Elementary graphs with respect to $(1, f)$-odd factors

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Abstract

This note concerns the $(1, f)$-odd subgraph problem, i.e. we are given an undirected graph G and an odd value function $f : V(G) \to \mathbb{N}$, and our goal is to find a spanning subgraph F of G with $\deg_F \leq f$ minimizing the number of even degree vertices. First we prove a Gallai–Edmonds type structure theorem and some other known results on the $(1, f)$-odd subgraph problem, using an easy reduction to the matching problem. Then we use this reduction to investigate barriers and elementary graphs with respect to $(1, f)$-odd factors, i.e. graphs where the union of $(1, f)$-odd factors form a connected spanning subgraph.

1 Introduction

In this paper we deal with a special case of the *degree prescribed subgraph problem*, introduced by Lovász [10]. This is as follows. Let G be an undirected graph and let $\emptyset \neq H_v \subseteq \mathbb{N}$ be a degree prescription for each $v \in V(G)$. For a spanning subgraph F of G define $\delta_H^F(v) = \min\{|\deg_F(v) - i| : i \in H_v\}$, and let $\delta_H^F = \sum\{\delta_H^F(v) : v \in V(G)\}$. The minimum δ_H^F among the spanning subgraphs F is denoted by $\delta_H(G)$. A spanning subgraph F is called ***H*-optimal** if $\delta_H^F = \delta_H(G)$, and it is an ***H*-factor** if $\delta_H^F = 0$, i.e. if $\deg_F(v) \in H_v$ for all $v \in V(G)$. The **degree prescribed subgraph problem** is to determine the value of $\delta_H(G)$.

An integer h is called a **gap** of $H \subseteq \mathbb{N}$ if $h \notin H$ but H contains an element less than h and an element greater than h. Lovász [12] gave a structural description on the degree prescribed subgraph problem in case H_v has no two consecutive gaps for all $v \in V(G)$. He showed that the problem is NP-complete without this restriction. The first polynomial algorithm was given by Cornuéjols [2]. It is implicit in Cornuéjols [2] that this algorithm implies a Gallai–Edmonds type structure theorem for the degree prescribed subgraph problem (first stated in [14]), which is similar to – but in some respects much more compact than – that of Lovász'.

The case when an odd value function $f : V(G) \to \mathbb{N}$ is given and $H_v = \{1, 3, 5, \ldots, f(v)\}$ for all $v \in V(G)$, is called the **$(1, f)$-odd subgraph problem**. We denote $\delta_H(G) = \delta_f(G)$. This problem is much simpler than the general case due to the fact that only parity requirements are posed. The $(1, f)$-odd subgraph problem was first investigated by Amahashi [1] who gave a Tutte type characterization of graphs having a $(1, 2k+1)$-odd factor. A Tutte type theorem for general odd value functions f was proved by Cui and Kano [3], and then a Berge type minimax formula on $\delta_H(G)$ by Kano and Katona [7]. A Gallai–Edmonds type theorem on the $(1, f)$-odd subgraph problem was given in [8] and [14].

In this note we show a new approach to the $(1, f)$-odd subgraph problem. Actually, it is worth allowing f to have also even values and defining H_v equal to $\{1, 3, \ldots, f(v)\}$ or $\{0, 2, \ldots, f(v)\}$, according to the parity of $f(v)$. We call this the **f-parity subgraph problem**. We show an easy reduction of the f-parity subgraph problem to the matching problem

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(the existence of such a reduction was already indicated in Lovász [12]), and we show that this reduction easily yields the above mentioned Gallai–Edmonds and Berge type theorems on the f-parity subgraph problem. Then we investigate barriers w.r.t. the f-parity subgraph problem. As another application, we explore the graphs for which the edges belonging to some f-parity factor form a connected spanning subgraph. We call such a graph an **f-elementary graph**. We generalize some results on matching elementary graphs (proved by Lovász [11]) to f-elementary graphs. An attempt putting the f-parity subgraph problem into the general context of graph packing problems can be found in [15].

The f-parity subgraph problem can be reduced to the $(1, f)$-odd subgraph problem by the following construction: for every vertex $v \in V(G)$ with $f(v)$ even, connect a new vertex w_v to v in G, define $f(w_v) = 1$ and increase $f(v)$ by 1. Now $\delta_f(G)$ remains the same.

To avoid minor technical difficulties we assume that $f > 0$. Almost all results of the paper would hold without this restriction, too. Note that if G is a nontrivial f-elementary graph then $f > 0$ always holds.

The constant function $f \equiv 1$ is simply denoted by $\mathbf{1}$. For $X \subseteq V(G)$ let $\Gamma(X) = \{y \in V(G) - X : \exists x \in X, xy \in E(G)\}$, let $f(X) = \sum\{f(x) : x \in X\}$ and let χ_X denote the function with $\chi_X(x) = 1$ if $x \in X$ and $\chi_X(x) = 0$ otherwise. $c(G)$ denotes the number of connected components of the graph G. $|\cdot|$ denotes the cardinality of a set, and $\mathbb{N}$ is the set of nonnegative integers. The graphs are undirected throughout.

2 Reduction to matchings

In this section we show a reduction of the f-parity subgraph problem to matchings, which will then be used to prove the Gallai–Edmonds type structure theorem on the f-parity subgraph problem. The auxiliary graph we use is defined below.

Definition 2.1. For the graph G and the function $f : V(G) \to \mathbb{N} \setminus \{0\}$ define G^f to be the following undirected graph. Replace every vertex $v \in V(G)$ by a new complete graph on $f(v)$ vertices, denoted by K_v, and for each pair of vertices $u, v \in V(G)$ adjacent in G, add all possible $f(u)f(v)$ edges between K_u and K_v. Let $V_v = V(K_v)$.

Observe that $G^{\mathbf{1}} = G$ and that $|V(G^f)| = f(V(G))$. $f > 0$ implies that $V_v \neq \emptyset$ for $v \in V(G)$. There is a strong connection between the maximum matchings of G^f and the f-parity optimal subgraphs of G. Note that the size of a maximum matching of G is just $|V(G)| - \delta_{\mathbf{1}}(G)$.

Lemma 2.2. *For every f-parity optimal subgraph F of G there exists a matching M of G^f such that $|V(M)| = f(V(G)) - \delta_f^F$. Moreover, if $\deg_F(w) \in \{\dots, f(w) - 3, f(w) - 1\}$ for a vertex $w \in V(G)$ then M can be chosen to miss a prescribed vertex $x \in V_w$.*

On the other hand, for every maximum matching M of G^f there exists a spanning subgraph F of G such that $\delta_f^F = f(V(G)) - |V(M)|$. Moreover, if M misses a vertex in V_w for some $w \in V(G)$ then F can be chosen such that $\deg_F(w) \in \{\dots, f(w) - 3, f(w) - 1\}$.

Hence $\delta_f(G) = \delta_{\mathbf{1}}(G^f)$.

Proof. Let F be an f-parity optimal subgraph of G. If $\deg_F(v) > f(v)$ for some $v \in V(G)$ then clearly $\delta_f^{F'} \leq \delta_f^F$ holds for the graph F' obtained from F by deleting an edge e incident to v. As F is f-parity optimal, e is not adjacent to w, so $\deg_{F'}(w) = \deg_F(w)$. Hence we assume that $\deg_F \leq f$, which implies that $\delta^F(v)$ is 1 or 0 for all v. Now it is easy to construct from F a matching of G^f missing exactly δ_f^F vertices, one in each V_v for the vertices v with $\deg_F(v) \not\equiv f(v) \bmod 2$. If w is such a vertex then M can be chosen to miss $x \in V_w$.

For the second part, let M be a maximum matching of G^f. If M contains two edges between K_u and K_v for some $u, v \in V(G)$, then replace them by two edges, one inside K_u and the other one inside K_v. Thus we may assume that M contains at most one edge between K_u and K_v for all distinct $u, v \in V(G)$. By contracting each K_u to one vertex u we get a spanning subgraph F of G with $\delta_f^F = f(V(G)) - |V(M)|$. Moreover, $\deg_F(w) \in \{\dots, f(w) - 3, f(w) - 1\}$ in case M misses a vertex in K_w. □

We define critical graphs w.r.t. the f-parity subgraph problem as in the matching case. If $f = \mathbf{1}$ the graphs defined below are called **factor-critical**.

Definition 2.3. Given a graph G and a function $f : V(G) \to \mathbb{N}$. G is called **f-critical** if for every $w \in V(G)$ there exists an f-parity optimal subgraph F of G such that $\deg_F(w) \in \{\dots, f(w)-3, f(w)-1\}$ and $\deg_F(v) \in \{\dots, f(v)-2, f(v)\}$ for all $v \neq w$.

By Lemma 2.2 G is f-critical if and only if G^f is factor-critical. The Gallai–Edmonds structure theorem for the f-parity subgraph problem follows from the classical Gallai–Edmonds theorem easily. We cite this latter result below.

Theorem 2.4. (Gallai, Edmonds)[4, 5, 6] *Let D consist of those vertices of the graph G which are missed by some maximum matching of G, let $A = \Gamma(D)$ and $C = V(G) - (D \cup A)$. Then*

1. *every component of $G[D]$ is factor-critical,*
2. *$|\{K : K$ is a component of $G[D]$ adjacent to $A'\}| \geq |A'| + 1$ for all $\emptyset \neq A' \subseteq A$,*
3. *$\delta_{\mathbf{1}}(G) = c(G[D]) - |A|$,*
4. *$G[C]$ has a perfect matching.*

A direct generalization of the above result is the version for the f-parity subgraph problem.

Theorem 2.5. [8, 14] *Let G be a graph and $f : V(G) \to \mathbb{N} \setminus \{0\}$ be a function. Let $D_f \subseteq V(G)$ consist of those vertices v for which there exists an f-parity optimal subgraph F of G with $\deg_F(v) \in \{\dots, f(v)-3, f(v)-1\}$. Let $A_f = \Gamma(D_f)$ and $C_f = V(G) - (D_f \cup A_f)$. Then*

1. *every component of $G[D_f]$ is f-critical,*
2. *$|\{K : K$ is a component of $G[D_f]$ adjacent to $A'\}| \geq f(A') + 1$ for all $\emptyset \neq A' \subseteq A_f$,*
3. *$\delta_f(G) = c(G[D_f]) - f(A_f)$,*
4. *$G[C_f]$ has an f-parity factor.*

Proof. Take the classical Gallai–Edmonds decomposition $V(G^f) = D \cup A \cup C$ of G^f. By symmetry, if V_v meets D then $V_v \subseteq D$. These vertices $v \in V(G)$ form D_f by Lemma 2.2. The other results follow from the construction and from Lemma 2.2. □

This proof implies:

Lemma 2.6. *For $X = D, A, C$ it holds that $X_f(G) = \{v \in V(G) : V_v \subseteq X(G^f)\}$, provided $f > 0$.*

From Theorem 2.5 the Berge type minimax formula on the f-parity subgraph problem follows in a few lines.

Definition 2.7. A connected component K of G is **f-odd** (**f-even**) if $f(V(K))$ is odd (even). Let f-*odd*(G) denote the number of f-odd components of G. For $Y \subseteq V(G)$ let $\mathrm{def}_f(Y) = f$-*odd*$(G - Y) - f(Y)$.

Theorem 2.8. [7] *If G is a graph and $f : V(G) \to \mathbb{N} \setminus \{0\}$ is a function then $\delta_f(G) = \max\{\mathrm{def}_f(Y) : Y \subseteq V(G)\}$.*

Proof. By virtue of Theorem 2.5, one only has to observe that if a graph K is f-critical then $f(V(K))$ is odd, and that if $f(V(K))$ is odd then K has no f-parity factor. □

We point out that up to this point $f = 0$ was excluded only for sake of convenience. Theorems 2.5 and 2.8 still hold in the general case. (If $f(v) = 0$ then join a pendant vertex u to v and define $f(u) = f(v) = 1$. Then construct G^f.) So we can define the canonical decomposition $D_f(G)$, $A_f(G)$, $C_f(G)$ for all f. However, Lemma 2.6 would fail.

Now we show how to use this approach to analyze barriers.

Definition 2.9. $Y \subseteq V(G)$ is called an *f-barrier* if $\operatorname{def}_f(Y) = \delta_f(G)$.

As f-critical graphs are f-odd, the canonical Gallai–Edmonds set A_f is an f-barrier. A **1**-barrier is just an ordinary barrier in matching theory. One can observe that if $Y \subseteq V(G^f)$ and $V_v \cap Y,\ V_v \setminus Y \neq \emptyset$ then $V_v \cap Y$ is adjacent to only one component of $G^f - Y$. Moreover, if Y is a barrier in G^f then each $X \subseteq Y$ is adjacent to at least $|X|$ odd components of $G^f - Y$ since otherwise $\operatorname{def}_{\mathbf{1}}(Y - X) > \operatorname{def}_{\mathbf{1}}(Y)$, which is impossible. Hence if Y is a barrier in G^f then $|Y \cap V_v| \in \{0, 1, f(v)\}$ for all $v \in V(G)$. It also follows that if $|Y \cap V_v| = 1$ and $V_v \setminus Y \neq \emptyset$ then $Y \setminus V_v$ is a barrier of G^f. Thus if Y is a barrier of G^f then $Y' = \{v \in V(G) : V_v \subseteq Y\}$ is an f-barrier of G. On the other hand, if Y' is an f-barrier of G then $\bigcup\{V_v : v \in Y'\}$ is clearly a barrier of G^f. Also the canonical Gallai–Edmonds barrier $A(G^f)$ of G^f has this form.

Definition 2.10. An f-barrier Y of G is called **strong** if the f-odd components of $G - Y$ are f-critical.

Also A_f is a strong f-barrier. Since a graph K is f-critical if and only if K^f is factor-critical, we have

Observation 2.11. $Y \subseteq V(G)$ is a strong f-barrier in G if and only if $\bigcup\{V_v : v \in Y\}$ is a strong **1**-barrier in G^f.

Király proved that the intersection of strong **1**-barriers is also a strong **1**-barrier [9]. This result holds for the f-parity subgraph problem as well.

Theorem 2.12. *The intersection of strong f-barriers is a strong f-barrier.*

Proof. Let Y_1, Y_2 be strong f-barriers of G. Then $Y_i' = \bigcup\{V_v : v \in Y_i\}$ are strong **1**-barriers of G^f, hence their intersection, which is just $\bigcup\{V_v : v \in Y_1 \cap Y_2\}$, is also a strong **1**-barrier by [9]. Thus $Y_1 \cap Y_2$ is a strong f-barrier of G. □

By Tutte's theorem, maximal matching barriers are strong. This remains true for f-barriers, too. Indeed, let Y be a maximal f-barrier of G and K be an f-odd component of $G - Y$. K has no f-parity factor so $C_f(K) \neq V(K)$ in its canonical Gallai–Edmonds decomposition. Hence either $D_f(K) = V(K)$ or $A_f(K) \neq \emptyset$. In the first case K is f-critical by Theorem 2.5, *1.*, and in the second case $Y \cup A_f(K)$ would be a larger f-barrier then Y, which is impossible. Thus all f-odd components of $G - Y$ are also f-critical, implying that Y is strong.

In the matching case it holds that the canonical Gallai–Edmonds barrier A is the intersection of all maximal barriers. This fails for the general case: take a triangle together with a pendant vertex of degree 1, and define $f \equiv \deg$. Here $A_f = \emptyset$ and there exists exactly one nonempty barrier.

However, the fact that in the matching case the canonical Gallai–Edmonds barrier A is the intersection of all strong barriers remains true by Observation 2.11 and the fact that A_f itself is strong.

3 f-elementary graphs

In this section we generalize some results on elementary graphs (presented in Lovász [11]) to the f-parity case.

Definition 3.1. Let G be a graph and $f : V(G) \to \mathbb{N}$. An edge $e \in E(G)$ is said to be **allowed** (or **f-allowed** if confusion may arise) if G has an f-parity factor containing e. Otherwise e is **forbidden**. G is said to be **f-elementary** if the allowed edges induce a connected spanning subgraph of G. G is **weakly f-elementary** if G_2 is f-elementary, where G_2 is the graph we get by replacing every edge $e \in E(G)$ by two parallel edges.

1-elementary graphs are simply called elementary. f-elementary graphs are weakly f-elementary, but not vice versa: $G = K_2$ with $f \equiv 2$ is weakly f-elementary but not f-elementary.

These classes coincide if $f = \mathbf{1}$. Note that the assumption $f > 0$ excludes only the singleton with $f = 0$ from the class of (weakly) f-elementary graph. Lemma 3.2 justifies why we introduced the weak version of f-elementary graphs.

Lemma 3.2. *G^f is elementary if and only if G is weakly f-elementary.*

Proof. Let M be a perfect matching of G^f. If M contains at least three edges between K_u and K_v for some $u, v \in V(G)$ then replace two of them by another two edges, one inside K_u and the other one inside K_v. So the number of edges of M between K_u and K_v decreased by 2. Repeted application of this process leads to a graph where the number of edge between any K_u and K_v is at most 2. This construction shows that if G^f is elementary then G is weakly f-elementary.

On the other hand, if G is weakly f-elementary then G^f is clearly elementary. □

The $f = \mathbf{1}$ special cases of the following two theorems can be found e.g. in Lovász and Plummer [13] (Theorems 5.1.3 and 5.1.6). Using our reduction these special cases together with Lemmas 2.6 and 3.2 imply both Theorem 3.3 and 3.4.

Theorem 3.3. *G is weakly f-elementary if and only if $\delta_f(G) = 0$ and $C_{f-\chi_w}(G) = \emptyset$ for all $w \in V(G)$.*

Proof. G is weakly f-elementary if and only if G^f is elementary by Lemma 3.2, and G^f is elementary if and only if $\delta_{\mathbf{1}}(G^f) = 0$ and $C(G^f - x) = \emptyset$ for all $x \in V(G^f)$ ([13], Theorem 5.1.3). Since $\delta_f(G) = \delta_{\mathbf{1}}(G^f)$, it is enough to prove that

$$\text{if } \delta_f(G) = 0,\ w \in V(G) \text{ and } x \in V_w \text{ then } C(G^f - x) = \emptyset \Longleftrightarrow C_{f-\chi_w}(G) = \emptyset. \tag{1}$$

As $G^f - x \simeq G^{f-\chi_w}$, if $f(w) \geq 2$ then (1) follows from Lemma 2.6. So assume that $f(w) = 1$. As $G^f - x \simeq (G-w)^{f-\chi_w}$, Lemma 2.6 implies that $C(G^f - x) = \emptyset \Longleftrightarrow C_{f-\chi_w}(G-w) = \emptyset$. $\delta_f(G) = 0$ and $f(w) = 1$, so it is easy to see that the $f - \chi_w$-parity optimal subgraphs of G are the f-parity factors of G and the $f - \chi_w$-parity optimal subgraphs of $G - w$ enlarged by w as an isolated vertex. Thus $D_{f-\chi_w}(G) = D_{f-\chi_w}(G-w)$ and hence $A_{f-\chi_w}(G) \setminus \{w\} = A_{f-\chi_w}(G-w)$. Now if $w \in X := A_{f-\chi_w}(G)$ then (1) clearly holds, while if $w \in C_{f-\chi_w}(G)$ then $\mathrm{def}_f^G(X) = \mathrm{def}_f^{G-w}(X) + 1 > 0$, which is impossible. □

Theorem 3.4. *G is weakly f-elementary if and only if f-odd$(G-Y) \leq f(Y)$ for all $Y \subseteq V(G)$, and if equality holds for some $Y \neq \emptyset$ then $G - Y$ has no f-even components.*

Proof. Call $Y \subseteq V(G)$ **f-bad** if either f-*odd*$(G-Y) > f(Y)$ or equality holds here and $G - Y$ has an f-even component. G is weakly f-elementary if and only if G^f is elementary (Lemma 3.2) if and only if G^f has no **1**-bad set ([13], Theorem 5.1.6). So we only have to prove that G has an f-bad set Y if and only if G^f has a **1**-bad set Y'. If $Y \subseteq V(G)$ is f-bad then $Y' = \bigcup\{V_v : v \in Y\}$ is **1**-bad in G^f. On the other hand, let $Y' \subseteq V(G^f)$ be **1**-bad in G^f. If $V_v \cap Y'$, $V_v \setminus Y' \neq \emptyset$ for some $v \in V(G)$ then let $x \in V_v \cap Y'$. Now x is adjacent to only one component of $G^f - Y'$ hence $Y' - x$ is also **1**-bad. So we can assume that Y' is a union of some V_v. Now $Y = \{v \in V(G) : V_v \subseteq Y'\}$ is f-bad in G. □

In the case of matchings the existence of a certain canonical partition of the vertex set was revealed by Lovász [11] (Lovász, Plummer [13], Theorem 5.2.2). We cite this result.

Definition 3.5. $X \subseteq V(G)$ is called **nearly f-extreme** if $\delta_{f-\chi_X}(G) = \delta_f(G) + |X|$. Besides, X is **f-extreme** if $\delta_f(G - X) = \delta_f(G) + f(X)$.

It is clear that $\delta_{f-\chi_X}(G) \leq \delta_f(G) + |X|$ and $\delta_f(G-X) \leq \delta_f(G) + f(X)$ for every $X \subseteq V(G)$. Nearly **1**-extreme and **1**-extreme sets coincide.

Theorem 3.6. (Lovász)[11] *If G is elementary then the maximal barriers of G form a partition $\mathcal{S}$ of $V(G)$. Moreover, it holds that*

1. *for $u, v \in V(G)$, the graph $G - u - v$ has a perfect matching if and only if u and v are contained in different classes of $\mathcal{S}$, (hence if $uv \in E(G)$ then uv is $\mathbf{1}$-allowed in G),*

2. *$S \in \mathcal{S}$ for some $S \subseteq V(G)$ if and only if $G - S$ has $|S|$ components, each factor-critical,*

3. *$X \subseteq V(G)$ is $\mathbf{1}$-extreme if and only if $X \subseteq S$ for some $S \in \mathcal{S}$.*

Lemma 3.2 implies the analogue of this result.

Theorem 3.7. *If G is weakly f-elementary then its maximal f-barriers form a subpartition $\mathcal{S}'$ of $V(G)$. Call the classes of $\mathcal{S}'$ **proper** and add all elements $v \in V(G)$ not in a class of $\mathcal{S}'$ as a **singleton** class yielding the partition $\mathcal{S}$ of $V(G)$. Now it holds that*

1. *for $u, v \in V(G)$, the graph G has an $f - \chi_{\{u,v\}}$-parity factor if and only if u and v are contained in different classes of $\mathcal{S}$ (hence if $uv \in E(G)$ then uv is f-allowed in G_2),*

2. *$S \in \mathcal{S}'$ for some $S \subseteq V(G)$ if and only if $G - S$ has $f(S)$ components, each f-critical,*

3. *$X \subseteq V(G)$ is nearly f-extreme (f-extreme, resp.) if and only if $X \subseteq S$ for some $S \in \mathcal{S}$ ($S \in \mathcal{S}'$, resp.).*

Proof. As we already observed, for every barrier Y of G^f it holds that $|Y \cap V_v| \in \{0, 1, f(v)\}$ for all $v \in V(G)$. G^f is elementary, hence its maximal barriers form a partition of $V(G^f)$ by Theorem 3.6. Thus, by symmetry, a maximal barrier of G^f is either the union of some V_v, or a singleton. If Y' is an f-barrier of G then $\bigcup\{V_v : v \in Y'\}$ is a barrier of G^f. On the other hand, if Y is a maximal barrier of G^f of the form $\bigcup V_v$ then $Y' = \{v \in V(G) : V_v \subseteq Y\}$ is clearly a maximal f-barrier of G. So these barriers Y' form the proper classes of $\mathcal{S}$, and for a singleton class $\{v\} \in \mathcal{S} - \mathcal{S}'$ it holds that each vertex $x \in V_v$ is a maximal barrier of G^f. Now the statement follows from Theorem 3.6, using $\delta_f(G) = \delta_{\mathbf{1}}(G^f)$ for *1.* and *3.*, and using the fact that a graph K is f-critical if and only if K^f is factor-critical for *2.* □

Remark 3.8. It follows from Theorem 3.7, *3.*, that $\mathcal{S}$ could be introduced as the partition $\{X \subseteq V(G) : X$ is a maximal nearly f-extreme set of $G\}$. Besides, if $X \subseteq V(G)$, $|X| \geq 2$ is maximal nearly f-extreme, then X is an f-barrier of G.

Corollary 3.9. *If G is f-elementary then $e \in E(G)$ is f-allowed if and only if e joins two classes of $\mathcal{S}$.*

Proof. Suppose that e joins u to v and let $g = f - \chi_{\{u,v\}}$. By Theorem 3.7, *1.*, we only have to prove that G has a g-parity factor if and only if e is f-allowed. Assume that G has a g-parity factor but e is not f-allowed. (The other direction is trivial.) If $G - e$ had a g-parity factor F then $F + e$ would be an f-parity factor of G, which is impossible. Thus by Theorem 2.8 there exists a set $Y \subseteq V(G)$ such that g-*odd*$(G - e - Y) > g(Y)$. G has a g-parity factor so by parity reasons g-*odd*$(G - e - Y) = g(Y) + 2$, and e runs between two g-odd components K_1 and K_2 of $G - e - Y$. But then clearly no edge entering $V(K_1) \cup V(K_2)$ is f-allowed in G. G is f-elementary thus $V(K_1) \cup V(K_2) = V(G)$, but then e is an f-forbidden cut edge. □

What happens if we increase $f(v)$ by 2? Let $f' = f + 2\chi_v$. First, G is still weakly f'-elementary. Note that all barriers of G^f disjoint from V_v remain a barrier also in $G^{f'}$. If v is a singleton in $\mathcal{S}$ w.r.t. f, then it is also a singleton w.r.t. f'. If v belongs to a proper class $S \in \mathcal{S}$ then S will not be an f-barrier of G any more, hence S is split to smaller, singleton and proper, classes of the new canonical partition.

Our last subject is generalizing bicritical graphs.

Definition 3.10. Let G be a graph and $f : V(G) \to \mathbb{N} \setminus \{0\}$ be a function. G is said to be ***f*-bicritical** if G has an $f - \chi_{\{u,v\}}$-parity factor for all pairs $u, v \in V(G)$.

Theorem 3.11. *If G is weakly f-elementary then the following statements are equivalent.*

1. *G is f-bicritical.*

2. All classes of $\mathcal{S}$ are singletons.

3. If $Y \subseteq V(G)$ and $|Y| \geq 2$ then f-odd$(G-Y) \leq f(Y) - 2$.

Proof. *1* $\Rightarrow$ *2* : Each edge in G_2 is allowed thus Theorem 3.7, *1.*, implies the equivalence.

2 $\Rightarrow$ *3* : Assume the contrary. By parity reasons, we have a set $Y \subseteq V(G)$ with $|Y| \geq 2$ such that f-*odd*$(G-Y) = f(Y)$. So Y is an f-barrier, which is contained in a set $S \in \mathcal{S}$ with $|S| \geq 2$.

3 $\Rightarrow$ *1* : Suppose G has no $g = f - \chi_{\{u,v\}}$-parity factor for some $u, v \in V(G)$. Thus there exists a set $Y \subseteq V(G)$ such that g-*odd*$(G-Y) > g(Y)$. Recall that G has an f-parity factor. If u or v belongs to a g-odd component K of $G-Y$ then Y is an f-barrier of G and K is an f-even component of $G-Y$, contradicting to Theorem 3.4. Hence both u and v belong to Y, thus $|Y| \geq 2$ and f-*odd*$(G-Y) = f(Y)$, a contradiction. □

Lovász [11] and Lovász, Plummer [13] developed a decomposition procedure for elementary graphs, showing that they build up from bipartite elementary graphs and from bicritical graphs. We mention that this procedure is possible to extend to weakly f-elementary graphs. Going one step further, the bipartite elementary graphs have a bipartite ear decomposition starting from an edge. Also this ear decomposition can be adapted to bipartite f-elementary graphs, hence further refining the decomposition procedure of weakly f-elementary graphs. (An f-elementary graph G is *bipartite f-elementary* if G is bipartite with color classes U and V and $f|_U = \mathbf{1}$.) We do not go into details.

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