_{G'}([Y']))$. If we follow P' starting from v', the first vertex of $[Y']$ on P' is a white one (remember that $G' - Y'$ has no H'-path). Furthermore, if we left a component of $[Y']$ via a black vertex, we enter the next component of $[Y']$ via a white one. This leads to the following:

Observation 10 *Either P' has two white vertices in the boundary of the same component of $[Y']$ or each component of $[Y']$ which shares a black vertex with P' has a white boundary vertex in P'.*

Let C' be the component of $[Y']$ containing the boundary of w', S' be a set of internally disjoint paths in G' connecting v' and w', and z' be the number of paths of S' containing a white vertex of the boundary of C'. By Observation 10 there are at least $|S'| - z' \geq 0$ pairs of white vertices – each of them contained in the boundary of a component of $[Y'] - C'$. Therefore, by Observation 9 we get $\mathrm{perm}_{G'}(X', Y') \geq yk + (|S'| - z') + \left\lfloor \frac{z'}{2} \right\rfloor \geq yk + \left\lfloor \frac{|S'|}{2} \right\rfloor$.

Since $k|H'| > \mathrm{perm}_{G'}(X',Y')$ and $|H'| - y = 1$ we get $k > \mathrm{perm}_{G'}(X',Y') - ky \geq \left\lfloor \frac{|S'|}{2} \right\rfloor$. Therefore, $z' < 2k$. In other words, in G' there are no $2k$ internally disjoint paths connecting v' and w'.

Consequently, in G there are no $2k$ internally disjoint paths connecting v and w. By Theorem 2 G has a set X of at most $2k-1$ vertices separating v from w. Let $Y = \emptyset$ and $H = \{v, w\}$. Then $H \subseteq H^*$ and (X,Y) is a total H-separator of G with $p_G(X,Y) < 2k = k|H|$, which is a contradiction that completes the proof. □

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