Matroids – the engineers' revenge

András Recski

Abstract Matroids are applied in electric engineering for over 30 years. These applications motivated the investigation of some new, pure matroidal questions. Such results are surveyed for readers with mathematical (rather than engineering) background. **Keywords:** matroids, balanced matroids, jump systems, half-plane property, Rayleigh-monotonicity, electrical networks

1 Introduction

Electrical engineering was perhaps the first area for application of graph theory. The pioneering results of Kirchhoff [11], 1847 and Maxwell [14], 1892 have been applied for a lot of problems of electric network theory in the first decades of the 20th century. The first "revenge" of engineers came around 1930 – for a proper understanding of the duality in electric network theory, the concept of the dual of planar graphs had to be clarified first, hence Whitney has introduced 2-isomorphism and abstract dual [26, 27, 28], and these concepts were perhaps the most important motivations for the discovery of matroids. Later matroid theory found several engineering applications, mostly between 1970–80.

We are currently witnessing the second "revenge" of the engineers: Since certain physical properties of electric networks are strongly related to the mathematical properties of the underlying graphs, and some of these latter properties can be generalized in a more or less straightforward way to matroids, network theory motivated some new, pure matroidal research in the last three decades.

In the present survey such results are collected for readers interested in matroid theory. They are arranged according to the physical concepts which inspired them,

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but although the subtitles suggest these physical concepts, the results are purely mathematical; no background in physics or engineering is required.

2 Preliminaries

Throughout the paper u stands for voltage difference and i stands for current. Suppose that an electric network consists of resistors and ideal voltage and current sources only. An ideal voltage source is described by the "equations" $\{u$ is given, i is arbitrary$\}$, an ideal current source is similarly described by $\{i$ is given, u is arbitrary$\}$, while a resistor is given by $u = Ri$ or $i = Yu$ where R and Y are called the resistance and the conductance of the resistor, respectively. The way how these elements are interconnected is described by the network graph where the edges correspond to the network elements. We may suppose that this graph is connected.

Recall Kirchhoff's laws that the signed sum of the voltages along any circuit of the network graph is zero and the signed sum of the currents along any cut set of the network graph is zero. The basic task of electric network analysis is that by applying Kirchhoff's laws every unknown quantity (voltage and current) should be expressed as the function of the the R or Y values and of the input (voltages of the voltage sources and currents of the current sources). This is posssible only if the subset of edges, corresponding to the voltage sources, is circuit free and the subset of edges, corresponding to the current sources, is cut set free. Moreover, these necessary conditions together are sufficent for the unique solvability of the network if all the resistors are passive elements, that is, if $R, Y > 0$ hold for every resistor.

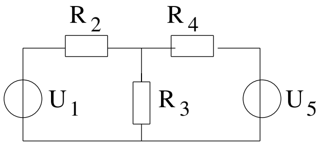
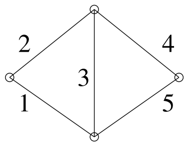


Fig. 1

Fig. 2



As an example consider the network in Figure 1. Its network graph is given in Figure 2 and, for example, the current of the resistor R_3 can be expressed as

$$i_3 = \frac{R_4 u_1 + R_2 u_5}{R_2 R_3 + R_2 R_4 + R_3 R_4} = \frac{Y_2 Y_3 u_1 + Y_3 Y_4 u_5}{Y_4 + Y_3 + Y_2}.$$

The other unknown quantities would be expressed by other expressions, however, all of them are fractions with the same denominator $W_R(G) = R_2 R_3 + R_2 R_4 + R_3 R_4$ or $W_Y(G) = Y_4 + Y_3 + Y_2$ (recall that we solve a system of linear equations and, by Cramer's rule, every unknown quantity can be expressed as a ratio of two determinants where the one in the denominator does not depend on the selection of the actual unknown quantity).

Since the short circuit ($u = 0$) and the open circuit ($i = 0$) are special cases of voltage and current sources, respectively, we shall always suppose that the edges corresponding to the voltage sources are contracted and those corresponding to the current sources are deleted, hence the network consists of resistors only. In what follows, this "new" network graph will be denoted by $G = (V,E)$. In case of Figure 1, G consists of three parallel edges.

A fundamental observation of Kirchhoff [11] and Maxwell [14] was that the common denominator $W_R(G)$ (respectively $W_Y(G)$) of every unknown quantity is simply $W_R(G) = \sum_T \prod_{i \notin T} R_i$ or $W_Y(G) = \sum_T \prod_{j \in T} Y_j$, where the summations are performed over the set of spanning trees T of G. These results are often called "topological formulae" in electric network theory, to emphasize that only the structure (the "topology") of the graph plays a role in these expressions.

A strongly related result of similar character is as follows:

Theorem 1. [14] *Suppose that resistors with conductances $Y_1, Y_2, \dots$ are interconnected along a connected graph G. Let $a,b \in V(G)$ be two specific points of G. Add a fictitious edge $e = \{a,b\}$ to G and denote the resulting graph by G' (if the two points are already adjacent in G then e is parallel to an old edge). The effective conductance measured between a and b in the network will be*

$$Y_{a,b} = \frac{W_Y(G' \backslash e)}{W_Y(G'/e)},$$

where, as before, $W_Y(G) = \sum_T \prod_{j \in T} Y_j$ and $\backslash$ and $/$ denote deletion and contraction of an edge, respectively.

Obviously, in the above theorem $G' \backslash e = G$ and we follow the convention that empty sums are 0 and empty products are 1. For example, the effective conductance of two parallel or two series resistors respectively (Figure 3) are

$$\frac{Y_1 + Y_2}{1} \qquad\qquad \frac{Y_1 Y_2}{Y_1 + Y_2} = \frac{1}{\frac{1}{Y_1} + \frac{1}{Y_2}}$$

(observe that the denominator in the first case is one since there is one "spanning tree" (namely the empty set) in a graph consisting of loops only).

Fig. 3

3 Energy

A trivial consequence of Theorem 1 is that if $a,b \in V(G), a \neq b$, in a connected graph G and the edges of the graph represent resistors with $Y_i > 0$ for every i then the effective conductance between a and b is also positive, hence, in particular, $W_Y(G) \neq 0$. The physical background (positive resistors are passive elements, thus their interconnection cannot produce energy) remains valid even if we use other network elements as well, like (positive) capacitors or inductors. Hence $W_Y(G) \neq 0$

remains true even if the Y_i's are complex numbers, provided that their real parts $\Re(Y_k)$ are positive for every k.

Let now $\mathscr{M}$ be an arbitrary matroid on the underlying set E and suppose that complex numbers Y_i are assigned to every element $e_i \in E$. We may define $W_Y(\mathscr{M}) = \sum_B \prod_{e_i \in B} Y_i$ where the summation is performed over the set of the bases B of the matroid. The matroid is said to have the *half-plane property* if $W_Y(\mathscr{M}) \neq 0$ holds provided that $\Re(Y_i) > 0$ for every $e_i \in E$.

Theorem 2. [7] *Every graphic matroid has the half-plane property.*

Although this is clear by the above physical considerations, the following purely mathematical *proof,* reproduced from [23], will be of importance.

Let $\mathscr{M}$ be the circuit matroid of a graph G. We may suppose that G is connected (otherwise its connected components can separately be studied), let n be the number of its vertices. Let A denote the reduced signed incidence matrix of G (obtained by deleting an arbitrary row). Since A has $n-1$ rows, using the theorem of Binet and Cauchy,

$$\det(AYA^*) = \sum_{S \subseteq E, |S|=n-1} \det A[S] \det A^*[S] \prod_{e_i \in S} Y_i,$$

where Y is a diagonal matrix containing the Y_i quantities and A^* stands for the adjoint (conjugate transpose) of the matrix A (which, in our case, is simply its transpose).

Let a subset S of the columns of A have cardinality $n-1$. Then $\det A[S] = 0$ if the S-edges contain a circuit and $\det A[S] = \pm 1$ if they form a tree. Hence

$$\det(AYA^*) = \sum_T \prod_{e_i \in T} Y_i = W_Y(\mathscr{M}).$$

Therefore all what we have to prove is that AYA^* is nonsingular if $\Re(Y_i) > 0$ for every i.

Let v be an arbitrary nonzero vector from the $(n-1)$-dimensional complex space. We shall prove that $AYA^*v \neq 0$. Recall that $A^*v \neq 0$ (since the rows of A were independent), hence at least one of its coordinates, say i_0, is nonzero. Finally

$$v^*AYA^*v = \sum_i Y_i |(A^*v)_i|^2,$$

and the real part of this sum is positive (since $\Re(Y_i) > 0$ for every i, their coefficients are non-negative and at least one of them (namely the i_0th) is positive). □

This result can be generalized in the following way:

Theorem 3. [7] *Every regular matroid has the half-plane property.*

Proof: Recall [22] that a matroid is regular (that is, representable over every field) if and only if it can be represented over the field $\mathbb{Q}$ of the rationals by a totally unimodular matrix A. A matrix is totally unimodular if all of its square submatrices have determinant 0 or ± 1. Since we may additionally suppose that the rows of the above matrix A are linearly independent, the above proof can be applied again. □

A slightly more careful analysis of the proof shows that actually we need even less. Matrix A was totally unimodular (that is, every square submatrix of it has determinant 0 or ± 1) but here we only need that if the determinant of a square submatrix is nonzero, its absolute value should be one. This property can be fulfilled by matrices over the field $\mathbb{C}$ of the complex numbers as well. In particular, a matrix is a $\sqrt[6]{1}$-matrix if all of its square submatrices are either zero or one of the sixth roots of unity. A matroid is called a *sixth-root-of unity matroid* if it can be represented by a $\sqrt[6]{1}$-matrix over $\mathbb{C}$.

Theorem 4. [7] *Every sixth-root-of-unity matroid has the half-plane property.* □

The importance of this class of matroids has probably not been fully recognized yet. A matroid is regular if and only if it can be represented over GF(2) and GF(3), see [22]; similarly, a matroid is sixth-root-of-unity if and only if it can be represented over GF(3) and GF(4), see [29].

These relations are summarized in Figure 4. Here $\boxed{2}$, $\boxed{3}$ and $\boxed{4}$ denote the sets of matroids representable over GF(2), GF(3) and GF(4), respectively, while $\boxed{A}$ is the set of regular matroids, $\boxed{A} \cup \boxed{B}$ is the set of the sixth-root-of-unity matroids and $\boxed{\text{HPP}}$ is the set of the matroids with the half-plane property.

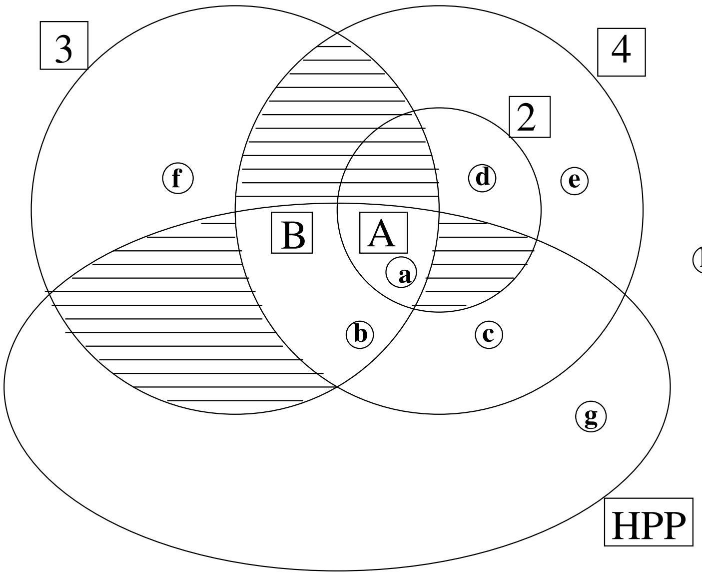


Fig. 4

Theorem 5. *The shaded regions in Figure 4 are empty subsets. The other eight subsets are non-empty, the lower case letters* $\mathbf{a}, \mathbf{b}, \ldots \mathbf{h}$ *in the other subsets are examples as follows:* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and* $\mathbf{g}$ *are the uniform matroids* $\mathscr{U}_{3,2}$, $\mathscr{U}_{4,2}$, $\mathscr{U}_{5,2}$ *and* $\mathscr{U}_{6,2}$, *respectively.* $\mathbf{d}$ *and* $\mathbf{f}$ *are the Fano matroid* $\mathscr{F}_7$ *and the anti-Fano matroid* $\mathscr{F}_7^-$, *respectively. Finally,* $\mathbf{e}$ *and* $\mathbf{h}$ *are the direct sums* $\mathscr{F}_7 \oplus \mathscr{U}_{4,2}$ *and* $\mathscr{F}_7^- \oplus \mathscr{U}_{5,2}$, *respectively.*

Proof: Fact 1. The first statement, that the shaded regions are empty, trivially follows from the results [7] that a binary matroid has the half-plane property if and only if it is regular and a ternary matroid has the half-plane property if and only if it is a sixth-root-of-unity matroid.

Fact 2. Every uniform matroid has the half-plane property and the Fano matroid does not have it [7].

Fact 3. The uniform matroid $\mathscr{U}_{k,2}$ is known to be representable over fields with cardinality at least $k-1$ only (folklore, see [13], for example).

Fact 4. The anti-Fano matroid is representable over fields of characteristic different from two and the Fano matroid over fields of characteristic two.

Fact 5. The class of matroids with the half-plane property is closed under, among others, minors and direct sum.

The positions of **a** and **b** are obvious: $\mathscr{U}_{3,2}$ is regular (in fact, graphic) and $\mathscr{U}_{4,2}$ is representable over every field except GF(2). The positions of **c**, **d** and **g** follow from Facts 2 and 3.

The anti-Fano matroid is ternary by Fact 4 and cannot have the half-plane property by Fact 1, leading to the position of **f**. Finally, the positions of **e** and **h** follow from the previous results and from Fact 5. □

4 Switches

Suppose that a network contains positive resistors and a switch. The effective conductance measured between two points can only increase if we turn the switch from OFF to ON position (or, more generally, if we increase the conductance of one resistor). This observation (sometimes called *Rayleigh-monotonicity*) may lead to a trivial algebraic relation, like $Y_1+Y_2 > Y_2$ in case of the first network of Figure 5, or to a less obvious one like

$$\frac{(Y_1+Y_2)(Y_3+Y_4)}{Y_1+Y_2+Y_3+Y_4} > \frac{Y_1Y_3}{Y_1+Y_3} + \frac{Y_2Y_4}{Y_2+Y_4}$$

in case of the second network of the same figure.

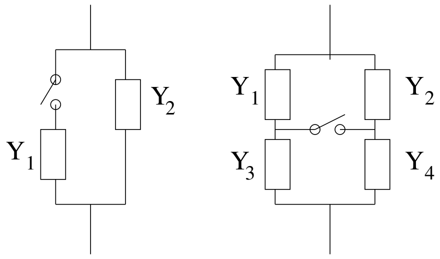


Fig. 5

Let again $\mathscr{M}$ be an arbitrary matroid on the underlying set E and suppose that positive real numbers Y_i are assigned to every element $e_i \in E$. As before, define $W_Y(\mathscr{M}) = \sum_B \prod_{e_i \in B} Y_i$ where the summation is performed over the set of the bases B

of the matroid. Applying Theorem 1 the effective conductance between two points can be expressed as the ratio of two such sums. Since turning the switch to ON position can be expressed as the contraction of an element, one can define *Rayleigh-matroids* as those where Rayleigh-monotonicity holds.

Graphic matroids are obviously Rayleigh (see a pure combinatorial proof in [6], using the matrix tree theorem [3, 4]), so are the matroids having the half-plane property (hence, in particular, all the sixth-root-of-unity matroids), see [8].

However, not all the binary matroids are Rayleigh, in fact, a binary matroid is Rayleigh if and only if it does not contain the matroid $\mathscr{S}_8$ as a minor [8, 9], where $\mathscr{S}_8$ is given by the matrix

$$\begin{pmatrix} 1&1&1&1&1&1&1&0 \\ 0&1&0&0&0&1&1&1 \\ 0&0&1&0&1&0&1&1 \\ 0&0&0&1&1&1&0&1 \end{pmatrix},$$

see also [21].

This is one of the smallest counterexamples in several senses: every matroid with at most seven elements is Rayleigh, see [8] and every matroid of rank at most three is Rayleigh, see [24].

It might be instructive to summarize the relations as

$$\textit{graphic} \subseteq \textit{regular} \subseteq \sqrt[6]{1} \subseteq \textit{HPP} \subseteq \textit{Rayleigh}.$$

All these classes are closed by taking duals and minors and all inclusions are proper:

Examples:
(1) The dual of the circuit matroid of a nonplanar graph is regular but nongraphic.
(2) The uniform matroid $\mathscr{U}_{4,2}$ is sixth-root-of-unity matroid [29] but nonregular. (So is the matroid $\mathscr{X}$ of the next section as well.)
(3) The uniform matroid $\mathscr{U}_{5,2}$ has the half-plane property [7] but is not a sixth-root-of-unity matroid (see Theorem 5).
(4) The Fano matroid does not have the half-plane property [7] but it is Rayleigh [8].

While not related directly to the electrical engineering applications, the concept of *balanced matroids* [9] should also be mentioned as a proper superset of the set of Rayleigh-matroids

$$\textit{graphic} \subseteq \textit{regular} \subseteq \sqrt[6]{1} \subseteq \textit{HPP} \subseteq \textit{Rayleigh} \subseteq \textit{balanced}.$$

Recent (mostly algebraic) research [2, 5] for properly understanding the relation of the last three classes revealed further connections with the theory of jump systems [1, 12, 20].

5 Control and Feedback

Networks consisting of resistors and ideal voltage and current sources can be described by graphs (see Section 2). Consider now a more complex linear network, containing controlled sources (see below) as well. The qualitative properties of such a network can be described by matroids, see [10, 15, 16], but these matroids will not always be graphic.

For example, both networks of Figure 6 contain two current sources; in the first network both sources are "usual" ones but in the second network I_2 is not, since its describing system of equations is $\{i_2 = ki_3, u_2$ is arbitrary$\}$ rather than $\{i_2$ is given, u_2 is arbitrary$\}$. Such a device is called a *current controlled current source.*

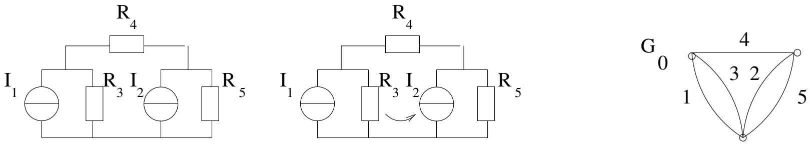


Fig. 6

In the first network (or in any network without controlled sources) the currents of certain devices can be prescribed independently of each other if and only if the subset of the corresponding edges contains no cut sets (that is, if the complement of this set is connected). Even the resistor R_4 could be replaced by a further independent current source I_4 since the edge set $\{1,2,4\}$ is cut set free. In general, the maximal subsets of edges whose currents can be prescribed independently, are just the complements of the spanning trees of the network graph G_0 (that is, they form the bases of the dual of the circuit matroid $\mathscr{G}_0$ of G_0).

However, in the second network at most two currents can be prescibed independently, for example, I_1 and I_4 (or any pair, except $\{2,3\}$). It is still true that the maximal subsets of edges whose currents can be prescribed independently form the bases of the dual of a matroid but this matroid is not $\mathscr{G}_0$ but the union (or sum) of matroids $\mathscr{G}_0 \vee \mathscr{A}$, see [10, 15, 16] where a subset of edges is independent in $\mathscr{A}$ if and only if it is either the empty set or one of the single-element sets $\{2\}, \{3\}$.

This matroid $\mathscr{X} = \mathscr{G}_0 \vee \mathscr{A}$ is not graphic (it is a single element series extension of the uniform matroid $\mathscr{U}_{4,2}$), in fact, this is the smallest example to illustrate that matroids, rather than graphs, are the proper tools for the study of the qualitative properties of active linear networks. The general question (when will the sum of two graphic matroids be graphic) seems to be difficult (see [13, 18] for some partial results). The following small "Kuratowski-type" observation, announced in [17] without proof, may be interesting from the physical point of view. Like in Kuratowski's theorem, series extension (or subdivision) allows iterations of single element series extensions as well.

Theorem 6. *The matroid $\mathscr{A}$ (describing the controll) is the circuit matroid of a graph consisting of two parallel edges e_2, e_3 and loops for every other edge $e_1, e_4, e_5, \ldots$. The sum $\mathscr{M}(G) \vee \mathscr{A}$ of this matroid and the circuit matroid of a graph*

G is graphic if and only if G does not contain the graph G_0 of Figure 6 or its series extension as a subgraph (with the two non-loop elements of $\mathscr{A}$ in the positions 2, 3).

Proof: Suppose that G contains a subgraph isomorphic to G_0 or to its series extension. Then the sum of $\mathscr{A}$ and the circuit matroid of this subgraph is a series extension of the uniform matroid $\mathscr{U}_{4,2}$, hence it is not graphic.

On the other hand, suppose that $\mathscr{M}(G) \vee \mathscr{A}$ is not graphic for some graph G containing two specific edges e_2, e_3 corresponding to the non-loop elements in $\mathscr{A}$. We may suppose that G is connected or, at least, e_2 and e_3 are in the same connected component of G for otherwise the sum would be the circuit matroid of the graph illustrated on the right in Figure 7.

Fig. 7

Moreover, one can see using very similar arguments that these edges must not be loops or parallel to each other. This latter is illustrated by Figure 8.

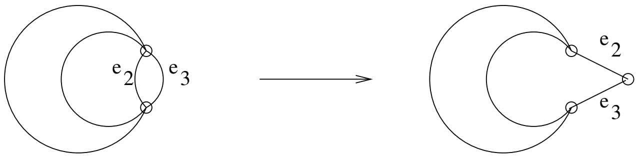


Fig. 8

If these edges were series (that is, if $\{e_2, e_3\}$ were a cut set) then the sum would be the circuit matroid of the graph of $G - \{e_2, e_3\}$ extended with these edges as bridges. Hence we may conclude that there is a circuit in G containing both of these edges and also there are circuits containing exactly one of these edges. Therefore G contains the graph of Figure 9 as a subgraph (where some of the paths P_1, P_2, P_4 and P_5 may be of length 0 as well).

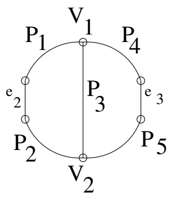


Fig. 9

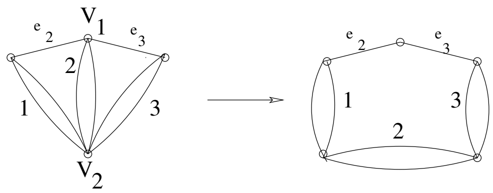


Fig. 10

Consider now two internal vertices from two distinct paths among $P_1, \ldots, P_5$. If all the paths connecting such pairs in G are covered by the vertices V_1, V_2 of Figure 9 then, within 2-isomorphism, G looks like the first graph of Figure 10 and then the sum is graphic, as illustrated by the second graph of Figure 10.

Hence we may suppose that there is a path connecting an internal vertex of, say, P_1 with an internal vertex of another path P_i without sharing any other vertex with the subgraph of Figure 9. Then one can find the requested subgraph in every case $i = 2, 3, 4$ or 5, see Figure 11. $\square$

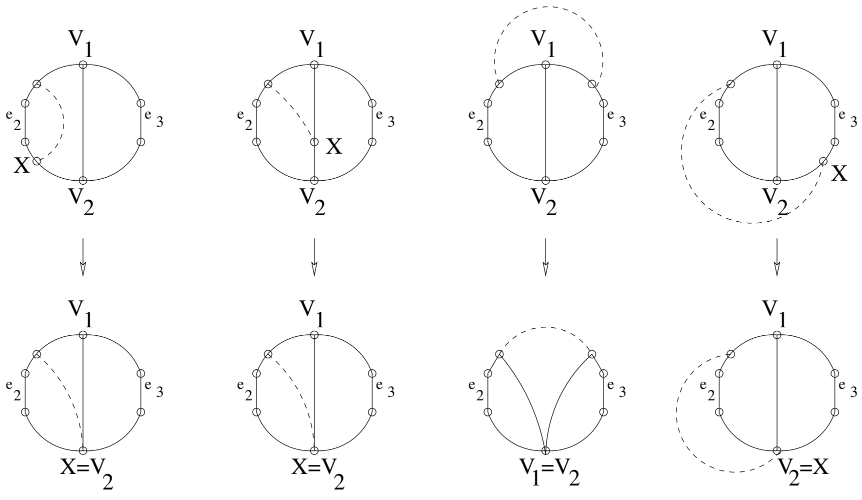


Fig. 11

One may interpret this result as follows: if a network contains a single controlled source then the describing matroid will not be graphic if and only if the network has a feedback (edge 4 in the graph of Figure 6).

Incidentally, the above matroid $\mathscr{X}$ is a sixth-root-of-unity matroid but it is not regular. This latter statement is obvious (it has $\mathscr{U}_{4,2}$ as a minor), for the former one observe that the following matrix coordinatizes it over GF(3) or GF(4) if x is an element in the appropriate field, differing both from 0 and 1.

$$\begin{pmatrix} 1 & x & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

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